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A Generalized Benders Decomposition Method for Mixed-Integer Nonlinear Programming: Theory and Application

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ABSTRACT

The Mixed-Integer Nonlinear Programming (MINLP) model is an optimization model involving integer and continuous variables with nonlinear objectives or constraints. One method to solve the MINLP model is the Generalized Benders Decomposition (GBD) method. The GBD method decomposes the problem into primal and master problems that are solved alternately until the optimal solution is found. This paper comprehensively explains how to solve the MINLP models using the GBD method, provides detailed proofs of theorems related to GBD that were not fully addressed in previous literature, and presents the application of the GBD method to solving real-world MINLP problems. The results show that the theorems related to GBD were successfully proven, and the practical MINLP problems were solved using the GBD method, demonstrating how the method can be effectively used to solve real-world MINLP problems. Overall, this research contributes theoretically and practically to understanding the GBD method and its application in solving the MINLP optimization problems.

Keywords: Optimization; Mixed-Integer Nonlinear Programming; Generalized Benders Decomposition

INTRODUCTION

Optimization is the process of finding the best solution for specific situations [1]. An optimization model is a mathematical representation of real-life situations aimed at decision-making through objective and constraint functions. The classes of optimization problems are categorized based on the types of decision variables and functions involved including linear programming problems, nonlinear programming problems, nonlinear programming with integer variables, mixed-integer nonlinear programming, and others[2]. The Mixed-Integer Nonlinear Programming (MINLP) model is an optimization model involving integer and continuous variables with nonlinear objective or constraint functions [3]. MINLP models are standard in various applications such as oil and gas [4], geothermal energy [5], natural gas [6], transportation [7], energy systems [8], and others.

The MINLP models are notoriously difficult to solve since they involve both discrete and continuous variables. For certain MINLP problems that are highly complex, commercial software

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may face difficulties or even fail to solve these problems [9]. The Generalized Benders Decomposition (GBD) method effectively addresses the complexity of MINLP problems. GBD enchances the Benders Decomposition (BD) method for mixed-integer linear problems (MILP) introduced by Benders [10]. This method decomposes the problem into more minor problems by separating the problem into a master problem, which contains difficult-to-solve integer variables, and a primal problem, which only involves continuous variables that are easier to solve. During the solution process, the primal and master problems are tackled in turns until the optimal solution is found.

Geoffrion [11] adapted the BD method for MINLP problems, known as the GBD method. The primary distinction between the GBD and BD approaches is that GBD uses nonlinear duality in the master problem so that GBD can solve the nonlinear problem, whereas BD applies linear duality. Floudas [12] improved GBD by adding binary variables and equality constraints. The disadvantage of this method modified by Floudas is that the constraints of the equation are not relaxed into the form of inequalities so that it may lead to a dead end or not obtaining an optimal solution. In addition, the disadvantage of GBD explained by Geoffrion [11] and Floudas [12] is that it does not consider the convex problem related to all variables and adds linear constraints to the master problem, which is the easiest way to implement.

Furthermore, Karbowski [13] refined the discussion on the GBD method for solving MINLP problems. In this work, Karbowski revisited the method by considering the relaxation of equality constraints and adding linear constraints to the master problem for cases involving convex objective functions and constraints. In [13], there are several theorems related to gbd. However, the proofs for three out of six theorems are not presented [13]. In addition, [13] did not provide examples of solving MINLP models using the GBD method. From the previous literature on the GBD method, Geoffrion [11] provides a step-by-step explanation of the technique, with a detailed proof for one theorem and an application of GBD. However, the other two theorems are not fully addressed. Floudas [12] also offers a guide and an example but does not present the proof for these three theorems. Karbowski presents the method thoroughly but does not provide evidence for these three theorems or an instance of its application.

The need for discussions on solving MINLP models using the GBD method has increased alongside the growing number of practical MINLP optimization models that can be addressed using this approach, such as telecommunication [15,16], transportation [16], energy [18,19], portfolio management [19], and others. Therefore, this paper explains how to solve MINLP models using the GBD method and provides detailed proofs of three theorems related to GBD, which were not fully addressed in previous literature. Additionally, the application of the GBD method is illustrated to demonstrate a step-by-step approach to solving the MINLP model, highlighting its effectiveness in addressing real-world MINLP problems. The organization of the paper is as detailed below: The Methods section discusses the MINLP model and the GBD method. The Results and Discussions section addresses the proofs of three theorems related to GBD and the application of the GBD method the application of the GBD method for solving MINLP problem. The Conclusions section provides the final remarks.

METHODS

Mixed-Integer Nonlinear Programming (MINLP)

The MINLP problem is an optimization problem characterized by nonlinear objective or constraint functions, and the decision variables include continuous and integer values [3]. Consider the problem (1)-(4) as expressed by [13]:

$$\min_{x,v} f(\mathbf{x}, \mathbf{v}) \tag{1}$$

$$s.t \quad \mathbf{g}(\mathbf{x}, \mathbf{v}) \leq \mathbf{0} \tag{2}$$

$$\mathbf{x} \in X \subseteq \mathbb{R}^n \tag{3}$$

$$\mathbf{v} \in V \subseteq \mathbb{Z}^q, \tag{4}$$

where $\mathbf{x}$ and $\mathbf{v}$ donate the vector of continuous and integer decision variables, respectively, within the feasible sets $X \subseteq \mathbb{R}^n$ and $\mathbf{v} \in V \subseteq \mathbb{Z}^q$. Here, n and q specifies the count of continuous and integer decision variables respectively. Function $f: \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$ is the objective function, and $\mathbf{g}: \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^m$ is a constraint functions vector that contains individual constraints $g_i(\mathbf{x}, \mathbf{v})$. The functions $f(\mathbf{x}, \mathbf{v})$and $g_i(\mathbf{x}, \mathbf{v})$ can be either linear or nonlinear. When considering V as $V \subseteq \mathbb{Z}^q$, problems (1)-(4) contain decision variables with both continuous and integer values, thereby classifying them as MINLP problems. The equality constraints can be transformed into inequality forms, becoming inequality pair [20]. For an equality constraint of the form $g_i(\mathbf{x}, \mathbf{v}) = 0$, it can be rewritten as an inequality pair $g_i(\mathbf{x}, \mathbf{v}) \leq 0$ and $g_i(\mathbf{x}, \mathbf{v}) \geq 0$.

Generalized Benders Decomposition (GBD)

GBD is a conceptual framework that contains steps for solving MINLP problems using decomposition principles[12]. This method decomposes the MINLP problem into smaller problems by separating the problem into a master problem, which contains integer variables that are difficult to solve, and a primal problem, which only involves continuous variables that are easier to solve. In the iterative process, the primal and the master problems are solved alternately until the objective function converges to the same or nearly the same value, indicating that the solution is close to the actual optimal solution.

Karbowski [13] explained the completion of the minimization MINLP model (1)-(4) with $\mathbf{v} \in V \subseteq \mathbb{Z}^q$ using the GBD Method. The primal problem is the solution of the initial problem (1)-(4) with a fixed **v**, i.e., $\mathbf{v} = \mathbf{v}^* \in V$. It can be rewritten as:

$$\min_{\mathbf{x} \in X} \{f(\mathbf{x}, \mathbf{v}^*): \mathbf{g}(\mathbf{x}, \mathbf{v}^*) \leq \mathbf{0}\}. \tag{5}$$

The master problem in GBD involves the variable **v**, which is more difficult to solve. The problem (1)-(4) are expressed in the following manner:

$$\min_{\mathbf{v} \in V} \left[\inf_{\mathbf{x} \in X} f(\mathbf{x}, \mathbf{v}) : \mathbf{g}(\mathbf{x}, \mathbf{v}) \leq \mathbf{0}\right]. \tag{6}$$

Let

$$z(\mathbf{v}) = \inf_{\mathbf{x} \in X} \{f(\mathbf{x}, \mathbf{v}): \mathbf{g}(\mathbf{x}, \mathbf{v}) \leq \mathbf{0}\}. \tag{7}$$

In problem (6) **v** must lie within the domain V_0 where $V_0 = \{\mathbf{v}: \exists \mathbf{x} \in X \ni \mathbf{g}(\mathbf{x}, \mathbf{v}) \leq \mathbf{0}\}$. The set V_0 is also known as the solvability set. Therefore, problems (6)-(7) are expressed as:

$$\min_{\mathbf{v} \in V \cap V_0} z(\mathbf{v}) \tag{8}$$

$$V_0 = \{\mathbf{v}: \exists \mathbf{x} \in X \ni \mathbf{g}(\mathbf{x}, \mathbf{v}) \leq \mathbf{0}\}. \tag{9}$$

The requirement $\mathbf{v} \in V \cap V_0$ is necessary to ensure that $z(\mathbf{v})$ is defined and has a valid results. Problem (8)-(9) are called as the master problem.

In [13], there are several theorems which related to V_o and $z(\mathbf{v})$. However, [13] this does not present proof for these theorems. The following section presents proofs for these theorems.

Theorem 1. *Assume X is a non-empty convex set and the function g is convex in X for each fixed value $\mathbf{v} \in V$. Also, assume $Z_{\mathrm{v}} = \{\mathbf{z} \in \mathbb{R}^m : \exists \mathbf{x} \in X \ni \mathbf{g}(\mathbf{x}, \mathbf{v}) \leq \mathbf{z}\}$ is closed for each fixed value $\mathbf{v} \in V$. Then $\mathbf{v}^* \in V$ lies in V_0 if and only if*

$$\inf_{\mathbf{x}\in X} L_f(\mathbf{x}, \mathbf{v}^*, \boldsymbol{\lambda}) \leq 0, \quad \forall \boldsymbol{\lambda} \in \Lambda, \tag{10}$$

Where

$$\Lambda = \left\{ \boldsymbol{\lambda} \in \mathbb{R}^m : \boldsymbol{\lambda} \geq \mathbf{0}, \sum_{j=1}^{m} \lambda_j = 1 \right\}, \tag{11}$$

$$L_f(\mathbf{x}, \mathbf{v}, \boldsymbol{\lambda}) = \boldsymbol{\lambda}^T \mathbf{g}(\mathbf{x}, \mathbf{v}). \tag{12}$$

Theorem 2. *Assume X is a non-empty convex set and the function g is convex in X for each fixed value $\mathbf{v} \in V$. Also, assume for $\mathbf{v}^*$, at least one of the following conditions holds:*

1. *$z(\mathbf{v}^*)$ is bounded and the problem (7) has an optimal vector of Lagrange multipliers;*
2. *$z(\mathbf{v}^*)$ is bounded, , $\mathbf{g}(\mathbf{x}, \mathbf{v}^*)$ and $f(\mathbf{x}, \mathbf{v}^*)$ are continuous in X, X is a closed set, and the set of optimal solutions for problem (7) with accuracy $\varepsilon \geq 0$ is non-empty and bounded,*

Then

$$z(\mathbf{v}) = \sup_{\boldsymbol{\lambda}\geq 0} \inf_{\mathbf{x}\in X} L_0(\mathbf{x}, \mathbf{v}, \boldsymbol{\lambda}), \qquad \forall \mathbf{v} \in V \cap V_0 \tag{13}$$

Where

$$L_0(\mathbf{x}, \mathbf{v}, \boldsymbol{\lambda}) = f(\mathbf{x}, \mathbf{v}) + \boldsymbol{\lambda}^T \mathbf{g}(\mathbf{x}, \mathbf{v}). \tag{14}$$

Using Theorem 1, Theorem 2, and the interpretation of supremum as the least upper limit, the master problem (8)-(9) can be expressed as:

$$\min_{\mathbf{v}\in V, \mu} \quad \mu \tag{15}$$

$$s.t \quad \inf_{\mathbf{x}\in X} \; L_0(\mathbf{x}, \mathbf{v}, \boldsymbol{\lambda}) \leq \mu, \qquad \forall \boldsymbol{\lambda} \geq \mathbf{0} \tag{16}$$

$$\inf_{\mathbf{x}\in X} \; L_f(\mathbf{x}, \mathbf{v}, \boldsymbol{\lambda}) \leq 0, \qquad \forall \boldsymbol{\lambda} \in \Lambda. \tag{17}$$

The master problem (15)-(17) has many constraints because constraint (16) holds for all $\boldsymbol{\lambda} \geq \mathbf{0}$ and constraint (17) holds for all $\boldsymbol{\lambda} \in \Lambda$. To solve it, relax the master problem by first involving only some of the constraints in (16) and (17). The master problem (15)-(17) in its relaxed form is as follows:

In the nonlinear version:

$$\min_{\mathbf{v}\in V, \mu} \quad \mu \tag{18}$$

$$s.t \quad L_0(\mathbf{x}^l, \mathbf{v}, \boldsymbol{\lambda}^l) \leq \mu, \qquad l \in K_0 \tag{19}$$

$$L_f(\mathbf{x}^l, \mathbf{v}, \boldsymbol{\lambda}^l) \leq 0, \quad l \in K_f \tag{20}$$

In the linear version:

$$\min_{\mathbf{v}\in V,\mu} \quad \mu \tag{21}$$

$$s.t \quad L_0(\mathbf{x}^l,\mathbf{v},\boldsymbol{\lambda}^l)+\frac{\partial L_0^T}{\partial \mathrm{v}}(\mathbf{x}^l,\mathbf{v}^l,\boldsymbol{\lambda}^l)(\mathbf{v}-\mathbf{v}^l)\le \mu, \quad l\in K_0 \tag{22}$$

$$L_f(\mathbf{x}^l,\mathbf{v},\boldsymbol{\lambda}^l)+\frac{\partial L_0^T}{\partial \mathbf{v}}(\mathbf{x}^l,\mathbf{v}^l,\boldsymbol{\lambda}^l)(\mathbf{v}-\mathbf{v}^l)\le 0, \quad l\in K_f \tag{23}$$

with $\mathbf{x}^l, \mathbf{v}, \boldsymbol{\lambda}^l$ being the optimal values at the l-th iteration obtained from solving the primal problem. The set of K_0 contains an iteration index where the optimal solution of the primal problem is successfully found, while the set of K_f contains an iteration index where the optimal solution of the primal problem is found through the feasibility problem.

The procedure for determining the optimal solution through the primal and master problem is as follows:

Initiation Stage:
Choosing an initial point $\mathbf{v}^0 \in V$ that is feasible for the primal problem (5), $K_0 = \emptyset, K_f = \emptyset, k = 0, UBD = \infty$, and a convergence tolerance $\varepsilon > 0$.

Iteration Stage:
Step 1: For a value $\mathbf{v} = \mathbf{v}^k$, solve the primal problem (5), set $k = k + 1$.

a) If the primal problem is feasible, note that $\mathbf{x}^k$ is the optimal value obtained, and the corresponding $\boldsymbol{\lambda}^k$ is the optimal Lagrange multiplier vector. Set $K_0 = K_0 \cup \{k\}$, which means adding a constraint of type (16) to the master problem. Revise $UBD = \min\{UBD, z(\mathrm{v}^k)\}$. If the upper bound (UBD) has improved, set the pair $(\mathbf{x}^k, \mathbf{v}^k)$ as the current most effective solution.
b) If the primal problem is infeasible, solve the feasibility problem. Note that $\mathbf{x}^k$ is the optimal solution obtained and $\boldsymbol{\lambda}^k$ is the corresponding optimal Lagrange multiplier vector. Set $K_f = K_f \cup \{k\}$, which means adding a constraint of type (17) to the master problem.

Step 2: Solving the relaxed master problem, in the nonlinear version, following (18)-(20), or in the linear version, following (21)-(23). Suppose $(\mathbf{v}^k, \boldsymbol{\lambda}^k)$ is the optimal solution of the relaxed master problem, then μ^k is a lower estimate of the initial problem. Set $LBD = \mu^k$. If $UBD - LBD \le \varepsilon$, the optimal solution is found, and the iteration stops. If $UBD - LBD > \varepsilon$, go back to Step 1.

RESULTS AND DISCUSSION

Proving Karbowski's Theorem

There are three theorems related to GBD in [13]. However, the proofs for these three theorems are not presented. Therefore, this study presents the proofs for these three theorems.

Theorem 1 addresses the dual representation of V_0 establishes a condition under which a particular integer decision variable leads to a solution that meets all constraints defined by $\mathbf{g}(\mathbf{x}, \mathbf{v})$. This provides a foundational criterion for the feasibility of solutions in optimization contexts where convex functions are involved.

Theorem 1. *Assume X is a non-empty convex set and the function g is convex in X for each fixed value $\mathbf{v} \in V$. Also, assume $Z_{\mathrm{v}} = \{\mathbf{z} \in \mathbb{R}^m : \exists \mathbf{x} \in X \ni \mathbf{g}(\mathbf{x}, \mathbf{v}) \le \mathbf{z}\}$ is closed for each fixed value $\mathbf{v} \in V$. Then $\mathbf{v}^* \in V$ lies in V_0 if and only if*

$$\inf_{\mathbf{x} \in X} L_f(\mathbf{x}, \mathbf{v}^*, \boldsymbol{\lambda}) \le 0, \quad \forall \boldsymbol{\lambda} \in \Lambda, \tag{24}$$

Where

$$\Lambda = \left\{ \boldsymbol{\lambda} \in \mathbb{R}^m : \boldsymbol{\lambda} \ge \mathbf{0}, \sum_{j=1}^{m} \lambda_j = 1 \right\}, \tag{25}$$

$$L_f(\mathbf{x}, \mathbf{v}, \boldsymbol{\lambda}) = \boldsymbol{\lambda}^T \mathbf{g}(\mathbf{x}, \mathbf{v}). \tag{26}$$

Proof.

(⇒) Suppose $\mathbf{v}^* \in V_0$, then $\exists \mathbf{x} \in X \ni \mathbf{g}(\mathbf{x}, \mathbf{v}^*) \le \mathbf{0}$. Therefore, $L_f(\mathbf{x}, \mathbf{v}^*, \boldsymbol{\lambda}) = \boldsymbol{\lambda}^T \mathbf{g}(\mathbf{x}, \mathbf{v}^*) \le \mathbf{0}, \forall \boldsymbol{\lambda} \in \Lambda$, because $\boldsymbol{\lambda}^T \ge 0$ and $\mathbf{g}(\mathbf{x}, \mathbf{v}^*) \le \mathbf{0}$. Thus, $\inf_{\mathbf{x} \in X} L_f(\mathbf{x}, \mathbf{v}^*, \boldsymbol{\lambda}) \le 0, \;\; \forall \boldsymbol{\lambda} \in \Lambda$.

(⇐) Suppose $\mathbf{v}^*$ satisfies (10)-(12),

$$\inf_{\mathbf{x} \in X} L_f(\mathbf{x}, \mathbf{v}^*, \boldsymbol{\lambda}) = \inf_{\mathbf{x} \in X} \boldsymbol{\lambda}^T \mathbf{g}(\mathbf{x}, \mathbf{v}^*) \le 0, \quad \forall \boldsymbol{\lambda} \in \Lambda \tag{27}$$

Then

$$\sup_{\boldsymbol{\lambda} \ge \mathbf{0}} \left[\inf_{\mathbf{x} \in X} \boldsymbol{\lambda}^T \mathbf{g}(\mathbf{x}, \mathbf{v}^*) \right] \le 0 \tag{28}$$

so that

$$\sup_{\boldsymbol{\lambda} \ge \mathbf{0}} \left[\inf_{\mathbf{x} \in X} \boldsymbol{\lambda}^T \mathbf{g}(\mathbf{x}, \mathbf{v}^*) \right] = 0, \tag{29}$$

Equation (29) represents the duality of the convex programming problem

$$\min_{\mathbf{x} \in X} \; 0^T \mathbf{x} \;\; s.t \; \mathbf{g}(\mathbf{x}, \mathbf{v}^*) \le \mathbf{0} \tag{30}$$

which has a minimum value of 0. Given that Z_v is closed and the value of the dual problem (29) is finite, the primal problem (30) is feasible[11], which means the constraint $\mathbf{g}(\mathbf{x}, \mathbf{v}^*) \le \mathbf{0}$ is satisfied, so $\mathbf{v}^* \in V_0$ □

Theorem 2 deals with the dual representation of $z(\mathbf{v})$. It states that under certain conditions, the optimal value $z(\mathbf{v}^*)$ can be expressed as a supremum of infima, providing a pathway to determine optimal strategies in MINLP. This relationship is crucial for establishing optimality conditions in scenarios where convexity is present.

Theorem 2. *Assume X is a non-empty convex set and the function g is convex in X for each fixed value $\mathbf{v} \in V$. Also, assume for $\mathbf{v}^*$, at least one of the following conditions holds:*

1. *$z(\mathbf{v}^*)$ is bounded and the problem (7) has an optimal vector of Lagrange multipliers;*
2. *$z(\mathbf{v}^*)$ is bounded, , $\mathbf{g}(\mathbf{x}, \mathbf{v}^*)$ and $f(\mathbf{x}, \mathbf{v}^*)$ are continuous in X, X is a closed set, and the set of optimal solutions for problem (7) with accuracy $\varepsilon \ge 0$ is non-empty and bounded,*

Then

$$z(\mathbf{v}) = \sup_{\boldsymbol{\lambda} \ge 0} \inf_{\mathbf{x} \in X} L_0(\mathbf{x}, \mathbf{v}, \boldsymbol{\lambda}), \qquad \forall \mathbf{v} \in V \cap V_0 \tag{31}$$

Where

$$L_0(\mathbf{x},\mathbf{v},\boldsymbol{\lambda}) = f(\mathbf{x},\mathbf{v}) + \boldsymbol{\lambda}^T\mathbf{g}(\mathbf{x},\mathbf{v}). \quad (32)$$

Proof. Considering equation (7). For $\mathbf{v} = \mathbf{v}^*$, equation (7) becomes

$$\begin{aligned} z(\mathbf{v}^*) = & \inf_{\mathbf{x}\in X} f(\mathbf{x},\mathbf{v}^*) \\ s.t \quad & \mathbf{g}(\mathbf{x},\mathbf{v}^*) \le \mathbf{0}. \end{aligned} \quad (33)$$

Given that X is non-empty and convex, f and g are convex on X, it is obtained

$$z(\mathbf{v}^*) = \inf_{\mathbf{x}\in X} L_0(\mathbf{x},\mathbf{v}^*,\boldsymbol{\lambda}) = \inf_{\mathbf{x}\in X}[\,f(\mathbf{x},\mathbf{v}^*) + \boldsymbol{\lambda}^T\mathbf{g}(\mathbf{x},\mathbf{v}^*)]. \quad (34)$$

The dual of the equation above can be written as:

$$\sup_{\boldsymbol{\lambda}\ge\mathbf{0}} \inf_{\mathbf{x}\in X} [\,f(\mathbf{x},\mathbf{v}^*) + \boldsymbol{\lambda}^T\mathbf{g}(\mathbf{x},\mathbf{v}^*)]. \quad (35)$$

(i) If condition 1 is satisfied

Since $z(\mathbf{v}^*)$ is bounded and there exists an optimal Lagrange multiplier vector for $z(\mathbf{v}^*)$. This Lagrange multiplier vector is also an optimal Lagrange multiplier for its dual, and the optimal solutions for the primal and dual will be the same.

(ii) If condition 2 is satisfied

Since X is closed, f and $\mathbf{g}$ are continuous on X, and $z(\mathbf{v}^*)$ is bounded, the dual and the primal values will be the same [11].

Based on (i) and (ii), the dual solution obtained will be the same as the primal solution, resulting in:

$$z(\mathbf{v}^*) = \sup_{\boldsymbol{\lambda}\ge\mathbf{0}} \inf_{\mathbf{x}\in X} [\,f(\mathbf{x},\mathbf{v}^*) + \boldsymbol{\lambda}^T\mathbf{g}(\mathbf{x},\mathbf{v}^*)]. \quad (36)$$

The value $\mathbf{v}^*$ applies to all possible fixed values of $\mathbf{v}$, hence

$$z(\mathbf{v}) = \sup_{\boldsymbol{\lambda}\ge\mathbf{0}} \inf_{\mathbf{x}\in X}[f(\mathbf{x},\mathbf{v}) + \boldsymbol{\lambda}^T\mathbf{g}(\mathbf{x},\mathbf{v})], \qquad \forall\mathbf{v}\in V\cap V_0 \quad (37)$$

or,

$$z(\mathbf{v}) = \sup_{\boldsymbol{\lambda}\ge\mathbf{0}} \inf_{\mathbf{x}\in X} L_0(\mathbf{x},\mathbf{v},\boldsymbol{\lambda}), \forall\mathbf{v}\in V\cap V_0, \quad (38)$$

where

$$L_0(\mathbf{x},\mathbf{v},\boldsymbol{\lambda}) = f(\mathbf{x},\mathbf{v}) + \boldsymbol{\lambda}^T\mathbf{g}(\mathbf{x},\mathbf{v}) \;\Box \quad (39)$$

Theorem 3 characterizes the projection of problems (1)-(4) into (8)-(9). It demonstrates the interconnectedness of solutions between the original MINLP problem and its projections.

Theorem 3. *For the projection from (1)-(4) to (8)-(9)., the following holds:*

1. *The problem (1)-(4) has no solution or is unbounded if and only if the same is true for the problem (8)-(9).*
2. *If $(\hat{\mathbf{x}},\hat{\mathbf{v}})$ is an optimal solution to the problem (1)-(4), then $\hat{\mathbf{v}}$ is an optimal solution to the problem (8)-(9).*
3. *If $\hat{\mathbf{v}}$ is an optimal solution to the problem (8)-(9), and $\hat{\mathbf{x}}$ attains the infimum in the problem (7) when $\mathbf{v} = \hat{\mathbf{v}}$ then $(\hat{\mathbf{x}},\hat{\mathbf{v}})$ is an optimal solution to the problem (1)-(4).*

Proof.

(i) It will be proven that the problem (1)-(4) has no solution or is unbounded if and only if the same holds for the problem (8)-(9).

(⇒) Suppose (1)-(4) has no solution. This means there is no $(\mathbf{x}, \mathbf{v})$ that satisfies the constraint $\mathbf{g}(\mathbf{x}, \mathbf{v}) \leq \mathbf{0}$. Consequently, $\forall \mathbf{v} \in V$, there is no $\mathbf{x} \in X$ that satisfies $\mathbf{g}(\mathbf{x}, \mathbf{v}) \leq \mathbf{0}$, resulting in $V_0 = \emptyset$. Therefore, the problem (8)-(9) has no solution. Suppose (1)-(4) is unbounded, this means that $\forall M > 0, \exists \mathbf{x} \in X, \mathbf{v} \in V \ni f(\mathbf{x}, \mathbf{v}) < M$. Since the problem (8) has the same objective function as the problem (1)-(4) with a smaller set V, i.e., $\mathbf{v} \in V \cap V_0$ which is a subset of V, it is clear that $\forall M > 0, \exists\ \mathbf{v} \in V \cap V_0 \ \ni z(\mathbf{v}) < M$, making the problem (8)-(9) also unbounded.

(⇐) Suppose (8)-(9) has no solution. This means there is no $\mathbf{v} \in V \cap V_0$. Consequently, there is no $(\mathbf{x}, \mathbf{v})$ that satisfies the constraint $\mathbf{g}(\mathbf{x}, \mathbf{v}) \leq \mathbf{0}$ solution, resulting in the problem (1)-(4). Suppose (8)-(9) is unbounded. This means $\forall M > 0, \exists \mathbf{x} \in X, \mathbf{v} \in V \ni f(\mathbf{x}, \mathbf{v}) < M$. Since $V_0 = \{\mathbf{v}: \exists \mathbf{x} \in X \ni \mathbf{g}(\mathbf{x}, \mathbf{v}) \leq \mathbf{0}\}$, there will be $\mathbf{x} \in X\ \forall \mathbf{v} \in V_0$ such that $f(\mathbf{x}, \mathbf{v}) < M$. Therefore, the problem (1)-(4) is also unbounded.

(ii) It will be proven that if $(\hat{\mathbf{x}}, \hat{\mathbf{v}})$ is an optimal solution for the problem (1)-(4), then $\hat{\mathbf{v}}$ is an optimal solution for the problem (8)-(9). Suppose $(\hat{\mathbf{x}}, \hat{\mathbf{v}})$ is an optimal solution for the problem (1)-(4), then $(\hat{\mathbf{x}}, \hat{\mathbf{v}})$ satisfies $\mathbf{g}(\hat{\mathbf{x}}, \hat{\mathbf{v}}) \leq \mathbf{0}$. By definition of $V_0, \hat{\mathbf{v}} \in V_0$. Since $(\hat{\mathbf{x}}, \hat{\mathbf{v}})$ is an optimal solution for the problem (1)-(4), we have $f(\hat{\mathbf{x}}, \hat{\mathbf{v}}) \leq f(\mathbf{x}, \hat{\mathbf{v}})$ for every $\mathbf{x} \in X$. Based on the definition of $z(\hat{\mathbf{v}})$, for every $\mathbf{v} \in V \cap V_0$ and $\mathbf{x} \in X$, $z(\hat{\mathbf{v}}) \leq f(\hat{\mathbf{x}}, \hat{\mathbf{v}}) \leq f(\mathbf{x}, \hat{\mathbf{v}})$. This means $\hat{\mathbf{v}}$ minimizes $z(\mathbf{v})$ over $V \cap V_0$, so $\hat{v}$ is an optimal solution for the problem (8)-(9).

(iii) If $\hat{\mathbf{v}}$ is an optimal solution for the problem (8)-(9) and $\hat{\mathbf{x}}$ reaches the infimum in problem (7) when $\mathbf{v} = \hat{\mathbf{v}}$, then $(\hat{\mathbf{x}}, \hat{\mathbf{v}})$ is an optimal solution for the problem (1)-(4). Suppose $\hat{\mathbf{v}}$ is an optimal solution for the problem (8)-(9), then $\hat{\mathbf{v}} \in V$ and $\hat{\mathbf{v}} \in V_0$. Since $\hat{\mathbf{v}} \in V_0, \exists \mathbf{x} \in X \ni \mathbf{g}(\mathbf{x}, \hat{\mathbf{v}}) \leq \mathbf{0}$. Also, suppose $\hat{\mathbf{x}}$ reaches the infimum in problem (7) when $\mathbf{v} = \hat{\mathbf{v}}$, then $\forall \mathbf{x} \in X, f(\hat{\mathbf{x}}, \hat{\mathbf{v}}) \leq f(\mathbf{x}, \hat{\mathbf{v}})$ and $(\hat{\mathbf{x}}, \hat{\mathbf{v}})$ satisfies $\mathbf{g}(\hat{\mathbf{x}}, \hat{\mathbf{v}}) \leq \mathbf{0}$. Therefore, $(\hat{\mathbf{x}}, \hat{\mathbf{v}})$ is an optimal solution for the problem (1)-(4). Based on (i), (ii), and (iii), Theorem 3 is proven □

The Application of the GBD Method for Solving MINLP Models

Consider the following problem:

A production company produces two varieties of products, Product A and Product B, measured in meters. The company must consider the production costs and available resources to produce these products. The production cost for each product depends on the number of meters produced and the choice of production method. The production cost for Product A is calculated as 0.3 times the square of the number of meters produced. The production cost for Product B is calculated as 1.5 times the square of the number of meters produced. Additionally, if the company decides to use a faster production method, there will be an additional fixed cost of 20 for Product A and 30 for Product B. The company has several constraints that must be considered:

- The total number of meters produced for Product A and Product B must be at least 170.
- The company can produce up to 150 meters of Product A and 100 meters of Product B.
- If the company chooses the faster production method for Product A, the number of meters produced must be at least 50 meters. Similarly, if the speedier production method is chosen for Product B, the total number of meters produced must be at

least 30.

- At least one of the products, either A or B, must use the faster production method.

How many meters of Product A and Product B should the company produce, and should it use the faster production method for each product, to minimize the total production cost?

Define the following variables:
x_1 : the quantity of Product A produced (in meters)
x_2 : the quantity of Product B produced (in meters)
v_1 : 1 if the faster production method is used for Product A, 0 otherwise
v_2 : 1 if the faster production method is used for Product B, 0 otherwise

The optimization model for the above problem is as follows:

$$\begin{aligned} &\min 0.3x_1^2 + 1.5x_2^2 + 20v_1 + 30v_2 \\ &s.t\ \ x_1 + x_2 \geq 170, \\ &\quad x_1 \leq 150, \\ &\quad x_2 \leq 100, \\ &\quad x_1 \geq 50v_1, \\ &\quad x_2 \geq 30v_2, \\ &\quad v_1 + v_2 \geq 1 \\ &\quad v_1, v_2 \in \{0,1\}, x \in \mathbb{R}. \end{aligned} \tag{40}$$

To simplify, transform problem (40) into the MINLP standard form (1)-(4) which only contains "≤" constraints. The problem (40) can be expressed as:

$$\min 0.3x_1^2 + 1.5x_2^2 + 20v_1 + 30v_2 \tag{41}$$

$$s.t\ \ 170 - x_1 - x_2 \leq 0, \tag{42}$$

$$x_1 - 150 \leq 0, \tag{43}$$

$$x_2 - 100 \leq 0, \tag{44}$$

$$50v_1 - x_1 \leq 0, \tag{45}$$

$$30v_2 - x_2 \leq 0, \tag{46}$$

$$1 - v_1 - v_2 \leq 0, \tag{47}$$

$$v_1, v_2 \in \{0,1\}, x \in \mathbb{R}. \tag{48}$$

The primal problem of problems (41)–(48) is part of the problem that depends only on the variable integer (**x**), with the value **v** already defined. Since constraint (47) only contains variable **v**, these can be removed and directly integrated into the master problem. The primal formulation obtained is:

$$\begin{aligned} &\min_{x_1, x_2} 0.3x_1^2 + 1.5x_2^2 + 20v_1 + 30v_2 \\ &s.t\ \ 170 - x_1 - x_2 \leq 0, \\ &\quad x_1 - 150 \leq 0, \\ &\quad x_2 - 100 \leq 0, \\ &\quad 50v_1 - x_1 \leq 0, \\ &\quad 30v_2 - x_2 \leq 0, \\ &\quad x \in \mathbb{R}, \end{aligned} \tag{49}$$

where v_1, v_2 are constants. The corresponding Lagrange function for the primal problem is:

$$L_0(\mathbf{x},\mathbf{v},\boldsymbol{\lambda}) = 0.3x_1^2 + 1.5x_2^2 + 20v_1 + 30v_2 + \lambda_1(170 - x_1 - x_2) + \lambda_2(x_1 - 150) \\ +\lambda_3(x_2 - 100) + \lambda_4(50v_1 - x_1 \leq 0) + \lambda_5(30v_2 - x_2) \tag{50}$$

The resulting relaxed master problem in nonlinear form follows (18)-(20), with the addition of constraint (47) as follows:

$$\begin{aligned} &\min_{\mathrm{v}\in V,\mu} \mu \\ &s.t\ \ L_0(\mathrm{x}^l, \mathrm{v}, \lambda^l) \leq \mu, \quad l \in K_0, \\ &\quad\ \ L_f(\mathrm{x}^l, \mathrm{v}, \lambda^l) \leq 0, \quad l \in K_f, \\ &\quad\ \ 1 - v_1 - v_2 \leq 0, \end{aligned} \tag{51}$$

Initiation Stage: Choosing an initial point $(v_1, v_2) = (1,1)$. Set $k = 0, K_0 = \emptyset, K_f = \emptyset, UBD = \infty$, and $\varepsilon = 0.1$.

Iteration Stage:
Iteration 1:

Step 1: Substituting $(v_1^1, v_2^1) = (1,1)$ into the primal problem (49). The optimal solution obtained is: $x_1^1 = 139.999, x_2^1 = 30, \lambda_1^1 = 83.999, \lambda_2^1 = 0,\ \lambda_3^1 = 0,\ \lambda_4^1 = 0, \lambda_5^1 = 6$, with an optimal value of 7280. Set $UBD = \min\{\infty, 7280) = 7280$ and $k = k + 1 = 1$. Since the optimal solution is directly obtained from the primal problem, set $K_0 = K \cup \{k\} = \{1\}$.
Step 2: Solve master problem:

$$\begin{aligned} &\min_{\mathrm{v}\in V,\mu} \mu \\ &s.t\ \ 7050 + 20v_1 + 210v_2 \leq \mu, \\ &\quad\ 1 - v_1 - v_2 \leq 0, \end{aligned} \tag{52}$$

The optimal solution obtained is: $(v_1^2, v_2^2) = (1,0), \mu = 7070$. Set $LBD = \mu = 7070$ and $UBD - LBD = 7280 - 7070 = 210 > 0.1$. Since $UBD - LBD > \varepsilon$, the iteration is continued.

Iteration 2:

Step 1: Substituting $(v_1^2, v_2^2) = (1,0)$ into the primal problem (49). The optimal solution obtained is: : $x_1^2 = 141.667, x_2^2 = 28.333, \lambda_1^2 = 85, \lambda_2^2 = 0,\ \lambda_3^2 = 0,\ \lambda_4^2 = 0, \lambda_5^1 = 0$, with an optimal value of 7245. Set $UBD = \min\{7280,7245\} = 7245$ and $k = k + 1 = 2$. Since the optimal solution is directly obtained from the primal problem, set $K_0 = K \cup \{k\} = \{1,2\}$.

Step 2: Solve the master problem:

$$\begin{aligned} &\min_{\mathrm{v}\in V,\mu} \mu \\ &s.t\ \ 7050 + 20v_1 + 210v_2 \leq \mu, \\ &\quad\ \ 7225 + 20v_1 + 30v_2 \leq \mu, \\ &\quad\ 1 - v_1 - v_2 \leq 0, \end{aligned} \tag{53}$$

The optimal solution obtained is: $(v_1^3, v_2^3) = (1,0), \mu = 7245$. Set $LBD = \mu = 7245$ and $UBD - LBD = 7245 - 7245 = 0 < 0.1$. Since $UBD - LBD \leq \varepsilon$, the iteration is stopped. The optimal solution obtained is $x_1 = 141.667, x_2 = 28.333, v_1 = 1, v_2 = 0$, achieve a result of the objective function of 7245.

Based on the optimal solution obtained, the company should produce 141.667 meters of Product A and 28.333 meters of Product B. To minimize costs, the faster production method should be applied to Product A, while it is unnecessary for Product B. By following this production plan, the total production cost will amount to 7245.

CONCLUSIONS

This paper explains how to solve the MINLP models using the GBD method. This approach breaks down the MINLP problem into smaller components by dividing it into a master problem, which includes challenging integer variables, and a primal problem, which only consists of continuous variables that are simpler to solve. The iterative process alternately solves the master and primal problems until the objective function converges to the same or nearly identical value, indicating that the solution is approaching the true optimal solution. This paper also provides detailed proofs of three theorems related to GBD, which were not fully addressed in previous literature. This contributes theoretically to understanding the GBD method in solving the MINLP optimization problems.

Furthermore, this paper presents the practical application of the GBD method in solving real-world MINLP problems. The results demonstrate that the GBD method consistently identifies optimal solutions that satisfy all constraints, showcasing its effectiveness in tackling complex MINLP challenges. This shows the potential of GBD to address various MINLP issues in the future.

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