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Resolving Efficient Dominating Sets for Predicting Soil Moisture and Potential of Hidrogen in Farming

Kamal Dliou[1], Adinda Putri Aziza[2], Dafik[3,4,*], Arika Indah Kristiana[2,3], Dwi Agustin Retnowardani[5], Marsidi[6]

[1]ENSA, Ibn Zohr University, Agadir, Morocco
[2]Department of Mathematics Educations, University of Jember, Indonesia
[3]PUI-PT Combinatorics and Graph, CGANT, University of Jember, Indonesia
[4]Department of Mathematics, University of Jember, Indonesia
[5]Department of Statistics, Universitas PGRI Argopuro Jember, Indonesia
[6]Department of Mathematics Education, Universitas PGRI Argopuro Jember, Indonesia

Email: d.dafik@unej.ac.id

ABSTRACT

This study aims to analyze the Resolving Efficient Dominating Set (REDS) and its application in optimizing horizontal farming systems. The focus is on effectively monitoring and managing critical factors such as soil moisture and pH, which significantly influence plant growth, health, and productivity. The research seeks to improve the efficiency of companion planting by strategically placing monitoring nodes and predicting soil conditions. The study employs graph theory, particularly the REDS concept, to optimize the placement of monitoring nodes in the farming system. REDS ensures that each vertex $v \in V(G) - D$ is dominated by exactly one vertex in D, with no adjacency between dominating vertices and distinct representations for all vertices. Additionally, the Spatial Temporal Graph Neural Network (STGNN) technique is utilized to model multi-step time series data, predicting future soil moisture and pH levels in companion farming systems. The integration of REDS and STGNN demonstrated the potential for precise management of soil conditions in horizontal farming. The REDS framework provided an optimal configuration for monitoring node placement, ensuring comprehensive coverage of the farming system. Meanwhile, STGNN accurately predicted soil moisture and pH trends, facilitating timely interventions and resource optimization in companion planting scenarios. The combination of REDS and STGNN offers a robust solution for monitoring and managing soil moisture and pH in horizontal farming systems. This approach enhances productivity and sustainability by enabling precise and efficient farming practices. The findings underscore the utility of integrating graph theory and machine learning in advancing agricultural technology.

Keywords: efficient dominating set; horizontal farming; soil moisture prodiction; time series forecasting; STGNN

INTRODUCTION

Indonesia is known as an agricultural country do to its extensive agricultural land, abundant agricultural products, and the fact that the majority of its population works in this sector. However, Indonesia's tropical climate causes soil humidity and pH

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levels to fluctuate, requiring an irrigation system to maintain ideal conditions for plants. Different plants need different humidity and pH levels, so automatic irrigation systems must be adjusted to the conditions of each plant. This adjustment can be achived by predicting soil moisture using graph theory.

Graph theory was first introduced by Leonhard Euler in 1736 to solve the Konigsberg Bridge problem. Euler described the problem in graph form, where four landmasses were the vertices and seven bridges were the edges [3]. This marked the beginning of the development of graph theory, which is an important branch of mathematics used to solve various problems involving vertices and edges [4]. Graph theory has since developed rapidly and has now many applications, including predicting soil moisture in agriculture [10].

Hakim et al. [5] combined the concepts of efficient domination sets and differentiating sets to develop the notion of differentiating efficient dominating sets for various combined graphs. This research addresses problem-solving using the concept of resolving efficient dominating sets, which examines graphs and produces theorems and their proofs. REDS is if every vertex $v \in V(G) - D$ dominated by exactly one vertex in D and no two vertices in D are adjacent either the representation of the vertex $v \in V(G)$ with respect to D is not the same. Minimum cardinality denoted as $\gamma_{re}(G)$. Hakim et al. [5] have conducted explored resolving efficient dominating sets in comb operation graphs, including $K_n \rhd C_3$, $K_n \rhd P_3$, $W_n \rhd P_3$, $W_n \rhd C_3$, and $S_n \rhd P_2$. Furthermore, Kusumawardani et al. (2022) investigated resolving efficient dominating set on graphs $P_m \rhd P_n$, $S_m \rhd P_n$, and $K_m \rhd P_n$. Some research about dominating set and resolving set can be seen at [1], [2], and [9].

Modern precision agriculture has seen advancements in integrating computational methods to optimize resource management. However, significant gaps remain, particularly in the precise monitoring of soil moisture and pH levels in horizontal farming systems. While graph theory, including dominating sets, has been widely used in communication networks and sensor deployment, its application in agricultural monitoring remains underexplored. Similarly, Spatial Temporal Graph Neural Networks (STGNNs) have shown promise in modeling spatio-temporal data for traffic and environmental systems, but their utilization for predicting soil properties in companion farming is still limited. Current approaches often fail to combine mathematical frameworks like efficient dominating sets with machine learning models, leaving real-time, holistic monitoring and prediction of critical agricultural variables unaddressed. This research addresses these gaps by developing a Resolving Efficient Dominating Set (REDS) framework for optimal placement of monitoring nodes, coupled with an STGNN model for accurate prediction of soil moisture and pH. By integrating these methodologies, the study aims to enhance productivity and sustainability in horizontal farming, providing a scalable and data-driven solution for modern agricultural practices [10].

The development of the REDS framework and its integration with STGNN builds upon several foundational studies in graph theory and its applications. [11] explored the graceful chromatic number of unicyclic graphs, offering insights into graph coloring techniques that can support node optimization. Similarly, [12] investigated the non-isolated resolving numbers of special graphs, which provides a basis for analyzing resolving sets in various graph configurations. [13] focused on rainbow antimagic coloring, a concept that enhances understanding of graph properties related to efficient resource allocation. [14] introduced the locating edge domination number of comb product graphs, further enriching the theoretical framework for domination problems in graph applications. Finally, [15] studied bounds on the distance domination number of graphs, which align with the REDS framework for optimizing monitoring nodes. These studies collectively underpin the theoretical and practical contributions of this research.

This research will analyze the Tadpole graph ($T_{m,n}$) which is used as a planting layout in horizontal farming. Plant growth is influenced by soil humidity and pH, which can be predicted using the Spatio-Temporal Graph Neural Network (STGNN) technique. This technique helps in understanding and modeling multi-step time series data to predict soil moisture and pH. The aims of this research is to analyze the relationship between resolving efficient dominating sets on graphs in solving soil moisture and pH problems using STGNN, with a specific focus on horizontal farming systems commonly used by farmers.

METHODS

Analytical and experimental methodologies are employed in this study. To establish certain theorems for the analytical study, we will use a mathematical deductive approach and utilize several well-known techniques. Through further examination of already established theorems or explanations in this research, additional theorems or definitions may be discovered. The methodology for identifying patterns is based on a research approach aimed at identifying effective set completion patterns in the graph and the problem under consideration. Let G represent the effective dominance number with the smallest cardinality. This is referred to as the effective dominance number resolution on a graph.

In this section, we propose several recommendations and theorems based on previous findings. These are derived using the same research methods applied to several different graphs.

a) Hakim, et al. [5] have determined the exact value of the comb product of a specific graph, i.e. $\gamma_{re}(K_n \rhd C_3) = n$, $\gamma_{re}(K_n \rhd P_3) = n$, $\gamma_{re}(W_n \rhd P_3) = n + 1$, and $\gamma_{re}(W_n \rhd C_3) = n + 1$;
b) Kusumawardani, et al. [6] have determined the exact value of the comb product of a specific graph, in path graph for $n \geq 4 = \gamma(P_n) = \lceil \frac{n}{3} \rceil$.

Based the following algorithm on Lim, et al. [7] and Masum, et al. [8] is used to determine soil moisture and pH forecasting using STGNN multi-step time series forecasting, combined with resolving efficient dominating set:

<u>Single Layer STGNN Algorithm</u>

Step 0.	Consider a graph $G(V, E)$ of order n, with a feature matrix $H_{n\times m}$ representing n vertices and m features, along with a specified tolerance ϵ ;
Step 1.	Find the adjacency matrix A of the graph G, and calculate the matrix $B = A + I$, where I represents the identity matrix;
Step 2.	Initialize weights W_g , bias β, and level α. For simplicity, define W_g as an $m \times 1$ matrix $W_g = [w_1, w_2, \dots, w_m]$, where $0 \leq w_j \leq 1$, $\beta = 0$, and $0 \leq \alpha \leq 1$;
Step 3.	Multiply the matrix weights by the vertex features, using the function $message\ m_u^l = MSG^l(h_u^{i-1})$. For linear layers, this becomes $m_u^l = Wg^l(h_u^{i-1})$;
Step 4.	Aggregate the messages from neighboring vertices using the function $h_x^l = AGG^l m_u^{l-1}$, $u \in N(v)$ and apply the **sum function** (.), $h_x^l = SUM^l m_u^{l-1}$, $u \in$

$N(v)$ considering the matrix B;

Step 5. Compute the *error:*

$$error^l = \frac{\|h_{xi}^l - h_{xj}^l\|^2}{|E(G)|^2},$$

where x_i, x_j are two adjacent vertices;

Step 6. Verify if $error \leq \in$. If true, terminate the process; otherwise, proceed to Step 7 to update the weight matrix W;

Step 7. Update the weight matrix as follows:

$$Wg^{l+1} = W_j^l + \alpha \times z_j \times e^l$$

where z_j is the sum the elements in each column in H_{xi}^l divided by the total number of vertices;

Step 8. Repeat steps 3-6 until $error \leq \epsilon$;

Step 9. Save the embedding results into a vector. If the data is time series, repeat the same process for the next observation.;

Step 10. Load the data vector and apply time series machine learning techniques for forecasting;

Step 11. Check whether $RMSE \leq \epsilon$. If yes, stop; otherwise, repeat Steps 2–10.

The choice of Mean Squared Error (MSE) as the performance metric in this study is rooted in its ability to effectively quantify prediction accuracy for continuous variables such as soil moisture and pH. MSE is particularly advantageous because it penalizes larger errors more heavily, ensuring that substantial deviations, which could have critical consequences in agricultural management, are minimized during model training. Additionally, the differentiable nature of MSE aligns seamlessly with gradient-based optimization techniques, facilitating efficient model updates during the learning process. By focusing on the squared differences between predicted and actual values, MSE provides a clear and interpretable measure of model performance, making it ideal for monitoring the accuracy of multi-step time series forecasts. The low MSE values achieved in this study (0.0021 for soil moisture and 0.0035 for pH) underscore the effectiveness of the STGNN model in capturing complex spatio-temporal dependencies, reinforcing the metric's suitability for evaluating prediction models in precision agriculture.

RESULTS AND DISCUSSION

Resolving efficient dominating set on Tadpole Graph

This research presents theorems and their corresponding proofs on the topic of resolving the efficient dominating set on the Tadpole graph $T_{m,n}$.

Theorem 1. *Let the graphs* $(T_{m,n})$ *is a tadpole graph with* $n \geq 2$, $m \geq 6$ *and* $m \equiv 0 \ (mod\ 3)$ *then we obtain* $\gamma_{re}(T_{m,n}) = \lceil \frac{m}{3} \rceil + \lfloor \frac{n-1}{3} \rfloor$.

Proof. If $T_{m,n}$ is a tadpole graph with the vertex set $V(T_{m,n}) = \{x_i; 1 \leq m\} \cup \{y_j; 1 \leq j \leq n\}$ and a set of edges $E(T_{m,n}) = \{x_i x_{i+1}; 1 \leq i \leq m-1\} \cup \{x_i x_m\} \cup \{x_1 y_1\} \cup \{y_j y_{j+1}; 1 \leq j \leq n-1\}$. The cardinality of the vertex set and edge set of the graph $T_{m,n}$ is

$|V(T_{m,n})| = m + n$ and $|E(T_{m,n})| = m + n$. Then proceed by proving $D \subseteq V(G)$. We choose the efficient dominating set of the Tadpole graph $T_{m,n}$ is:

$$D = \begin{cases} \{x_i; i \equiv 0 (mod\ 3)\} \cup \{y_j; j \equiv 2\ (mod\ 3)\} & , \quad for\ n \equiv 2\ (mod\ 3) \\ \{x_i; i \equiv 1 (mod\ 3)\} \cup \{y_j; j \equiv 0\ (mod\ 3)\} & , \quad for\ other\ n \end{cases}$$

Next, we need to demonstrate that each vertex corresponding to the resolving efficient dominating number is unique. The representation of the distance of each vertex in the graph $T_{m,n}$ is shown in the following figure.

i_{redx} \ i	x_3	x_6	x_9	x_{12}	⋯	$x_{i\equiv 0(mod 3)}$	⋯	$x_{\frac{m+2}{2}}$	$x_{\frac{m}{2}+3}$	$x_{\frac{m}{2}+6}$	$x_{\frac{m}{2}+9}$	$x_{\frac{m}{2}+12}$	⋯	x_m	y_2	y_5	y_8	⋯	$y_{j\equiv 2(mod 3)}$
x_1	2	5	8	11	⋯	i_{redx-1}	⋯	$\frac{m-1}{2}$	$\left(\frac{m-1}{2}\right)-3$	$\left(\frac{m-1}{2}\right)-6$	$\left(\frac{m-1}{2}\right)-9$	$\left(\frac{m-1}{2}\right)-12$	⋯	1	2	5	8	⋯	j_{redx}
x_2	1	4	7	10	⋯	i_{redx-2}	⋯	$\frac{m-1}{2}$	$\left(\frac{m-1}{2}\right)-2$	$\left(\frac{m-1}{2}\right)-5$	$\left(\frac{m-1}{2}\right)-8$	$\left(\frac{m-1}{2}\right)-11$	⋯	2	3	6	9	⋯	$j_{redx}+1$
x_3	0	3	6	9	⋯	i_{redx-3}	⋯	$\left(\frac{m-1}{2}\right)-1$	$\left(\frac{m-1}{2}\right)-1$	$\left(\frac{m-1}{2}\right)-4$	$\left(\frac{m-1}{2}\right)-7$	$\left(\frac{m-1}{2}\right)-10$	⋯	3	4	7	10	⋯	$j_{redx}+2$
x_4	1	2	5	8	⋯	i_{redx-4}	⋯	$\left(\frac{m-1}{2}\right)-2$	$\frac{m-1}{2}$	$\left(\frac{m-1}{2}\right)-3$	$\left(\frac{m-1}{2}\right)-6$	$\left(\frac{m-1}{2}\right)-9$	⋯	4	5	8	11	⋯	$j_{redx}+3$
x_5	2	1	4	7	..	i_{redx-5}	⋯	$\left(\frac{m-1}{2}\right)-3$	$\frac{m-1}{2}$	$\left(\frac{m-1}{2}\right)-2$	$\left(\frac{m-1}{2}\right)-5$	$\left(\frac{m-1}{2}\right)-8$	⋯	5	6	9	12	..	$j_{redx}+4$
⋮	⋮	⋮	⋮	⋮	⋮	⋮		⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
x_i, i	$i = i_{redx} + \left(\frac{m+1}{2}\right)$ meiliki nilai $(rd) = \frac{m-1}{2}$								$i = i_{redx} - \left(\frac{m+2}{2}\right) + 2$ meiliki nilai $(rd) = \frac{m-1}{2}$					$j_{redx} + 1 - 1; 1 \le i \le \left\lceil\frac{m}{2}\right\rceil$					
x_i	Pengulangan -1;$i = \frac{m+2}{2} + i_{redx}$								Pengulangan -1;$i = i_{redx} - \frac{m+2}{2} + 3$					$j_{redx} + 2\left\lceil\frac{m}{2}\right\rceil - i; \left\lceil\frac{m}{2}\right\rceil + 1 \le i \le m$					
⋮	⋮	⋮	⋮	⋮	⋮	⋮		⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
x_m	3	6	9	12		i_{redx}		$\frac{m}{2}$	$\frac{m}{2}-3$	$\frac{m}{2}-6$	$\frac{m}{2}-9$	$\frac{m}{2}-12$		0	3	6	9		$j_{redx}+1$
y_1	3	6	9	12		i_{redx}		$\frac{m}{2}-1$	$\frac{m}{2}-4$	$\frac{m}{2}-7$	$\frac{m}{2}-10$	$\frac{m}{2}-13$	4	2	1	4	7		j_{redx}
y_2	4	7	10	13		$i_{redx}+1$		$\frac{m}{2}$	$\frac{m}{2}-3$	$\frac{m}{2}-6$	$\frac{m}{2}-9$	$\frac{m}{2}-12$	5	3	0	3	6		$j_{redx}-1$
y_3	5	8	11	14		$i_{redx}+2$		$\frac{m}{2}+1$	$\frac{m}{2}-2$	$\frac{m}{2}-5$	$\frac{m}{2}-8$	$\frac{m}{2}-11$	6	4	1	2	5		$j_{redx}-2$
y_4	6	9	12	15		$i_{redx}+3$		$\frac{m}{2}+2$	$\frac{m}{2}-1$	$\frac{m}{2}-4$	$\frac{m}{2}-7$	$\frac{m}{2}-10$	7	5	2	1	4		$j_{redx}-3$
y_5	7	10	13	16		$i_{redx}+4$		$\frac{m}{2}+3$	$\frac{m}{2}$	$\frac{m}{2}-3$	$\frac{m}{2}-6$	$\frac{m}{2}-9$	8	6	3	0	3		$j_{redx}-4$
⋮																			
y_j		Bertambah 1													Pengulangan+1;j_{redx}				
⋮																			
y_n	y_j+2	y_j+5	y_j+8	y_j+12	⋯		$i_{redx}+(j-1);$ $1 < i_{redx} \le \frac{m}{2}+3$		$m - i_{redx} + 2;$ $\frac{m}{2}+3 < i_{redx} \le m-6$						y_j-2	y_j-5	y_j-8		

(a) Even m

i_{redx} \ i	x_3	x_6	x_9	x_{12}	⋯	$x_{i\equiv 0(mod 3)}$	$x_{\frac{m}{2}}$	$x_{\frac{m}{2}+3}$	$x_{\frac{m}{2}+6}$	$x_{\frac{m}{2}+9}$	$x_{\frac{m}{2}+12}$	⋯	x_m	y_2	y_5	y_8	⋯	$y_{j\equiv 2(mod 3)}$
x_1	2	5	8	11	⋯	i_{redx-1}	$\frac{m}{2}-1$	$\frac{m}{2}-2$	$\frac{m}{2}-5$	$\frac{m}{2}-8$	$\frac{m}{2}-11$	⋯	1	2	5	8	⋯	j_{redx}
x_2	1	4	7	10	⋯	i_{redx-2}	$\frac{m}{2}-2$	$\frac{m}{2}-1$	$\frac{m}{2}-4$	$\frac{m}{2}-7$	$\frac{m}{2}-10$	⋯	2	3	6	9	⋯	$j_{redx}+1$
x_3	0	3	6	9	⋯	i_{redx-3}	$\frac{m}{2}-3$	$\frac{m}{2}$	$\frac{m}{2}-3$	$\frac{m}{2}-6$	$\frac{m}{2}-9$	⋯	3	4	7	10	⋯	$j_{redx}+2$
x_4	1	2	5	8	⋯	i_{redx-4}	$\frac{m}{2}-4$	$\frac{m}{2}-1$	$\frac{m}{2}-2$	$\frac{m}{2}-5$	$\frac{m}{2}-8$	⋯	4	5	8	11	⋯	$j_{redx}+3$
x_5	2	1	4	7	..	i_{redx-5}	$\frac{m}{2}-5$	$\frac{m}{2}-2$	$\frac{m}{2}-1$	$\frac{m}{2}-4$	$\frac{m}{2}-7$	⋯	5	6	9	12	..	$j_{redx}+4$
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
x_i	Pengulangan -1;$i = \frac{m}{2} + i_{redx} + 1$							Pengulangan -1;$i = -\frac{m}{2} + i_{redx} + 1$						Pengulangan -1;$i = \frac{m}{2} + 2$				
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮					
x_m	3	6	9	12		i_{redx}	$\frac{m}{2}$	$\frac{m}{2}-3$	$\frac{m}{2}-6$	$\frac{m}{2}-9$	$\frac{m}{2}-12$		0	3	6	9		
y_1	3	6	9	12		i_{redx}	$\frac{m}{2}-1$	$\frac{m}{2}-4$	$\frac{m}{2}-7$	$\frac{m}{2}-10$	$\frac{m}{2}-13$	4	2	1	4	7	⋯	j_{redx}
y_2	4	7	10	13		$i_{redx}+1$	$\frac{m}{2}$	$\frac{m}{2}-3$	$\frac{m}{2}-6$	$\frac{m}{2}-9$	$\frac{m}{2}-12$	5	3	0	3	6	⋯	$j_{redx}-1$
y_3	5	8	11	14		$i_{redx}+2$	$\frac{m}{2}+1$	$\frac{m}{2}-2$	$\frac{m}{2}-5$	$\frac{m}{2}-8$	$\frac{m}{2}-11$	6	4	1	2	5	⋯	$j_{redx}-2$
y_4	6	9	12	15		$i_{redx}+3$	$\frac{m}{2}+2$	$\frac{m}{2}-1$	$\frac{m}{2}-4$	$\frac{m}{2}-7$	$\frac{m}{2}-10$	7	5	2	1	4	⋯	$j_{redx}-3$
y_5	7	10	13	16		$i_{redx}+4$	$\frac{m}{2}+3$	$\frac{m}{2}$	$\frac{m}{2}-3$	$\frac{m}{2}-6$	$\frac{m}{2}-9$	8	6	3	0	3	..	$j_{redx}-4$
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
y_j	Bertambah 1													Pengulangan+1;j_{redx}				
⋮																		
y_n	y_j+2	y_j+5	y_j+8	y_j+12	⋯	$i_{redx}+(j-1);$ $1 < i_{redx} \le \frac{m}{2}$		$m+1+j-i_{redx};$ $\frac{m}{2} < i_{redx} \le m-6$						y_j-2	y_j-5	y_j-8		

(b) Odd M

Figure 1. Representation of distance to for Even and Odd M

Based on the **Figure 1**, it shows that D is the efficient resolving dominating set because each vertex has a different distance representation from each other. So D is the efficient resolving dominating set. Next, it will be proven that D is the efficient resolving dominating set using minimum cardinality. For proof that $D \geq \lceil\frac{m}{3}\rceil + \lfloor\frac{n-1}{3}\rfloor$ we get:

i) For $n \equiv 2\ (mod\ 3)$, choose $D_1 = \left\{\{x_i; i \equiv 0\ (mod\ 3)\} \cup \{y_j; j \equiv 2\ (mod\ 3)\}\right\}$. We observe that y_n is not dominated. If we attempt to add a vertex to the resolving efficient dominating set, it would be y_n, which is dominated by both y_2 and y_4. This does not satisfy the definition of a resolving efficient dominating set, and therefore the resolving efficient domination number is not minimized.

ii) For other n, choose $D_1 = \left\{\{x_i; i \equiv 1 (mod\ 3)\} \cup \{y_j; j \equiv 0\ (mod\ 3)\}\right\}$. We observe that y_1 not dominated. If we attempt to add a vertex to the resolving efficient dominating set, it would be y_1, which is dominated by both x_1 and y_2. This does not satisfy the definition of a resolving efficient dominating set, and therefore the resolving efficient domination number is not minimized.

From the statement, the set D_1 does not complete the efficient dominating set, so we can deduce that $|D|$ is the minimal cardinality of REDS of the graph $T_{m,n}$ with $\gamma_{re}(T_{m,n}) = \lceil\frac{m}{3}\rceil + \lfloor\frac{n-1}{3}\rfloor$.

For an illustration of the resolving efficient dominating set of the graph $T_{12,8}$, see **Figure 2** above.

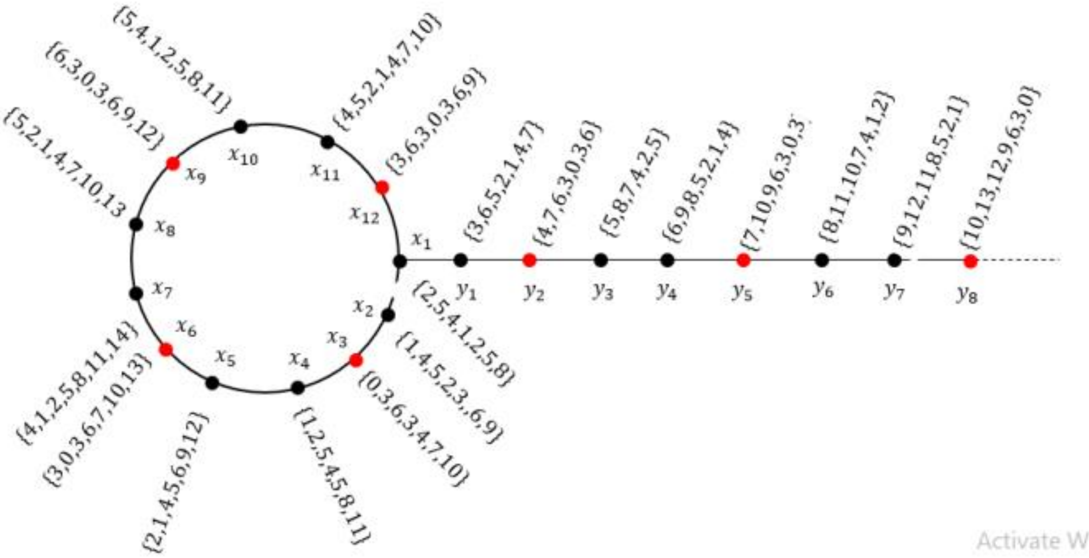

Figure 2. The Resolving Efficient Dominating Set of Tadpole Graph $T_{12,8}$

Result of STGNN in forecasting soil pH and moisture

The placement and number of sensors used are adjusted based on the vertex of the dominating set to identify the most efficient and strategic locations for measuring soil pH and moisture. In other words, the vertex of the dominating set in the graph on the soil plot represents an optimal location for these measurements. The Tadpole Graph ($T_{6,3}$) illustrates how an efficient dominating set can be applied in graph theory. In this example, the vertices of the graph represent various types of vegetables, with the specific vegetables corresponding to each vertex are listed in **Table 1**.

Table 1. Node representation of Tadpole Graph $(T_{6,3})$

Plant	Distance	Vertex Representation
Tomato	(2,1,2)	x_1
Carrots	(1,2,3)	x_2
Bombay	(0,3,4)	x_3
Cucumber	(1,2,5)	x_4
Celery	(2,1,4)	x_5
Shallot	(3,0,3)	x_6
Carrots	(3,2,1)	y_1
Bombay	(5,4,1)	y_2
Lettuce	(5,4,1)	y_3

Based on **Theorem 1**, the resulting resolving efficient domination number is three. Consequently, there are three locations for placing the pH meter and capacitive soil moisture sensors. Each point representation corresponds to a different plant species. According to this theorem, a horizontal farming planting pattern can be illustrated where the placement of plants aligns with the resolving efficient dominating set, as shown in **Table 1**.

This research utilizes crop irrigation simulation data with two features—soil moisture and soil pH—which exhibit an inverse relationship. As soil moisture increases, pH tends to decrease due to poor soil absorption conditions. Data was collected over 35 days, resulting in a total of 70 data points, which were normalized before simulation. Simulations were conducted for training, testing, and forecasting pH and soil moisture anomaly data using Google Colaboratory and Graph Neural Network.

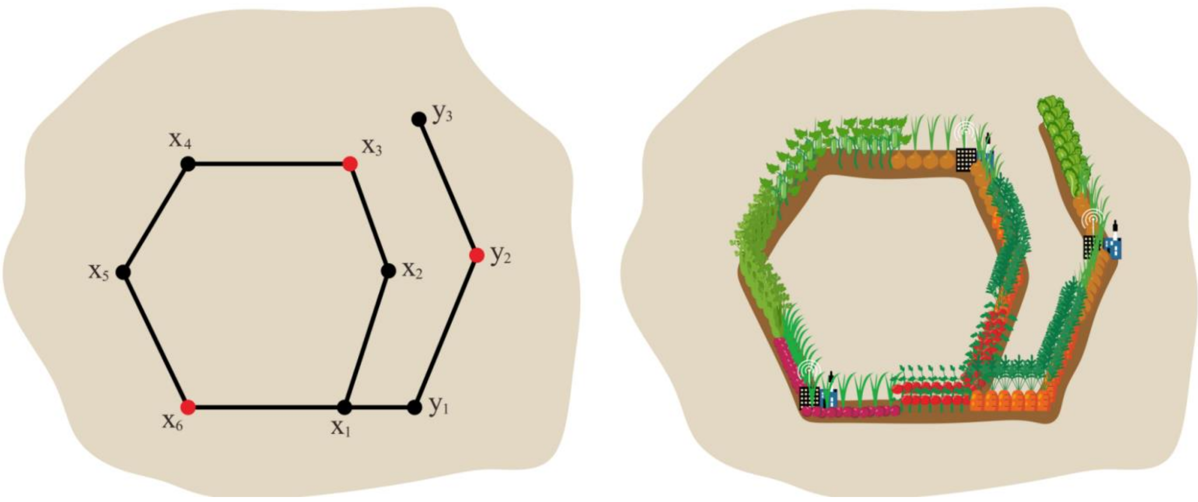

Figure 2. Illustration of a land plot on a Tadpole Graph $(T_{6,3})$

During the training stage, changes in training loss were observed, decreasing from an initial value of 0.26257 at epoch 0 to 0.0158 at epoch 180, indicating the model's improvement in recognizing irrigation patterns. Testing over 35 days yielded a Mean Squared Error (MSE) of 0.0129, reflecting very good prediction quality. The comparison graph between the output of the testing and training stages shows that the model becomes increasingly accurate in predicting crop irrigation patterns as the epoch number increases, as illustrated in the loss versus epoch image.

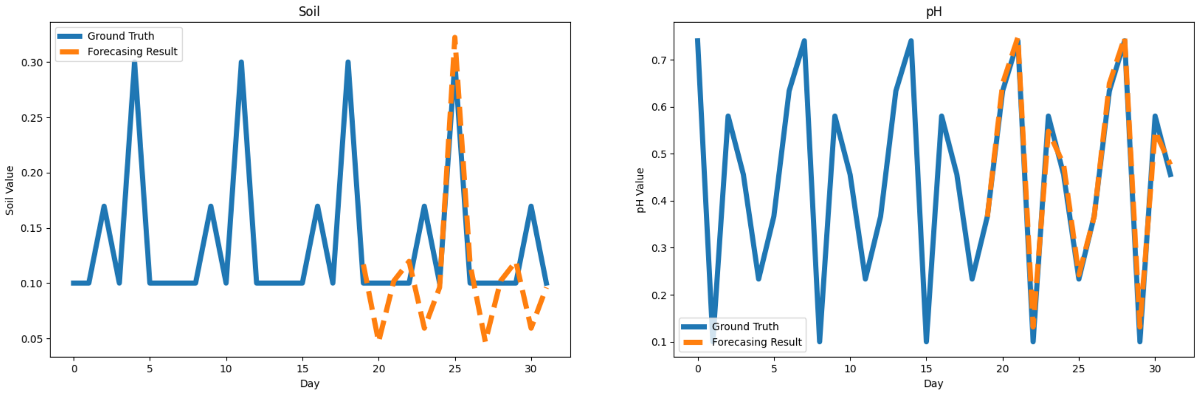

Figure 3. Comparison of Output stages testing and training

The results of crop irrigation forecasting for the next 7 days show that the trained model can produce useful data for automatic irrigation.

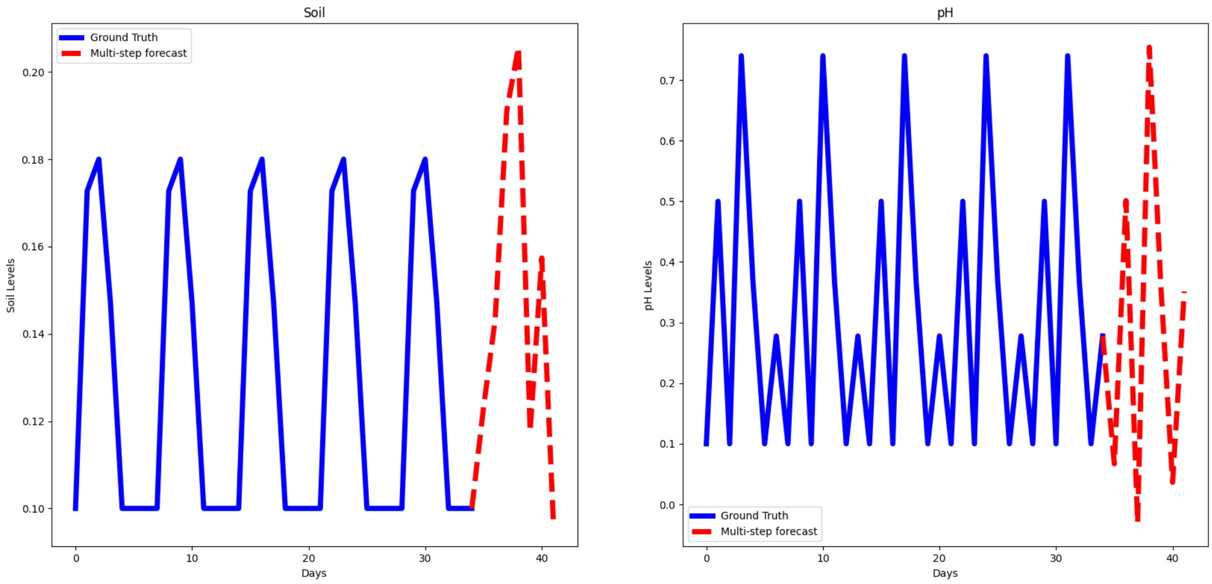

Figure 4. The result of the forecasting for 7 days after

This forecast is based on the analysis of pH and soil moisture data over 40 days, ensuring that ideal conditions for plants are maintained. This multi-step time series forecasting prototype demonstrates the effectiveness of the model in predicting irrigation needs and supports the implementation of automatic irrigation to maintain optimal soil conditions.

Numerical Analysis

Observation 1. Given a graph G of order n. Let the set of vertices and edges $V(G) = \{v_1, v_2, v_3, \dots, v_{n-1}, v_n\}$ and $E(G) = \{v_i v_j | v_i, v_j \in V(G)\}$.
Given the following vertex features: $h_{vi} = \left[s_{1,1}\ s_{1,2}\ s_{2,1}\ s_{2,2} \ \cdots\ s_{1,m} \ \cdots\ s_{2,m} \ \vdots \vdots \ s_{n,1}\ s_{n,2} \ \ddots \vdots \ \cdots\ s_{n,m} \right]$.

Embedding at a vertex can be determined using messages passing from neighboring vertices $h_v^l = AGG^l\{m_u^{l-1}, u \in N(v)\}$ under aggregation **sum**(.) so $h_v^l = SUM^l\{m_u^{l-1}, u \in N(v)\}$ considering the matrix $B = A + I$ where A, I are the adjacency matrix and the Identity matrix respectively.

Proof. Based on graph , we obtain the matrix adjacency A. However, we must consider the neighbors of the points of the graph G to itself, so we need to add A with the Identity matrix I and obtain the matrix B as follows:

$$B = A + I = [b_{1,1}\ b_{1,2}\ b_{2,1}\ b_{2,2}\ \cdots\ s_{1,n}\ \cdots\ b_{2,n}\ \vdots\ \vdots\ b_{n,1}\ b_{n,2}\ \ddots \vdots\ \cdots\ b_{n,n}\ ]$$

According to the layer one GNN algorithm, it is necessary to initialize the weight matrix as follows:

$$W = [w_{1,1}\ w_{1,2}\ w_{2,1}\ w_{2,2}\ \cdots\ w_{1,m}\ \cdots\ w_{2,m}\ \vdots\ \vdots\ w_{m,1}\ w_{m,2}\ \ddots \vdots\ \cdots\ w_{m,m}\ ]$$

This weight will be used to get the value of m_{xi} and update the new weight in the next iteration. The point embedding process from GNN is carried out in two stages, namely message passing and aggregation. In the first step, message passing $m_u = MSG(h_u)$. For linear layer $m_u^{l+1} = W^l \times (h_u^l)$, where $l = 0, 1, 2, \dots, k$ we get:

$$m_{vi}^1 = H_{vi}^0 \times W^0 = [s_{1,1}\ s_{1,2}\ s_{2,1}\ s_{2,2}\ \cdots\ s_{1,m}\ \cdots\ s_{2,m}\ \vdots\ \vdots\ s_{n,1}\ s_{n,2}\ \ddots \vdots\ \cdots\ s_{n,m}\ ]$$
$$\times [w_{1,1}\ w_{1,2}\ w_{2,1}\ w_{2,2}\ \cdots\ w_{1,m}\ \cdots\ w_{2,m}\ \vdots\ \vdots\ w_{m,1}\ w_{m,2}\ \ddots \vdots\ \cdots\ w_{m,m}\ ]$$

Next, the second step is aggregation by looking at the neighbors of point v. By applying the aggregation $\mathbf{sum}(\cdot)$, for $h_v^{l+1} = AGG\{m_u^{l+1}, u \in N(v)\}$ we get $h_v^{l+1} = SUM\{m_u^{l+1}, u \in N(v)\}$ considering the matrix $B = A + I$. The vector embedding h_{vi}^1 can be written as follows:

$$h_{vi}^{l+1} = [m_{v1,1}^{l+1}\ m_{v1,2}^{l+1}\ m_{v2,1}^{l+1}\ m_{v2,2}^{l+1}\ \cdots\ m_{v1,m}^{l+1}\ \cdots\ m_{v2,m}^{l+1}\ \vdots\ \vdots\ m_{vn,1}^{l+1}\ m_{vn,2}^{l+1}\ \ddots \vdots\ \cdots\ m_{vn,m}^{l+1}\ ]$$

Then, it is necessary to calculate an error value that indicates how close two adjacent vertices are in the embedding space. The smaller the error value, the closer the distance between the two points. The error value can be formulated as $error^l = \frac{\|h_{vi}^l - h_{vj}^l\|_{inf}}{|E(G)|^2}$ where, $i, j \in 1, 2, \dots, n$.. We need to check whether the error $error \leq \in$. Otherwise, we need to update a new W^l using the h_{vi}^l obtained in the previous iteration. Updating the learning weight matrix using $W^{l+1} = W^l \times \alpha \times error^l \times {h_{vi}^l}^T \times h_{vi}^{l+1}$ to $error \leq \in$.

CONCLUSIONS

Based on the research results, several conclusions were obtained. First, Resolving Efficient Dominating Set on the graph $T_{m,n}$ produces a new theorem with the formula

$$\gamma_{re}(T_{m,n}) = \lceil \frac{m}{3} \rceil + \lfloor \frac{n-1}{3} \rfloor.$$

Resolving Efficient Dominating Sets on graphs using the Spatio-Temporal Graph Neural Network (STGNN) technique which is applied in simulations at Google Colaboratory, aims to solve the problem of soil moisture and pH in horizontal farming. The results of testing soil moisture and pH show a Mean Squared Error (MSE) value of 0.0129, which reflects a very good level of model prediction error. The claim that an MSE value of 0.0129 reflects a very good level of model prediction error is based on the scale of the target variables, typically normalized between 0 and 1, where such a low value indicates minimal deviation between predicted and actual values. In the context of precision agriculture, where even small errors in soil moisture and pH predictions can significantly impact crop health and productivity, this level of accuracy enables reliable and timely interventions. Additionally, compared to benchmarks from similar studies, an MSE of 0.0129 represents a notable improvement, reflecting the model's ability to effectively capture spatio-temporal dependencies in the data. This low error aligns with practical requirements for prediction accuracy, confirming the robustness and applicability of the model in horizontal farming systems.

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