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Correspondence Between Torsion and Annihilator Graph of Modules Zn Over Commutative Rings

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ABSTRACT

Graph theory approaches have become an important tool for studying algebraic structures such as rings and modules. Specifically, graphs associated with modules provide valuable insights into their internal properties. In this paper, we investigate the relationship between the torsion graph and the annihilator graph of the module $\mathbb{Z}_n$ over a commutative ring, focusing on their structural properties based on the prime factorization of n. We identify the torsion elements and annihilators of $\mathbb{Z}_n$, then construct and analyze the corresponding graphs. It investigates the fundamental characteristics of these graphs and establishes theoretical results regarding their connectivity and completeness. Through this approach, we highlight specific algebraic conditions on n that implies isomorphism or partial correspondence between the graphs. Our results reveal a strong connection between two distinct graph constructions, offering a broader understanding of algebraic graph theory and its applications in module theory.

Keywords: annihilator graph; torsion graph; module over commutative rings

INTRODUCTION

Graph theory is an important branch of mathematics that studies how objects are connected through a network of nodes and edges. It originated in Euler's journal on the Seven Bridges of Königsberg in 1736 [1]. Since then, graph theory has developed into a useful field with wide applications in computer science, biology, social sciences, and engineering [2].

Formally, a graph $G = (V, E)$ is a pair of sets that consists of the vertex set V and edge set $E \subseteq V^2$ [3]. This abstraction allows a systematic study of the structural properties and combinatorial characteristics of networks [1]. A simple graph is a graph without loops (edges connecting a vertex to itself) or multiple edges between the same pair of vertices [4]. Graph G is connected if there exists a path for every pair of vertices in G [1]. A complete graph with n vertices, denoted by K_n, is a simple graph where every pair of distinct vertices is adjacent. A bipartite graph is a graph whose vertices can be partitioned into two disjoint sets V_1 and V_2 such that every edge connects a vertex from

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V_1 to V_2, with no edges between vertices within the same set. Furthermore, a complete bipartite graph is a bipartite graph in which every vertex in V_1 is adjacent to every vertex in V_2 [1]. A complete bipartite graph is denoted by K_{n_1,n_2} if $|V_1| = n_1$ and $|V_2| = n_2$. An empty graph is a graph with $V = \emptyset$ and $E = \emptyset$, simply write as $\emptyset$ [3].

Over the years, graph theory has been developed in various ways, one of which is to relate it to algebraic structures such as rings. This idea has led to graph theory-based models that reflect algebraic properties. The earliest and most influential idea is the zero-divisor graph of a ring proposed by Beck [5] in 1988. He defined all elements of a ring as vertices, and two distinct vertices are adjacent if their product is zero. His work aimed to study such graphs in terms of coloring problems.

Later, Anderson and Livingston [6] refined Beck's approach by restricting the set of vertices to nonzero divisors of the commutative ring, resulting in a more focused structure and preserving the essential behavior of the ring. This version, now the standard definition, denoted by $\Gamma(R)$, has become a cornerstone in the study of graph-based rings. A further extension of this idea was introduced by Behboodi [7] by developing the graph of the zero divisors of a module, generalizing to the case of rings and opening up a wider class of algebraic objects for graph-theoretic investigation.

Based on the basic idea of zero divisor graphs, researchers have developed new graph models to capture various aspects of ring and module theory. One such model is the annihilator graph of ring R is denoted by $AG(R)$, introduced by Badawi [10]. This graph uses nonzero zero divisors of the ring as vertices and connects two distinct elements if their union annihilator is not equal to their product annihilator. Subsequent studies have extended and deepened this model. Nikmehr [9] explored further structural aspects of $AG(R)$, examining properties like connectivity and completeness under various algebraic conditions. Dutta [10] focused on finite commutative rings and identified specific classes where the annihilator graph is complete or contains particular substructures. Barati [11] continued this line of inquiry by analyzing the annihilator graph in broader classes of commutative rings and refining known results. Furthermore, Nikandish [12] investigated coloring properties of the annihilator graph, providing insight into chromatic numbers and clique structures.

The concept was further extended to modules by Hamidizadeh [13] and Nozari [14]. Nozari's approach defines the annihilator graph of an R-module M, denoted by $AG(M)$, using the set of elements in R that annihilate some nonzero element of the module M. Let M be an R-module, the annihilator of x in M is a subset of M defined as $Ann_M(x) = \{m \in M \mid xm = 0_M\}$ and annihilator of M in R defined as $Ann_R(M) = \{s \in R \mid sM = 0\}$. The vertex set of these annihilator graph is $Z_R(M) \backslash Ann_R(M)$ where $s \in Z_R(M)$ if $s \in R$ and there exists a nonzero element $m \in M$ such that $sm = 0_M$. The adjacency condition similarly reflects a difference in the union and product-based annihilators of elements in the module M which means two different vertices x and y is adjacent if $Ann_M(x) \cup Ann_M(y) \neq Ann_M(xy)$. This model provides a deeper understanding of the interaction between module elements and their annihilators from a graph-theoretic perspective.

While zero-divisor and annihilator graphs emphasize the behavior of elements in a ring or module with respect to multiplication and annihilation, another compelling graph-theoretic structure arises from torsion theory is the torsion graph of a module. First introduced by Ghalandarzadeh and Malakooti Rad [15], the torsion graph captures the interplay between torsion elements in a module via their shared annihilators. Let M be an R-module, where R is a commutative ring with identity. An element $m \in M$ is a torsion element if there exists a nonzero $r \in R$ such that $rm = 0$. The set of nonzero torsion elements is the vertex set of the torsion graph. Two distinct torsion elements x

and y are adjacent in the torsion graph if and only if the intersection of their annihilators in R is nontrivial, i.e., $Ann_R(x) \cap Ann_R(y) \neq 0$ where $Ann_R(x) = \{r \in R \mid rx = 0\}$. This construction yields a simple undirected graph that reflects both the module structure and the behavior of annihilation within the ring.

The introduction of the torsion graph brought a new dimension to the study of algebraic structures through graph theory. In their foundational work [15], Ghalandarzadeh and Malakooti Rad investigated fundamental properties of the torsion graph, such as connectivity, isolated vertices, and graph invariants under certain module conditions. The development continued in [16], where Ghalandarzadeh analyzed connected subgraphs of the torsion graph and provided criteria for when such subgraphs occur, relating them to properties of the module and its annihilators. Further extending this work, Rad [17] examined graph invariants such as diameter and girth, offering insights into the global structure of the torsion graph and conditions under which it is connected or possesses short cycles.

While the zero-divisor graph, annihilator graph, and torsion graph have each been studied in depth, the connections between these graph types have received relatively little attention. This paper addresses that gap by investigating the structural relationship between the torsion graph and the annihilator graph when the module is $\mathbb{Z}_n$ over itself. The main objective is to determine conditions under which these graphs coincide, differ, or reveal unique structural properties, and to establish a clear correspondence between them in the context of $\mathbb{Z}_n$.

The ring $\mathbb{Z}_n$, the integers modulo n, is a good setting for this kind of analysis. As a finite commutative ring with identity, it provides a manageable yet structurally rich environment. Its algebraic behavior depends on the prime factorization of n, and it admits decomposition into a product of local rings via the Chinese Remainder Theorem, which helps in clarifying the structure of associated graphs. Moreover, torsion elements and annihilators in $\mathbb{Z}_n$ can be studied through elementary number theory and ideal theory, making the exploration both theoretically insightful and computationally accessible.

By focusing on $M = R$ and $M = \mathbb{Z}_n$, we explore the fundamental properties of the torsion and annihilator graphs and identify their points of intersection and divergence. This work contributes to the field by offering a clearer understanding of how different algebraic graph models relate within a concrete and widely applicable ring structure, laying groundwork for future generalizations to broader classes of modules and rings.

METHODS

In this study, we use a mathematical proof approach to explore the properties of the torsion graph and annihilator graph of the module $\mathbb{Z}_n$ over a commutative ring, then we compare them to investigate the relationship between them. Our primary method is direct proof by selecting two arbitrary vertices and showing the adjacency between them based on the graph definition. In the particular situation where the structure of $\mathbb{Z}_n$ requires separate consideration of different types of elements, then we apply proof by cases to handle all possibilities.

In addition, we also use a literature-based approach. We draw on known results about torsion elements, annihilators, and graph constructions in module theory to help our analysis and explain when and why the torsion graph and annihilator graph align or differ. This combination of direct argument and theoretical background allows us to clearly understand the correspondence between the two graphs. Firstly, it is necessary to review the formal definition of the torsion graph and annihilator graph.

Definition 1 [15] Let R be a ring and M be an R-module. The torsion graph of M, denoted by $\Gamma_R(M)$ is a simple undirected graph with the vertex set containing the nonzero torsion element of M and two distinct vertices x and y is adjacent if $Ann_R(x) \cap Ann_R(y) \neq 0$.

Definition 2 [14] Let R be a ring and M be an R-module. The annihilator graph of M, denoted by $AG(M)$ is a simple undirected graph with the vertex set is $Z_R(M) \backslash Ann_R(M)$ and two different vertices x and y is adjacent if $Ann_M(x) \cup Ann_M(y) \neq Ann_M(xy)$.

In this paper, we focus on the structural properties based on the prime factorization of n. Furthermore, the prime factorization is divided into some cases for n is a prime number, n is a power of a prime number, n is a multiplication of two and three distinct prime numbers. For each case, the research methodology involves the following procedures:

i. Identify the vertex set of the torsion graph and annihilator graph for the module $M = \mathbb{Z}_n$.
ii. Investigate the adjacency between each vertex in the $\Gamma_R(M)$ and $AG(M)$.
iii. Determine the graph type of the $\Gamma_R(M)$ and $AG(M)$.
iv. Analyze the correspondence between $\Gamma_R(M)$ and $AG(M)$.

RESULTS AND DISCUSSION

In the first case, if $M = \mathbb{Z}_p$ for any prime number p then $AG(M)$ and $\Gamma_R(M)$ is an empty graph. Since $T(\mathbb{Z}_p) = \{\bar{0}\}$, then $V = \emptyset$, that implies $AG(M)$ and $\Gamma_R(M)$ is an empty graph.

Secondly, we investigate the torsion graph and annihilator graph for $M = \mathbb{Z}_{p^k}$ for any prime number p and positive integer $k \geq 2$.

Theorem 3 Let $M = \mathbb{Z}_{p^k}$ is an R-module for any prime number p and positive integer $k \geq 2$. Then

a. Torsion graph $\Gamma_R(M)$ is a complete graph K_n
b. Annihilator graph $AG(M)$ is a complete graph K_n

where $n = p^{k-1} - 1$.

Proof.

Since $R = \mathbb{Z}_{p^k}$ then torsion element of M has form $a\bar{p}$ for positive integer a or we can write

$$T(M) = \{a\bar{p} \mid a \in \mathbb{N}\}$$

Since $a\bar{p} \in M$ then we have the number of vertices is

$$|V| = |T(M)^*| = \frac{|\mathbb{Z}_{p^k}|}{p} - 1 = p^{k-1} - 1$$

Let x and y be two arbitrary different elements in V where $x = a_1\bar{p}$ and $y = a_2\bar{p}$ for positive integer a_1 and a_2, then

$$\overline{p^{k-1}}x = \overline{p^{k-1}}(a_1\bar{p}) = a_1(0) = 0$$

It means that $Ann_R(x)$ contain $\overline{p^{k-1}}$ such that $p^{k-1} \in Ann_R(x) \cap Ann_R(y)$. Therefore, for any $x, y \in V$ satisfy $Ann_R(x) \cap Ann_R(y) \neq 0$ or x and y is adjacent. Proved that $\Gamma_R(M)$ is complete graph K_n. For the annihilator graph, consider the following vertex set.

$$V = \{a\bar{p} \mid a \in \mathbb{N}\} = \left\{a\bar{p}, a\overline{p^2}, \dots, a\overline{p^{k-1}} \mid a \in \mathbb{N}, a < p\right\}$$

Let x and y be two arbitrary different elements in V where $x = a_1\overline{p^{k_1}}$ and $y = a_2\overline{p^{k_2}}$ for positive integer $a_1, a_2 < p$ and $k_1, k_2 < k$ then

$$Ann_M(x) = Ann_M\left(a_1\overline{p^{k_1}}\right) = \overline{p^{k-k_1}}M$$

and $Ann_M(y) = p^{k-k_2}M$. Without loss of generality, suppose that $k_1 > k_2$ then

$$\begin{aligned} Ann_M(xy) &= Ann_M\left(a_1a_2\overline{p^{k_1+k_2}}\right) \\ &= \begin{cases} \overline{p^{k-(k_1+k_2)}}M, & k_1 + k_2 \leq k \\ \overline{p^{2k-(k_1+k_2)}}M, & k_1 + k_2 > k \end{cases} \end{aligned}$$

Since $0 < k_1, k_2 < k$ then $p^{k-k_1}M \neq \overline{p^{k-(k_1+k_2)}}M$ and $p^{k-k_1} \neq \overline{p^{2k-(k_1+k_2)}}M$ such that $Ann_M(xy) \neq Ann_M(x) \cup Ann_M(y)$. Therefore, x and y is adjacent for all x, y in V and $AG(M)$ is a complete graph K_n. ■

Based on Theorem 3, the torsion graph and the annihilator graph of the module is same when n is a prime power number. Furthermore, we can see an example regarding this correspondence in the following torsion and annihilator graph of module $\mathbb{Z}_8$ and $\mathbb{Z}_{16}$ over itself.

Example 4 Let $M_1 = \mathbb{Z}_8$ is a $\mathbb{Z}_8$-module and $M_2 = \mathbb{Z}_{16}$ is a $\mathbb{Z}_{16}$-module. Note that M_1 and M_2 are $\mathbb{Z}_{p^k}$ for $p = 2$ with $k = 3$ and $k = 4$, respectively. Based on Theorem 3, we have $AG(M) = \Gamma_R(M)$ is complete graph K_n. Moreover, the graph $AG(M_1) = AG(\mathbb{Z}_8) = K_{n_1}$ and $AG(M_2) = AG(\mathbb{Z}_{16}) = K_{n_2}$ where $n_1 = 2^{3-1} - 1 = 3$ and $n_2 = 2^{4-1} - 1 = 7$.

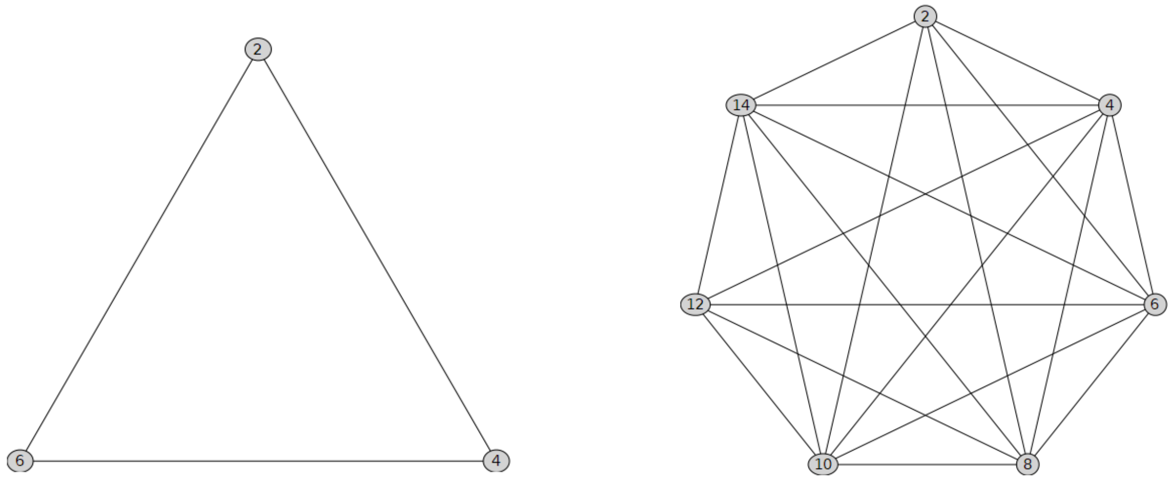

Figure 1. The annihilator and torsion graph of $\mathbb{Z}_8$ and $\mathbb{Z}_{16}$.

After that, we investigate the torsion and annihilator graph for $\mathbb{Z}_n$ where n is the multiplication of two different prime numbers p_1 and p_2.

Theorem 5 Let $M = \mathbb{Z}_{p_1p_2}$ is an R-module for prime p_1 and p_2 where $p_1 \neq p_2$. Then

a. Annihilator graph $AG(M)$ is a complete bipartite graph K_{n_1,n_2}.
b. Torsion graph $\Gamma_R(M)$ is a union of two complete graph $K_{n_1} \cup K_{n_2}$.

where $n_1 = p_2 - 1$ and $n_2 = p_1 - 1$.

Proof.
Since $M = R = \mathbb{Z}_{p_1p_2}$ then $a_1\overline{p_1}$ and $a_2\overline{p_2}$ are torsion elements of M. Suppose that

$$T(M) = \{a_1\overline{p_1}, a_2\overline{p_2} \mid a_1, a_2 \in \mathbb{N}\}$$

such that we have the vertex set is

$$V = \{a_1\overline{p_1}, a_2\overline{p_2} \mid a_1, a_2 \in \mathbb{N}, a_1 < p_2, a_2 < p_1\}$$

Then, split the vertex set into two partitions V_1 and V_2

$$V_1 = \{a_1\overline{p_1} \mid a_1 \in \mathbb{N}, a_1 < p_2\}$$
$$V_2 = \{a_2\overline{p_2} \mid a_2 \in \mathbb{N}, a_2 < p_1\}$$

The cardinality of the partitioned vertex set is $|V_1| = p_2 - 1$ and $|V_2| = p_1 - 1$. After that, we will prove that $AG(M)$ is a complete bipartite graph.

a. Without loss of generality, let x and y are two arbitrary different elements in V_1 where $x = a\overline{p_1}$ and $y = b\overline{p_1}$ for some positive integer $a, b < p_2$ then

$$Ann_M(x) = Ann_M(a\overline{p_1}) = \overline{p_2}M$$
$$Ann_M(y) = Ann_M(b\overline{p_1}) = \overline{p_2}M$$
$$Ann_M(xy) = Ann_M\left(ab\overline{p_1^2}\right) = \overline{p_2}M$$

such that $Ann_M(xy) = \overline{p_2}M = Ann_M(x) \cup Ann_M(y)$. Therefore, x, y is not adjacent for all $x, y \in V_1$ and x, y is not adjacent for all $x, y \in V_2$.

b. Let x and y be arbitrary element in V_1 and V_2, respectively, where $x = a\overline{p_1}$ and $y = b\overline{p_2}$ for some positive integer $a < p_2$ and $b < p_1$ then

$$Ann_M(x) = Ann_M(a\overline{p_1}) = \overline{p_2}M$$
$$Ann_M(y) = Ann_M(b\overline{p_2}) = \overline{p_1}M$$
$$Ann_M(xy) = Ann_M(ab\overline{p_1p_2}) = Ann_M(\overline{0}) = M$$

Since $\overline{1} \in M$, $\overline{1} \notin \overline{p_1}M$, and $\overline{1} \notin \overline{p_2}M$ then $Ann_M(xy) \neq Ann_M(x) \cup Ann_M(y)$. Hence, x and y is adjacent for all $x \in V_1$ and $y \in V_2$.

Therefore, $AG(M)$ is a complete bipartite graph K_{n_1,n_2} where $n_1 = |V_1|$ and $n_2 = |V_2|$. Now, we will prove that the torsion graph for V_1 and V_2 are complete graphs and there is no edge form V_1 and V_2.

a. Without loss of generality, let x and y are two arbitrary different elements in V_1 where $x = a\overline{p_1}$ and $y = b\overline{p_1}$ then

$$Ann_R(x) = Ann_R(a\overline{p_1}) = \overline{p_2}R$$
$$Ann_R(y) = Ann_R(b\overline{p_1}) = \overline{p_2}R$$

such that $Ann_R(x) \cap Ann_R(y) \neq 0$. Hence x, y is adjacent for any $x, y \in V_1$ and $x, y \in V_2$.

b. Let x and y be arbitrary element in V_1 and V_2, respectively, where $x = a\overline{p_1}$ and $y = b\overline{p_2}$ then

$$Ann_R(x) = Ann_R(a\overline{p_1}) = \overline{p_2}R$$
$$Ann_R(y) = Ann_R(b\overline{p_2}) = \overline{p_1}R$$

Clear that $Ann_R(x) \cap Ann_R(y) = 0$, then x and y is not adjacent.
Therefore, $\Gamma_R(M)$ is a union of the complete graph K_{n_1} and K_{n_2} where $n_1 = |V_1|$ and $n_2 = |V_2|$ with $V_1 \cap V_2 = \emptyset$. ■

Example 6 Let $M_1 = \mathbb{Z}_{10}$ is a $\mathbb{Z}_{10}$-module and $M_2 = \mathbb{Z}_{15}$ is a $\mathbb{Z}_{15}$-module. Note that the vertex set is $V_1 = \{\bar{2}, \bar{4}, \bar{5}, \bar{6}, \bar{8}\}$ and $V_2 = \{\bar{3}, \bar{5}, \bar{6}, \bar{9}, 1\bar{0}, 1\bar{2}\}$ where the partition is $V_{11} = \{\bar{2}, \bar{4}, \bar{6}, \bar{8}\}$, $V_{12} = \{\bar{5}\}$, $V_{21} = \{\bar{3}, \bar{6}, \bar{9}, 1\bar{2}\}$, and $V_{22} = \{\bar{5}, 1\bar{0}\}$. The annihilator graph for M_1 and M_2 is complete bipartite $K_{4,1}$ and $K_{4,2}$, respectively. Meanwhile the torsion graph for M_1 and M_2 is $K_4 \cup K_1$ and $K_4 \cup K_2$, respectively.

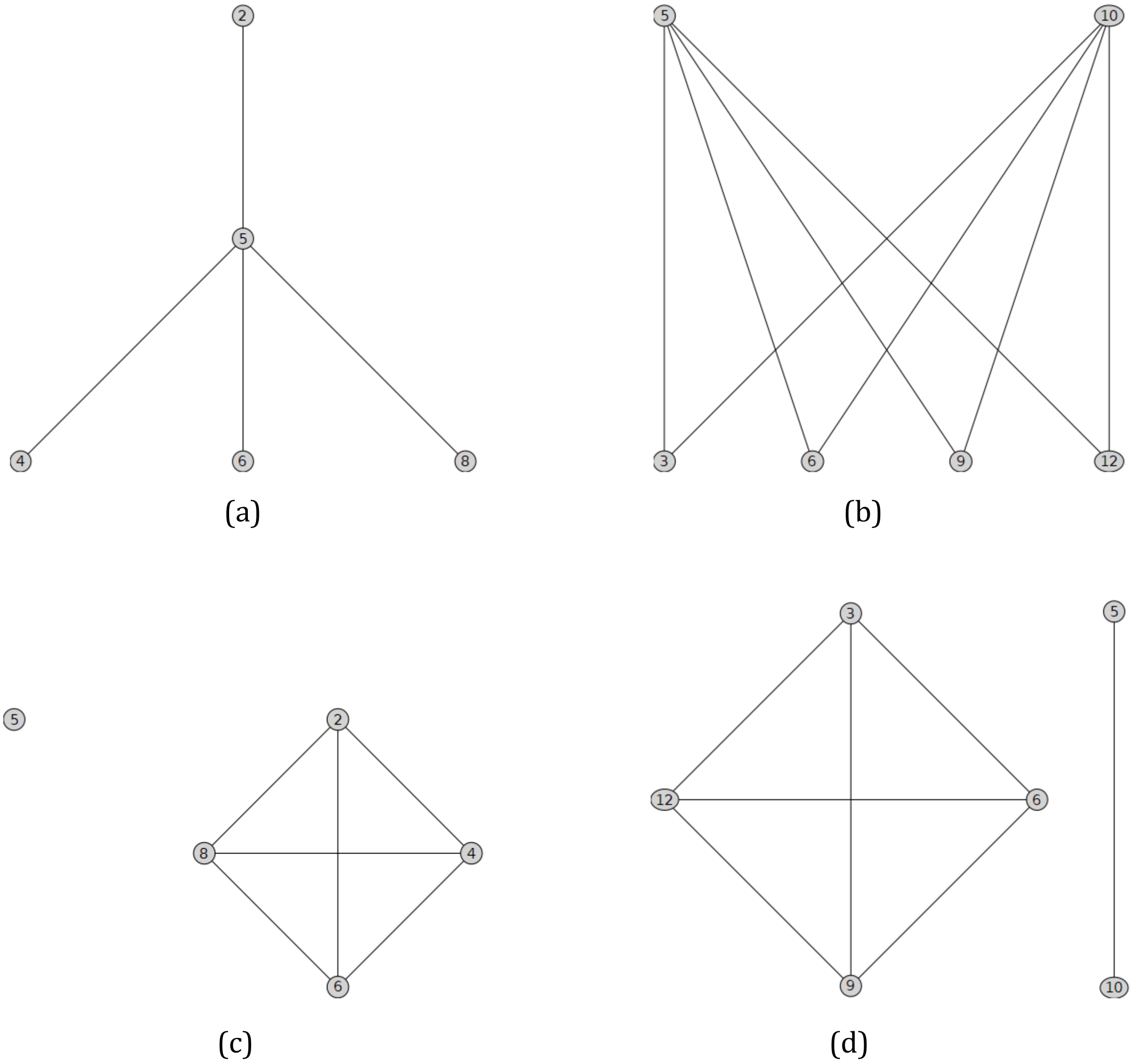

Figure 2. (a) The annihilator graph of $\mathbb{Z}_{10}$, (b) the annihilator graph of $\mathbb{Z}_{15}$, (c) the torsion graph of $\mathbb{Z}_{10}$, and (d) the torsion graph of $\mathbb{Z}_{15}$.

Based on the Theorem 5 and Example 6, the torsion and annihilator graph have a unique relation which is the union of the edge sets is $V \times V$, while the intersection of them is empty set. Therefore, if n multiplication of two distinct prime numbers, this relationship can be represented in the following corollary.

Corollary 7 If $M = \mathbb{Z}_{p_1p_2}$ is an R-module where p_1 and p_2 are two different prime numbers, then $AG(M) = \overline{\Gamma_R(M)}$.

In the next theorem, we will discuss $\mathbb{Z}_n$ where n is the multiplication of three distinct prime numbers. Moreover, we will investigate the connectedness of the annihilator and torsion graph.

Theorem 8 Let $M = \mathbb{Z}_n$ is an R-module where $n = p_1p_2p_3$ is the multiplication of three different prime numbers and $V' = V \setminus \bigcup_{i<j} V_{ij}$ where $V_{ij} = \{a\overline{p_i}\overline{p_j} \mid a \in \mathbb{N}\}$. Let G_1 and G_2 are induced subgraphs of $AG(M)$ and $\Gamma_R(M)$, respectively.

a. If $V(G_1) = V'$ then G_1 is a complete 3-partite graph K_{n_1,n_2,n_3}.
b. If $V(G_2) = V'$ then G_2 is a union of three complete graph $K_{n_1} \cup K_{n_2} \cup K_{n_3}$.
c. If $V(G_1) = \bigcup_{i<j} V_{ij}$ then G_1 is a complete 3-partite graph K_{m_1,m_2,m_3}.
d. If $V(G_2) = \bigcup_{i<j} V_{ij}$ then G_2 is a complete graph K_m.

Proof.
Suppose that the vertex set can be partitioned into

$$V = \left(\bigcup_{i=1}^{3} V_i\right) \cup \left(\bigcup_{i<j} V_{ij}\right)$$

where $V_i = \{a\overline{p_i} \mid a \in \mathbb{N}, a \neq m\overline{p_j}, m \in \mathbb{N}, j \neq i\}$ and $V_{ij} = \{a\overline{p_i}\overline{p_j} \mid a \in \mathbb{N}\}$. Clear that, we have $V_i \cap V_j = \emptyset, V_{ij} \cap V_{jk} = \emptyset, V_{ij} \cap V_{ik} = \emptyset$ for any different index i, j, k and $V_i \cap V_{jk} = \emptyset$ for all index $i, j, k \in \{1,2,3\}$. Let $V' = V \setminus \bigcup_{i<j} V_{ij}$ then $V' = V_1 \cup V_2 \cup V_3$.

a. Suppose that $V(G_1) = V' = V_1 \cup V_2 \cup V_3$. Let x, y are two arbitrary different elements in V_i for $i = 1,2,3$ such that $x = a\overline{p_i}$ and $y = b\overline{p_i}$ for some natural number a and b that is not multiple of p_j for all $j \neq i$. Note that

$$Ann_M(x) = Ann_M(a\overline{p_i}) = \overline{p_j}\overline{p_k}M$$
$$Ann_M(y) = Ann_M(b\overline{p_i}) = \overline{p_j}\overline{p_k}M$$
$$Ann_M(xy) = Ann_M\left(ab\overline{p_i^2}\right) = \overline{p_j}\overline{p_k}M$$

for any distinct index i, j, k. Hence, $Ann_M(x) \cup Ann_M(y) = Ann_M(xy)$ then x and y is not adjacent. After that, let x and y be an arbitrary element in V_i and V_j, respectively, for $i \neq j$ such that $x = a\overline{p_i}$ and $y = b\overline{p_j}$ for some natural number a and b that is not multiple of p_{k_1} and p_{k_2}, respectively, for all $k_1 \neq i$ and $k_2 \neq j$. Note that

$$Ann_M(x) = Ann_M(a\overline{p_i}) = \overline{p_j}\overline{p_k}M$$
$$Ann_M(y) = Ann_M\left(b\overline{p_j}\right) = \overline{p_i}\overline{p_k}M$$
$$Ann_M(xy) = Ann_M\left(ab\overline{p_i}\overline{p_j}\right) = \overline{p_k}M$$

for any distinct index i, j, k. Hence, $Ann_M(x) \cup Ann_M(y) \neq Ann_M(xy)$ then x is adjacent with y. Therefore, G_1 is complete 3-partite graph K_{n_1,n_2,n_3} where $n_i = |V_i|$ for $i = 1,2,3$.

b. Suppose that $V(G_2) = V' = V_1 \cup V_2 \cup V_3$. Let x, y are two arbitrary different elements in V_i for $i = 1,2,3$ such that $x = a\overline{p_i}$ and $y = b\overline{p_i}$ for some natural number a and b that is not multiple of p_j for all $j \neq i$. Note that

$$Ann_R(x) = Ann_R(a\overline{p_i}) = \overline{p_j}\overline{p_k}R$$
$$Ann_R(y) = Ann_R(b\overline{p_i}) = \overline{p_j}\overline{p_k}R$$

for any distinct index i, j, k. Hence, $Ann_R(x) \cap Ann_R(y) \neq 0$ then x is adjacent with y. Therefore, V_i in G_2 induce a complete graph K_{n_i} with $n_i = |V_i|$ for $i = 1,2,3$. Next, let x and y be an arbitrary element in V_i and V_j, respectively, for $i \neq j$ such that $x = a\overline{p_i}$ and $y = b\overline{p_j}$ for some natural number a and b that is not multiple of p_{k_1} and p_{k_2}, respectively, for all $k_1 \neq i$ and $k_2 \neq j$. Note that

$$Ann_R(x) = Ann_R(a\overline{p_i}) = \overline{p_j}\overline{p_k}R$$
$$Ann_R(y) = Ann_R(b\overline{p_j}) = \overline{p_i}\overline{p_k}R$$

for any distinct index i, j, k. It is clear that $\overline{p_j}\overline{p_k}R \cap \overline{p_i}\overline{p_k}R = 0$, then x and y is not adjacent. Hence, G_2 is a union of three complete graph $K_{n_1} \cup K_{n_2} \cup K_{n_3}$ with $n_i = |V_i|$.

c. Suppose that $V(G_1) = \bigcup_{i<j} V_{ij} = V_{12} \cup V_{13} \cup V_{23}$. Without loss of generality, let x, y are two arbitrary different elements in V_{12} such that $x = a\overline{p_1p_2}$ and $y = b\overline{p_1p_2}$ for some natural number a and b. Note that

$$Ann_M(x) = Ann_M(a\overline{p_1p_2}) = \overline{p_3}M$$
$$Ann_M(y) = Ann_M(b\overline{p_1p_2}) = \overline{p_3}M$$
$$Ann_M(xy) = Ann_M\left(ab\overline{p_1^2p_2^2}\right) = \overline{p_3}M$$

Hence, $Ann_M(x) \cup Ann_M(y) = Ann_M(xy)$ such that x and y is not adjacent. Next, without loss of generality, let x, y are two arbitrary different elements in V_{12} and V_{13}, respectively, such that $x = a\overline{p_1p_2}$ and $y = b\overline{p_1p_3}$ for some natural number a and b.

$$Ann_M(x) = Ann_M(a\overline{p_1p_2}) = \overline{p_3}M$$
$$Ann_M(y) = Ann_M(b\overline{p_1p_3}) = \overline{p_2}M$$
$$Ann_M(xy) = Ann_M\left(ab\overline{p_1^2p_2p_3}\right) = M$$

Clear that $Ann_M(x) \cup Ann_M(y) \neq Ann_M(xy)$ such that x is adjacent with y. Therefore, G_1 is a complete 3-partite graph K_{m_1,m_2,m_3} where $m_1 = |V_{12}|, m_2 = |V_{13}|$, and $m_3 = |V_{23}|$.

d. Suppose that $V(G_2) = \bigcup_{i<j} V_{ij} = V_{12} \cup V_{13} \cup V_{23}$. Without loss of generality, let x, y are two arbitrary different elements in V_{12} such that $x = a\overline{p_1p_2}$ and $y = b\overline{p_1p_2}$ for some natural number a and b. Note that

$$Ann_R(x) = Ann_R(a\overline{p_1p_2}) = \overline{p_3}R$$
$$Ann_R(y) = Ann_R(b\overline{p_1p_2}) = \overline{p_3}R$$

Hence, $Ann_R(x) \cap Ann_R(y) \neq 0$ such that x and y is adjacent. After that, without loss of generality, let x, y are two arbitrary different elements in V_{12} and V_{13}, respectively, such that $x = a\overline{p_1p_2}$ and $y = b\overline{p_1p_3}$ for some natural number a and b.

$$Ann_R(x) = Ann_R(a\overline{p_1p_2}) = \overline{p_3}R$$
$$Ann_R(y) = Ann_R(b\overline{p_1p_3}) = \overline{p_2}R$$

Clear that $Ann_R(x) \cap Ann_R(y) \neq 0$ because $\overline{p_2p_3}$ contains in both of them. Hence, x is adjacent with y. Therefore, G_2 is a complete graph K_m where $m = \left|\bigcup_{i<j} V_{ij}\right|$. ■

In the Theorem 8, the connection between the torsion and annihilator graph has similar characteristic with the relationship in the Theorem 5 for the induced subgraph based on the vertex set partition. To clarify their relationship, we partition the vertex set into two subsets. Hence, we obtain the following graph relation.

Corollary 9 Let $M = \mathbb{Z}_n$ is an R-module where $n = p_1p_2p_3$ is the multiplication of three different prime numbers and $V' = V \setminus \cup_{i<j} V_{ij}$ where $V_{ij} = \{a\overline{p_i}\overline{p_j} \mid a \in \mathbb{N}\}$. Let G_1 and G_2 are induced subgraphs of $AG(M)$ and $\Gamma_R(M)$, respectively. If $V(G_1) = V' = V(G_2)$ then $G_1 = \overline{G_2}$.

Although we have the partial relation between them, we don't know whether they are connected graph or disjoint graph. Therefore, we investigate the connectedness for each graph based on the previous results.

Theorem 10 Let $M = \mathbb{Z}_n$ is an R-modules where $n = p_1p_2p_3$ is the multiplication of three different prime numbers. Then $AG(M)$ and $\Gamma_R(M)$ are connected graphs.

Proof.
First, consider that the vertex set V can be partitioned as in the proof of Theorem 8 then suppose that $V' = V \setminus \cup_{i<j} V_{ij}$ and $V'' = \cup_{i<j} V_{ij}$. Let $AG(M')$ and $AG(M'')$ are induced subgraphs of $AG(M)$ with the vertex set are V' and V'', respectively. Based on Theorem 8, we have $AG(M')$ and $AG(M'')$ are complete 3-partite graphs. Let u and v are arbitrary elements in V' and V'', then choose $x = a\overline{p_1}$ in V' and $y = b\overline{p_2p_3}$ in V'' for some natural number a and b such that $Ann_M(x) = \overline{p_2p_3}M$, $Ann_M(y) = \overline{p_1}M$, and $Ann_M(xy) = M$. Thus, $Ann_M(x) \cup Ann_M(y) \neq Ann_M(xy)$ which mean x is adjacent with y. Since $AG(M')$ and $AG(M'')$ are complete 3-partite graph then there exists a path from u to x and from v to y. Finally, since x and y is adjacent then there exists a path from u to v which mean $AG(M)$ is connected. On the other side, let $\Gamma_R(M')$ and $\Gamma_R(M'')$ are induced subgraphs of $\Gamma_R(M)$ with the vertex set are V' and V'', respectively, then we have $\Gamma_R(M')$ is a union of three complete graphs and $\Gamma_R(M'')$ is a complete graph based on Theorem 8. We will show that for any V_i, there exist vertices $x_i \in V_i$ such that x_i is adjacent with a vertex in V''. Without loss of generality, choose $x = \overline{p_1} \in V_1$ and $y = \overline{p_1p_2} \in V_{12}$ such that $Ann_R(x) = \overline{p_2p_3}R$ and $Ann_R(y) = \overline{p_3}R$. Hence, $Ann_R(x) \cap Ann_R(y) \neq 0$ because $\overline{p_2p_3} \in Ann_R(x) \cap Ann_R(y)$, it means that x and y is adjacent. Therefore, $V_1, V_2,$ and V_3 are connected with V''. Finally, $\Gamma_R(M)$ is connected. ■

Example 11 Let $M = \mathbb{Z}_{30}$ is a $\mathbb{Z}_{30}$-module and let $p_1 = 2, p_2 = 3,$ and $p_3 = 5$. Note that the vertex set V can be partitioned into $V_1 = \{\overline{2}, \overline{4}, \overline{8}, \overline{14}, \overline{16}, \overline{22}, \overline{26}, \overline{28}\}$, $V_2 = \{\overline{3}, \overline{9}, \overline{21}, \overline{27}\}$, $V_3 = \{\overline{5}, \overline{25}\}$, $V_{12} = \{\overline{6}, \overline{12}, \overline{18}, \overline{24}\}$, $V_{13} = \{\overline{10}, \overline{20}\}$, and $V_{23} = \{\overline{15}\}$.

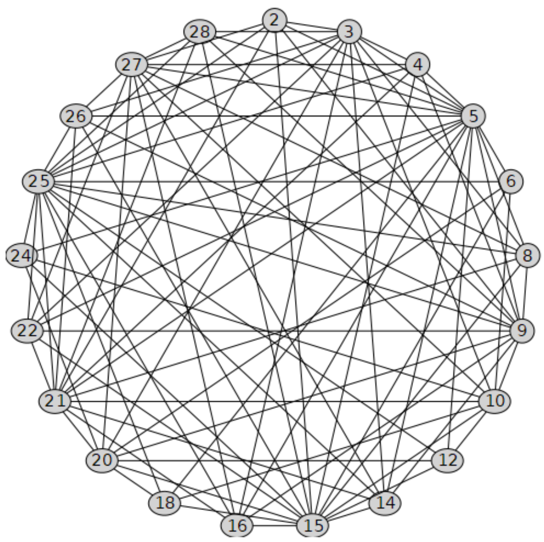

Figure 3. The annihilator graph of $\mathbb{Z}_{30}$

CONCLUSIONS

This research highlights the strong correspondence between the two associated graphs of the module $\mathbb{Z}_n$ over itself: the torsion graph and the annihilator graph. Although their exact relationship varies depending on the prime factorization of n, they consistently exhibit a high degree of structural connection. In some cases, the graphs are identical; in others, they are complementary or share partially overlapping substructures. This variation reflects how the properties of the ring influence the graphical expression of module-theoretic properties, especially the torsion elements and the annihilator of the modules.

The correspondence identified between these graphs contributes to bridging the gap between module theory and graph theory, offering new tools for interpreting algebraic structures through combinatorial frameworks. Such insights can be valuable in simplifying complex algebraic relationships or uncovering hidden symmetries in module categories.

Future research can extend these findings by continuing the investigation across broader prime factorizations of n, with the aim of systematically characterizing the correspondence for all possible forms of n, where n is any finite product of prime powers. Moreover, the analysis may be broadened by considering $\mathbb{Z}_n$ as a module over a commutative ring R where $R \neq \mathbb{Z}_n$, thereby exploring how these graph correspondences evolve in more general module and ring contexts.

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