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Application of Mixture of Weibull and Pareto (IV) Distribution to Health and Environmental Data

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Abstract

Weibull and Pareto distributions are widely used in several areas, including lifetime data modelling and reliability analysis. In real-life practice, these distributions may not capture the various distributional properties of certain datasets. The use of finite mixture models enhances the performance of these distributions in terms of adaptability and accuracy. This study focuses on the Mixture Weibull–Pareto (IV) distribution proposed by [1], which has been used in modelling insurance claims and has shown superior performance compared to other distributions. The current study applies this distribution to health and environmental datasets. The results show that the Mixture Weibull–Pareto (IV) distribution performs better than several competing distributions based on the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), and Kolmogorov–Smirnov (KS) goodness-of-fit test.

Keywords: Weibull distribution, Pareto distribution, Weibull–Pareto (IV), Mixture model, Density function

1 Introduction

Modelling lifetime and reliability data often requires probability distributions that are flexible enough to accommodate a wide range of hazard rate shapes and tail behaviours. Among such models, the Weibull distribution (WD) has become a standard tool in reliability engineering, survival analysis, and related fields because of its intuitive parameters and ease of interpretation [2]. Nevertheless, the classical WD is restricted to monotone hazard rate functions and therefore may provide an inadequate fit for datasets exhibiting non–monotone, bathtub-shaped, or unimodal hazard structures [3]. This limitation has motivated extensive research on generalizations of the WD and on mixture formulations that enhance its flexibility.

A large number of Weibull-based extensions have been proposed in the literature. Examples include the flexible Weibull extension [4] and the exponentiated Weibull distribution [5], both of which enrich the range of possible hazard shapes. Marshall and Olkin introduced an extended Weibull model to generate more complex failure-rate patterns, while [6] developed a modified Weibull extension specifically designed to capture bathtub-shaped hazard functions. Further families have been constructed by embedding the Weibull distribution in broader generator schemes, such as the Alpha Power Transformed Weibull–G family [7], the Beta–Weibull distribution [8], the Kumaraswamy–Weibull distribution [9], the Beta–modified Weibull distribution [10], the

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additive modified Weibull model [11], the Marshall–Olkin Weibull–Burr XII distribution [12], and the Topp–Leone modified Weibull model [13]. These constructions demonstrate the central role of the WD as a building block for more sophisticated lifetime models.

In parallel, mixture models have been widely used to capture heterogeneity and multimodality in lifetime data. Mixtures of Weibull components, in particular, provide additional flexibility through the superposition of different hazard shapes. Illustrative examples include two–component Weibull mixtures [14], mixture Weibull distributions for failure-time data [15], the log–logistic Weibull distribution [16], the half–logistic generalized Weibull distribution [17], the Weibull log–logistic (exponential) model [18], the Weibull–Lindley distribution [19], and the three-parameter Weibull inverse Rayleigh distribution [20]. Such models are particularly appealing when the underlying population consists of latent sub-populations with different failure mechanisms.

The Pareto distribution (PD), originally proposed by Vilfredo Pareto in the context of income and wealth modelling, is another fundamental model, especially suited for heavy–tailed data and extreme events. Its ability to accommodate large outliers and strong right–tail behaviour has led to numerous generalizations. Among these are the Beta–Pareto distribution [21], the Weibull–Pareto distribution [22], the Beta–exponentiated Pareto distribution [23], the extended odd Weibull–Pareto distribution [24], and a generalized version of the new Weibull–Pareto distribution [25]. These models have proven effective in fields where heavy-tail behaviour is prominent, such as finance, insurance, hydrology, and telecommunications.

Both the Weibull and Pareto families have therefore established themselves as workhorses for modelling skewed and heavy-tailed phenomena. Combining their strengths within a single framework is a natural way to increase modelling flexibility. In this direction, [1] introduced the Mixture Weibull–Pareto (IV) distribution, hereafter abbreviated as MWP(IV)D. This model blends a Weibull component with a Pareto (IV) component through a finite mixture structure, allowing it to accommodate a rich variety of shapes and tail behaviours. In the original study, the MWP(IV)D demonstrated competitive performance in modelling insurance claim data, outperforming several existing alternatives in terms of standard goodness-of-fit criteria.

The present paper extends the investigation of the MWP(IV)D by examining its usefulness in two applied contexts: health and environmental data. Specifically, we consider a dataset of breast cancer survival times and a dataset of flood exceedances from the Wheaton River. In both cases, the MWP(IV)D is compared with classical and generalized models based on the Weibull and Weibull–Pareto distributions.

The remainder of the article is organized as follows. Section 2 presents the theoretical formulation of the MWP(IV) distribution, including its density, cumulative distribution, survival, and hazard functions, as well as key shape characteristics and maximum likelihood estimation. Section 3 reports a simulation study and two real-data applications to assess the empirical performance of the model. Section 4 discusses the main findings and outlines directions for future research.

2 The MWP(IV) Distribution

In this section, we present the Mixture Weibull–Pareto (IV) distribution, denoted by MWP(IV)D. Following [1], consider a finite mixture of two continuous distributions with density

$$f(x) = pf_1(x) + (1-p)f_2(x), \tag{1}$$

where $0 < p < 1$ is the mixing proportion, and $f_1(x)$ and $f_2(x)$ are probability density functions (pdfs) of the component distributions. In the present case, $f_1(x)$ corresponds to a Weibull distribution and $f_2(x)$ corresponds to a Pareto (IV) distribution.

We denote the parameter vector of the mixture model by (μ, σ, k, p), where μ is the location parameter, $\sigma > 0$ is a scale parameter, $k > 0$ is a shape parameter, and p is the mixing weight.

Then the pdf of the MWP(IV)D can be written as

$$f(x \mid \mu, \sigma, k) = p\, f_1(x \mid \mu, \sigma, k) + (1-p)\, f_2(x \mid \mu, \sigma, k). \tag{2}$$

A functional representation for $f(x \mid \mu, \sigma, k)$ proposed in [1] is

$$f(x \mid \mu, \sigma, k) = p\left[1 - \exp\left\{-\left(\frac{x-\mu}{\sigma}\right)^{k}\right\}\right] + (1-p)\frac{1}{\sigma}\left[1 + k\left(\frac{x-\mu}{\sigma}\right)^{-1-\frac{1}{k}}\right], \tag{3}$$

for $x > \mu$, $\sigma > 0$, $k > 0$, and $0 < p < 1$.

2.1 Cumulative Distribution Function

The cumulative distribution function (CDF) associated with (3) is given by

$$F_{\mathrm{MWP(IV)D}}(x) = P(X \le x) = \int_{-\infty}^{x} f(t \mid \mu, \sigma, k)\, dt. \tag{4}$$

Using the mixture form in (2), we can equivalently write

$$F_{\mathrm{MWP(IV)D}}(x \mid \mu, \sigma, k) = p\, F_1(x \mid \mu, \sigma, k) + (1-p)\, F_2(x \mid \mu, \sigma, k), \tag{5}$$

where F_1 and F_2 denote the CDFs of the Weibull and Pareto (IV) components, respectively.

2.2 Survival Function

The survival function is defined as the complement of the CDF,

$$S(x) = 1 - F(x). \tag{6}$$

For the MWP(IV)D, using (5), we obtain

$$S_{\mathrm{MWP(IV)D}}(x \mid \mu, \sigma, k) = p\, S_1(x \mid \mu, \sigma, k) + (1-p)\, S_2(x \mid \mu, \sigma, k), \tag{7}$$

where S_1 and S_2 are the survival functions of the Weibull and Pareto (IV) components, respectively.

A closed-form expression for the survival function proposed in [1] can be written in the form

$$S_{\mathrm{MWP(IV)D}}(x \mid \mu, \sigma, k) = p\exp\left\{-\left(\frac{x-\mu}{\sigma}\right)^{k-1}\right\} + (1-p)\left(\frac{x-\mu}{\sigma}\right)^{-2-\frac{2}{k}}, \qquad x > \mu. \tag{8}$$

2.3 Hazard Function

The hazard function (HF) of a continuous distribution is defined as the ratio of the pdf to the survival function,

$$h(x) = \frac{f(x)}{S(x)} = \frac{f(x)}{1 - F(x)}. \tag{9}$$

For the MWP(IV)D, combining (3) and (8), we obtain

$$h_{\mathrm{MWP(IV)D}}(x \mid \mu, \sigma, k) = \frac{f(x \mid \mu, \sigma, k)}{S_{\mathrm{MWP(IV)D}}(x \mid \mu, \sigma, k)}, \qquad x > \mu. \tag{10}$$

2.4 Skewness and Kurtosis

Skewness and kurtosis are important measures that characterize the shape and tail behaviour of a distribution. Let $\mu = E(X)$ and $\sigma^2 = \text{Var}(X) = E(X^2) - \mu^2$. The (standardized) skewness and kurtosis can be defined, respectively, as

$$\theta_1 = \frac{E\big[(X-\mu)^3\big]}{\sigma^3}, \tag{11}$$

$$\theta_2 = \frac{E\big[(X-\mu)^4\big]}{\sigma^4}. \tag{12}$$

These moments can be obtained from the pdf in (3) once closed-form expressions or numerical approximations for $E(X^r)$, $r = 1, 2, 3, 4$, are available.

2.5 Maximum Likelihood Estimation

Following [1], let $X_1, \ldots, X_n$ be a random sample from the MWP(IV)D with pdf (3). Let θ_1 and θ_2 denote the parameter vectors associated with the Weibull and Pareto (IV) components, respectively, and write

$$\theta_1 = (\alpha, \delta, \sigma)', \qquad \theta_2 = (\beta, \gamma)', \qquad \theta = (\theta_1, \theta_2, p)'.$$

Denote the component densities by $f_1(x \mid \theta_1)$ and $f_2(x \mid \theta_2)$.

The likelihood function based on the observed sample $\mathbf{x} = (x_1, \ldots, x_n)'$ is

$$L(\theta) = \prod_{i=1}^{n} \left[p f_1(x_i \mid \theta_1) + (1-p) f_2(x_i \mid \theta_2)\right]. \tag{13}$$

In the context of the EM-type representation discussed in [1], latent indicators $\mu_i \in \{0, 1\}$ can be introduced and the complete-data likelihood can be written as

$$L_c(\theta) = \prod_{i=1}^{n} \left[f_1(x_i \mid \theta_1)^{\mu_i} f_2(x_i \mid \theta_2)^{1-\mu_i} p^{\mu_i} (1-p)^{1-\mu_i}\right]. \tag{14}$$

Maximization of the (observed) log-likelihood $\ell(\theta) = \log L(\theta)$ can be carried out numerically to obtain the maximum likelihood estimates (MLEs) of the parameters.

2.6 Model Selection Measures

To assess the goodness of fit and compare competing models, we employ the Akaike Information Criterion (AIC), the Bayesian Information Criterion (BIC), and the Kolmogorov–Smirnov (KS) statistic.

Let $\hat{\theta}$ denote the MLE of θ, and $\ell(\hat{\theta})$ the maximized log-likelihood. If p denotes the total number of free parameters in the model and n the sample size, then

$$\text{AIC} = -2\ell(\hat{\theta}) + 2p, \tag{15}$$

$$\text{BIC} = -2\ell(\hat{\theta}) + p\log(n). \tag{16}$$

The one-sample KS statistic is defined as

$$D_n = \sup_x \left|F_n(x) - F(x; \hat{\theta})\right|, \tag{17}$$

where $F_n(x)$ is the empirical distribution function and $F(x; \hat{\theta})$ is the fitted CDF of the model, such as the MWP(IV)D given in (5).

3 Model Application

In this section, we first present a simulation study to assess the estimation performance and goodness-of-fit of the Mixture Weibull–Pareto (IV) distribution (MWP(IV)D). We then illustrate the practical usefulness of the model by applying it to real datasets.

3.1 Simulation Study

Following the estimation procedure in [1], let $X_1, X_2, \ldots, X_n$ be a random sample of size n drawn from the density in (3). Let $\theta_1 = (\alpha, \delta, \sigma)'$ and $\theta_2 = (\beta, \gamma)'$ denote the parameter vectors associated with the component densities $f_1(x)$ and $f_2(x)$, respectively. The maximum likelihood estimators (MLEs) of the model parameters are obtained by maximizing the log-likelihood function with respect to all unknown parameters.

In the parametrization adopted in [1], if the support of the distribution is such that $x_i \geq \theta$ for all i, the MLE of the location parameter θ is the sample minimum,

$$\hat{\theta} = \min_i x_i.$$

The performance of the MLEs was investigated via a simulation study for several sample sizes, namely $n = 50, 100, 200, 500$, and 1000, and for different parameter sets $(\alpha, \delta, \sigma, \theta)$.

Table 1 summarizes the estimated parameters and the corresponding Akaike Information Criterion (AIC) values for three competing models: the Weibull–Weibull distribution (WWD), the Weibull–Pareto distribution (WPD), and the proposed MWP(IV) distribution. For each sample size, the MWP(IV) model attains the smallest AIC value, indicating a better fit among the considered alternatives.

Figure 1 illustrates the shapes of the MWP(IV)D pdf for various parameter combinations, with the mixing proportion fixed at $p = 0.4, 0.5, 0.6$, and 0.7.

3.2 Application of MWP(IV) to Real Datasets

In this subsection, the MWP(IV) distribution is applied to two real datasets in order to evaluate its empirical performance and to compare it with two competing models: (i) the Weibull distribution (WD), and (ii) the Weibull–Pareto distribution (WPD).

For each dataset, model parameters are estimated via maximum likelihood, and goodness-of-fit is assessed using the Akaike Information Criterion (AIC), the Bayesian Information Criterion (BIC), and the Kolmogorov–Smirnov (KS) statistic (with associated p-values).

Breast Cancer Survival Times

The first dataset consists of survival times (in months) of breast cancer patients, originally reported by Lee (1992) and reproduced in [26]. The sample contains $n = 100$ uncensored observations, with survival times ranging from 0.3 to 78.0 months. The ordered survival times are reported in (Table 2).

Table 3 reports the maximum likelihood estimates (MLEs) of the model parameters together with the AIC, BIC, and KS statistics for the WD, WPD, and MWP(IV)D models.

Figure 2 displays the fitted densities for the three models together with the empirical distribution of the breast cancer survival times, while Figure 3 presents the total time on test (TTT) plot, which provides additional insight into the underlying hazard behaviour.

Table 1: Simulation results: parameter estimates and AIC values for WWD, WPD, and MWP(IV) models under different sample sizes.

Sample size	Parameter	WWD	WPD	MWP(IV)
$n = 50$	$\hat{a}$	0.405	5.101	0.721
	$\hat{b}$	2.672	1.478	0.156
	$\hat{\alpha}$	5.313	5.602	0.086
	$\hat{\beta}$	27.653	1.525	0.981
	$\hat{\mu}$	-4.374	-0.230	-0.034
	AIC	4082.258	4035.758	3997.092
$n = 100$	$\hat{a}$	0.996	7.356	0.987
	$\hat{b}$	3.595	1.159	0.242
	$\hat{\alpha}$	2.310	9.122	0.028
	$\hat{\beta}$	40.191	2.742	1.367
	$\hat{\mu}$	-11.879	-0.198	-0.240
	AIC	9533.286	9426.776	9387.932
$n = 200$	$\hat{a}$	1.769	11.958	0.886
	$\hat{b}$	1.650	0.917	0.205
	$\hat{\alpha}$	1.2921	11.378	0.016
	$\hat{\beta}$	94.3416	11.6811	9.653
	$\hat{\mu}$	-22.965	-2.341	-0.087
	AIC	21818.960	21626.260	21507.140
$n = 500$	$\hat{a}$	0.748	2.054	0.763
	$\hat{b}$	0.415	12.307	0.170
	$\hat{\alpha}$	3.046	2.3610	0.008
	$\hat{\beta}$	461.985	2.545	1.079
	$\hat{\mu}$	-56.255	-0.852	-0.034
	AIC	63677.740	63537.080	62847.640
$n = 1000$	$\hat{a}$	0.933	4.662	0.813
	$\hat{b}$	0.795	1.707	0.175
	$\hat{\alpha}$	2.440	6.063	0.004
	$\hat{\beta}$	759.899	17.966	13.727
	$\hat{\mu}$	-111.914	-1.197	-2.850
	AIC	141195.100	140246.600	139632.300

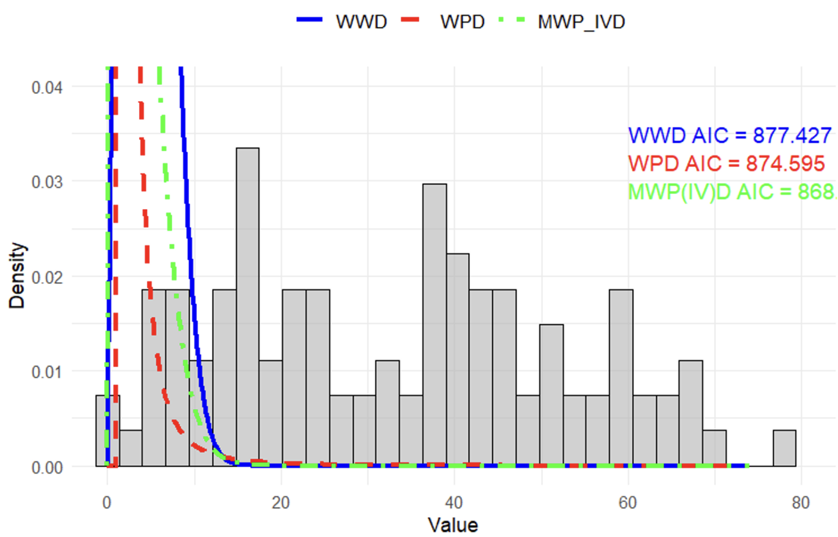
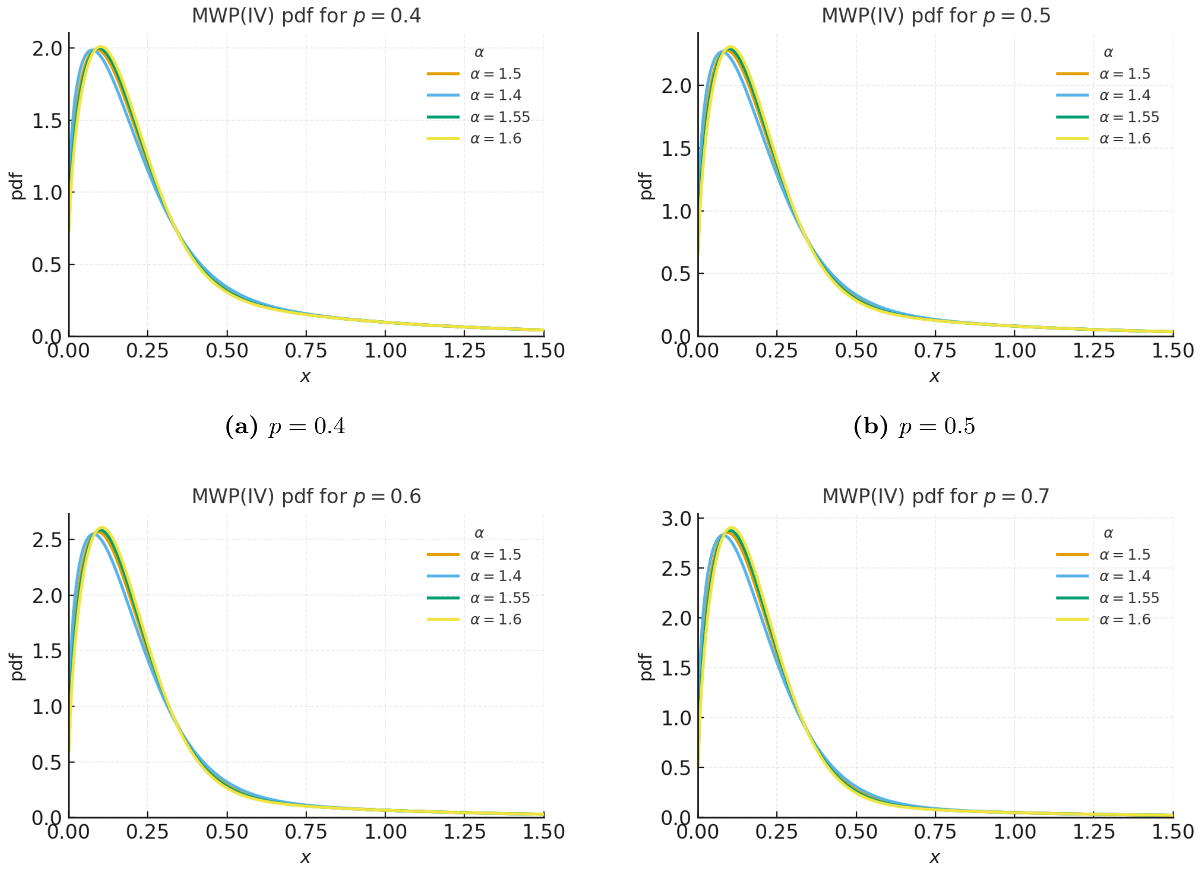


Figure 2: Fitted models for the breast cancer survival times data with corresponding AIC values.



(a) $p = 0.4$ **(b)** $p = 0.5$

(c) $p = 0.6$ **(d)** $p = 0.7$

Figure 1: Probability density functions of the MWP(IV) distribution for different values of the mixing proportion p and shape parameter α (with $\sigma = 0.2$, $\delta = 0.5$, $\beta = 9$, $Y = 5$).

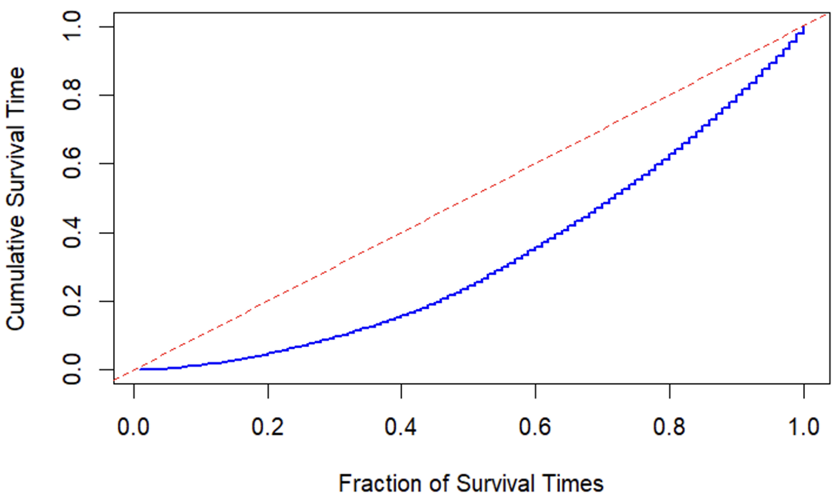


Figure 3: TTT plot for the breast cancer survival times data.

Exceedances of Wheaton River Flood Data

The second dataset consists of exceedances of daily mean flow from the Wheaton River, originally reported in [21]. The data exhibit strong right skewness and a pronounced heavy tail, making them a suitable benchmark for extreme–value and flood–risk modelling. The sample contains $n = 72$ positive observations, with exceedance values ranging from 0.1 to 64.0 (in appropriate

Table 2: Ordered survival times (in months) for the breast cancer dataset.

0.3	0.3	4.0	5.0	5.6	6.2	6.3	6.6	6.8	7.4
7.5	8.4	8.4	10.3	11.0	11.8	12.2	12.3	13.5	14.4
14.4	14.8	15.5	15.7	16.2	16.3	16.5	16.8	17.2	17.3
17.5	17.9	19.8	20.4	20.9	21.0	21.0	21.1	23.0	23.4
23.6	24.0	24.0	27.9	28.2	29.1	30.0	31.0	31.0	32.0
35.0	35.0	37.0	37.0	37.0	38.0	38.0	38.0	39.0	39.0
40.0	40.0	40.0	41.0	41.0	41.0	42.0	43.0	43.0	43.0
44.0	45.0	45.0	46.0	46.0	47.0	48.0	49.0	51.0	51.0
51.0	52.0	54.0	55.0	56.0	57.0	58.0	59.0	60.0	60.0
60.0	61.0	62.0	65.0	65.0	67.0	67.0	68.0	69.0	78.0

Table 3: MLEs, AIC, BIC, and KS statistics for the breast cancer survival times data [26].

Parameter / Criterion	WWD	WPD	MWP(IV)D
$\hat{a}$	1.0341	7.4227	1.0185
$\hat{b}$	0.8641	0.8995	0.3640
$\hat{\alpha}$	1.9155	4.4893	0.0325
$\hat{\beta}$	45.9588	15.5317563	–
$\hat{\mu}$	-4.7113	-4.6347	-0.5900
AIC	877.4268	874.5950	868.5098
BIC	890.4527	887.6208	878.8902
KS (p-value)	0.0970 (0.3034)	0.0912 (0.3769)	0.0696 (0.7177)

flow units). For completeness, the ordered exceedances are reported in Table 4.

Table 4: Ordered exceedances of Wheaton River flood levels.

0.1	0.3	0.4	0.4	0.6	0.6	0.6	0.7	1.0	1.1
1.1	1.4	1.5	1.7	1.7	1.7	1.7	1.9	2.2	2.2
2.5	2.5	2.5	2.7	2.8	3.4	3.6	4.2	5.3	5.3
5.6	7.0	7.3	8.5	9.0	9.3	9.7	9.7	10.4	10.7
11.0	11.6	11.9	12.0	13.0	13.3	14.1	14.1	14.4	14.4
15.0	16.8	18.7	20.1	20.2	20.6	21.5	22.1	22.9	25.5
25.5	27.0	27.1	27.4	27.5	27.6	30.0	30.8	36.4	37.6
39.0	64.0								

The parameter estimates and fit measures for the three competing models (WWD, WPD, and MWP(IV)D) are summarized in Table 4.

Figure 4 shows the fitted densities superimposed on the empirical distribution of the flood exceedances together with the AIC values, while Figure 5 provides the corresponding TTT plot.

4 Discussion

As the sample size n increases, the estimates $\hat{\beta}$ tend to decrease. According to [22], overestimation of β may occur when the minimum observation in a sample is substantially larger than the true population minimum, especially when the sample size is small.

The mixing parameter p plays a crucial role in characterizing the balance between central behaviour and tail behaviour in the MWP(IV)D. To investigate this effect, we conducted a simulation study for several values of p. In each case, the sample size was $n = 1000$, and the simulation was repeated 100 times. For illustration, the remaining parameters were fixed at $\alpha = 1.5$, $\delta = 1$, $\sigma = 0.5$, and $\theta = 1.5$, and we generated data from the MWP(IV)D using these values.

Table 5: MLEs, AIC, BIC, and KS statistics for the Wheaton River flood exceedances data [21].

Parameter / Criterion	WWD	WPD	MWP(IV)D
$\hat{a}$	0.7351	2.8756	0.7422
$\hat{b}$	0.7129	0.9693	2.6981
$\hat{\alpha}$	1.1456	4.3277	0.0190
$\hat{\beta}$	15.5925	3.6271	0.0491
$\hat{\mu}$	0.0266	-0.1205	–
AIC	512.5343	511.5384	508.0321
BIC	523.9180	522.8518	517.0829
KS (p-value)	0.0883 (0.6280)	0.0716 (0.8545)	0.0826 (0.7100)

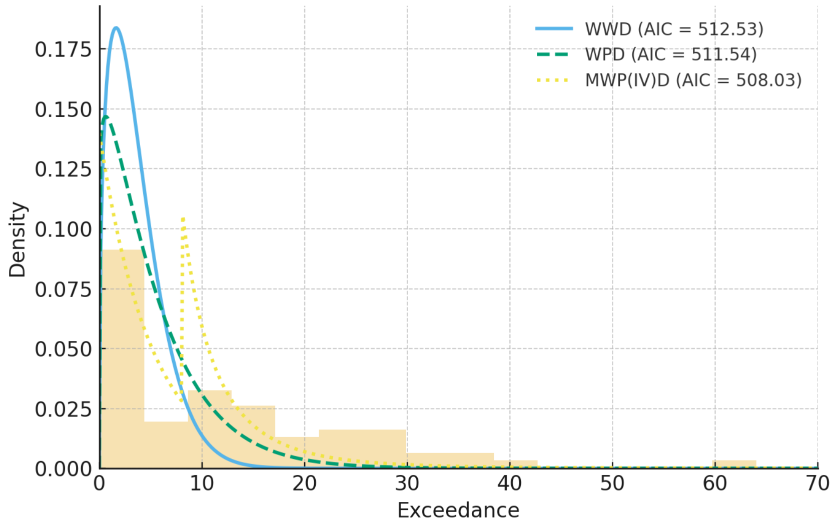


Figure 4: Fitted models for the Wheaton River flood exceedances data with corresponding AIC values.

Figure 1 displays the simulated probability density functions (pdfs) of the Mixture Weibull–Pareto (IV) distribution for different choices of the mixing proportion p. The plots highlight the remarkable flexibility of the MWP(IV)D in capturing a variety of shapes simply by changing p. When $p = 0.4$, the distribution exhibits a monotonically decreasing shape with a pronounced right tail, making it suitable for modelling highly skewed data or heavy-tailed phenomena such as extreme-value datasets. As p increases to 0.5, 0.6, and 0.7, the shape gradually shifts from heavy-tailed to lighter-tailed, with a more prominent peak and less dominant tail behaviour.

These plots clearly demonstrate that p primarily controls the trade-off between peak and tail: lower values of p emphasize the tail, which is important in reliability, risk analysis, and other applications where extreme outcomes are of interest; higher values of p reduce skewness and yield a more centrally concentrated distribution, making the model suitable for datasets with a more balanced spread. In addition, the other parameters influence the rate of decay and overall shape, enabling the MWP(IV)D to mimic unimodal, near-symmetric, and highly skewed patterns, and, for appropriate parameter choices, even more complex shapes.

The simulation results strongly suggest that the MWP(IV)D is capable of reproducing a wide range of data behaviours, from heavy-tailed to more moderate distributions. This flexibility underscores its potential as a powerful modelling tool for real-world datasets with complex patterns, skewness, or large tails.

After the simulation study, the MWP(IV)D was applied to two real datasets: (i) survival times of patients with breast cancer, and (ii) exceedances of flood levels from the Wheaton River. In each case, the proposed model was compared with two alternatives: the Weibull–Weibull distribution (WWD) and the Weibull–Pareto distribution (WPD).

For the breast cancer survival data, Table 3 presents the parameter estimates along with

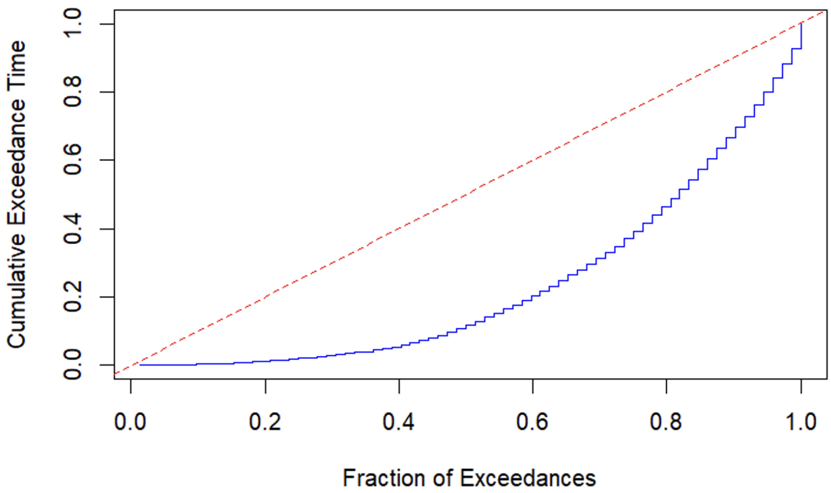


Figure 5: TTT plot for the Wheaton River flood exceedances data.

the AIC, BIC, and KS statistics. The AIC values are 877.4268 (WWD), 874.5950 (WPD), and 868.5098 (MWP(IV)D), while the corresponding BIC values are 890.4527, 887.6208, and 878.8902, respectively. The KS statistics and p-values are 0.0970 (0.3034) for WWD, 0.0912 (0.3769) for WPD, and 0.0696 (0.7177) for MWP(IV)D. Hence, the MWP(IV)D achieves the lowest AIC and BIC values and the largest KS p-value, indicating the best overall fit among the three models according to these criteria.

Figure 2 illustrates the fitted densities overlaid on the empirical distribution of the cancer survival data. The WWD curve fits well at very early survival times but then decays too rapidly, underestimating the density for survival times above approximately 10 and remaining close to zero across most of the histogram range. The WPD captures the early high density somewhat better but still drops off too quickly and fails to follow the distribution over the full range. The MWP(IV)D provides a more flexible fit, capturing a portion of the initial density while also offering improved behaviour in the right tail; nevertheless, it still does not perfectly match the empirical histogram across the entire range. Thus, although the AIC and BIC values favour the MWP(IV)D, some discrepancy between the fitted model and the empirical distribution remains, particularly in the mid-to-upper range of survival times.

The total time on test (TTT) plot in Figure 3 exhibits a non-linear, non-monotonic pattern, revealing variability in the underlying hazard rate over time. The initial convexity of the TTT curve suggests a decreasing hazard rate at early times, implying a lower failure risk initially. As the curve becomes concave, the hazard rate increases at longer survival times, a pattern consistent with many clinical settings, where the risk of adverse events often rises as time progresses. This behaviour indicates that a flexible model capable of accommodating non-monotone hazard rates, such as the MWP(IV)D, is particularly appropriate for this dataset.

For the Wheaton River flood exceedances data, Table 5 summarizes the fit results. The AIC values are 512.5343 (WWD), 511.5384 (WPD), and 508.0321 (MWP(IV)D); the corresponding BIC values are 523.9180, 522.8518, and 517.0829. The KS statistics and p-values are 0.0883 (0.6280) for WWD, 0.0716 (0.8545) for WPD, and 0.0826 (0.7100) for MWP(IV)D. Based on AIC and BIC, the MWP(IV)D provides the best fit to the flood exceedances, with the WPD as a close alternative and the WWD performing comparatively worse.

Figure 4 shows that the MWP(IV)D captures the distribution of exceedances well, particularly in the upper tail region, where accurate modelling of extreme events is crucial for risk assessment. The WPD also offers a reasonable approximation but tends to be less precise in the most extreme values. The WWD, with higher AIC and BIC and less agreement in the tail, appears less suitable for this dataset.

The TTT plot for the Wheaton River data (Figure 5) reveals important features of the extreme-value behaviour. The initial steep portion of the curve indicates a relatively high rate of exceedances at moderate levels, corresponding to frequent but less severe flood events. As the curve flattens, it reflects the presence of a heavier tail and rarer but more extreme exceedances. The MWP(IV)D, which can flexibly account for such changing hazard behaviour, is therefore a natural candidate for modelling flood exceedances, capturing both moderate and extreme events within a unified framework.

Overall, the empirical analyses demonstrate that the MWP(IV)D is not only competitive but often superior to the WWD and WPD in terms of AIC, BIC, and KS statistics, particularly in applications where tail behaviour and non-monotonic hazard rates are important.

5 Conclusion

In this work, we have studied a Pareto-type family of distributions constructed from the Weibull distribution, referred to as the Mixture Weibull–Pareto (IV) distribution (MWP(IV)D). The model combines the flexibility of the Weibull distribution with the heavy-tailed nature of the Pareto (IV) distribution through a mixing mechanism. The parameters of the MWP(IV)D were estimated using the maximum likelihood approach, following the methodology in [1].

A simulation study highlighted the flexibility of the MWP(IV)D in generating a wide variety of shapes and tail behaviours by appropriate choices of the mixing proportion and component parameters. The model was then applied to oncology (breast cancer survival times) and environmental (Wheaton River flood exceedances) datasets. In both cases, the MWP(IV)D provided an excellent fit relative to competing models, as indicated by AIC, BIC, and KS statistics, and it proved particularly useful in capturing non-monotone hazard rates and heavy-tail behaviour.

Future research may focus on enhancing the robustness and flexibility of the MWP(IV)D, for example by introducing additional shape parameters, exploring alternative estimation techniques (such as Bayesian or robust inference), or extending the model to regression and multivariate frameworks. Such developments would further broaden the scope of applications, as envisaged in [1].

CRediT Authorship Contribution Statement

Tolulope Olubunmi Adeniji: Conceptualization, Methodology, Formal Analysis, Investigation, Data Curation, Software, Visualization, Writing–Original Draft Preparation. **Akinwumi Sunday Odeyemi:** Methodology, Validation, Supervision, Writing–Review & Editing.

Declaration of Generative AI and AI-assisted Technologies

Generative AI and AI-assisted technologies were used during the preparation of this manuscript strictly for language polishing, structural editing, and assistance in generating illustrative figures. All statistical analyses, modelling decisions, and scientific interpretations were conceived and verified by the authors, who take full responsibility for the content of the article.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data and Code Availability

All empirical data analyzed in this study (breast cancer survival times and Wheaton River flood exceedances) are reproduced in the manuscript from previously published sources cited in the reference list. The simulated datasets and computer codes used to fit the models and generate the numerical results are available from the corresponding author on reasonable request (e-mail: tolbum17@gmail.com).

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