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Systematic Review of Robust Mixed-Integer Linear Programming with Benders Decomposition for Facility Location Problems

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ABSTRACT

The robust Mixed-Integer Linear Programming (MILP) model addresses uncertainties in linear optimization problems involving integer and continuous variables and can be solved using the Benders Decomposition method. One significant application area is facility location problems, which frequently encounter uncertainties on demand, costs, and capacities. This study fills a gap by conducting a systematic literature review (SLR) on solving robust MILP models using the Benders Decomposition method, focusing on their application to facility location problems. It aims to examine the state-of-the-art, identify commonly addressed issues, and analyze frequently used uncertainty sets. Using the Preferred Reporting Items for Systematic Review and Meta-Analysis (PRISMA) method, the SLR review publications from the last five years in Scopus, Science Direct, and Dimensions databases, with bibliometric analysis using VOSviewer and RStudio. Our findings reveal a gap in the research on robust MILP models for facility location with ellipsoidal uncertainty sets using Benders Decomposition. The method is widely applied to robust MILP problems in energy, logistics, supply chains, and scheduling, with interval uncertainty sets being the most common. This area offers significant potential for further exploration.

Keywords: SLR, Robust Optimization, Uncertainty Sets, Mixed-Integer Linear Programming, Benders Decomposition, Facility Location Problem, Optimization in Logistics.

INTRODUCTION

Optimization is an effort to achieve the best results in a given situation [1]. The optimization model expresses optimization problems by determining the decision variables, objective functions, and constraint functions [2]. One such model is mixed-integer linear programming (MILP), where some variables must take integer values [1].

In real-world problems, MILP often involves uncertain parameters, known as uncertainty. Robust optimization is a key method for addressing such uncertainty [3], where uncertain parameters are assumed to lie within specific uncertainty sets, such as box/interval, ellipsoidal, or polyhedral sets [4]. Robust MILP models have been widely applied to solve practical problems, including inventory routing [5], stock portfolio optimization [6], and notably, facility location problems[7].

Facility location problems are critical due to their strategic importance, influencing industry operational decisions [8]. These problems often involve uncertain parameters,

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such as customer demand, facility capacities, and other factors [9]. To effectively manage such uncertainties, robust MILP models offer a systematic approach to ensure feasible and reliable decisions.

Given their combinatorial nature and uncertain parameters, solving robust MILP models for facility location problems is computationally demanding. One promising method for tackling this complexity is Benders Decomposition. This approach divides the original problem into smaller, more manageable subproblems: a continuous subproblem to handle real-valued variables and an integer subproblem to address discrete decision variables. By iteratively solving these subproblems, Benders Decomposition significantly enhances computational efficiency and scalability, especially for large-scale problems.

Despite its potential, systematic literature review (SLR) focusing on robust MILP models solved using Benders Decomposition, particularly in facility location problems, remains limited. Table 1 summarizes existing review articles relevant to this study to establish the context and identify gaps.

Table 1. Previous Review Article

Paper	Robust Optimization	MILP	Benders Decomposition	Facility location
[10]	✓	-	-	-
[11]	✓	-	-	-
[12]	✓	-	-	✓
[13]	-	✓	✓	-
[14]	-	-	-	✓
[15]	-	-	-	✓
This paper	✓	✓	✓	✓

The review reveals that while articles review [10] and [11] discuss robust optimization applications, they do not address MILP or Benders Decomposition. References [12] examines facility location models considering uncertainty, focusing on stochastic and robust optimization approaches. Article [13] highlights MILP solution techniques, including Benders Decomposition, but does not apply these to facility location problems. Meanwhile, articles [14] and [15] review facility location models but overlook robust MILP and decomposition methods.

From this analysis, no existing articles provide an SLR specifically addressing robust MILP models solved using Benders Decomposition for facility location problems. This study aims to fill that gap and contribute to the field by providing a focused review. The objectives of this research are to identify the state of the art in solving robust MILP models using the Benders decomposition method in facility location problems, analyze research trends in this area, examine the most commonly used uncertainty sets, and explore the types of problems that have been modeled as robust optimization problems and solved using Benders Decomposition.

METHODS

This systematic literature review (SLR) involves two main steps: conducting a systematic search using the Preferred Reporting Items for Systematic Review and Meta-Analysis (PRISMA) method to address RQ.1 through 4 and performing bibliometric analysis to answer RQ.2.

PRISMA Methods

A systematic search followed the steps outlined in the PRISMA method, which provides a structured guideline for conducting SLR [16]. The PRISMA method has been proven to enhance the quality of SLR [17] in terms of methodology and outcomes. comprises four stages: identification, screening, eligibility, and inclusion.

In the identification stage, literature was searched using relevant keywords, as shown in Table 2, by combining at least two. The search was conducted on Scopus, Science Direct, and Dimensions databases with the following limitations: publication year between 2020 and 2024, document type as either articles or conference papers, English language, and sources from journals or conference proceedings. This process resulted in 486 articles, with details presented in Table 3.

Table 2. Keyword Used

Code	Keyword
A	“optimization model” AND "integer" AND "linear"
B	“robust counterpart” OR “robust optimization”
C	“benders decomposition”
D	"facility location"

Table 3. Search Results on the databases

Type	Code	Scopus	Science Direct	Dimensions
I	A	1703	731	1336
II	B	6458	2155	5057
III	C	1213	485	1114
IV	D	2263	650	1961
V	A AND C	22	14	15
VI	B AND C	130	63	90
VII	D AND C	53	29	46
VIII	A AND B AND C	9	8	7
IX	A AND B AND C AND D	0	0	0
	Total types V to IX	214	114	158

In the second stage (Screening), duplicate articles were identified and removed using Mendeley reference management software. This process resulted in the exclusion of 280 duplicate articles, leaving 206 articles for further evaluation.

The third stage is eligibility, where articles are selected based on their titles, abstracts, and full texts. During the selection based on titles and abstracts, 71 articles were excluded for not addressing robust optimization and Benders decomposition simultaneously, leaving 135 articles. From these, 39 articles that employed robust optimization and classical Benders decomposition were selected, while 96 articles focusing on either robust optimization or advanced forms of Benders decomposition were excluded. In the full-text evaluation, six articles were removed due to lack of access, and 19 articles were excluded for being irrelevant to the topic. This process resulted in 14 articles being included in the final stage.

As detailed in the steps outlined earlier, the search process based on the PRISMA methodology is summarized in Figure 1, which illustrates the flow of articles through the stages of identification, screening, eligibility, and inclusion. This figure provides a clear overview of how articles were processed and selected at each stage.

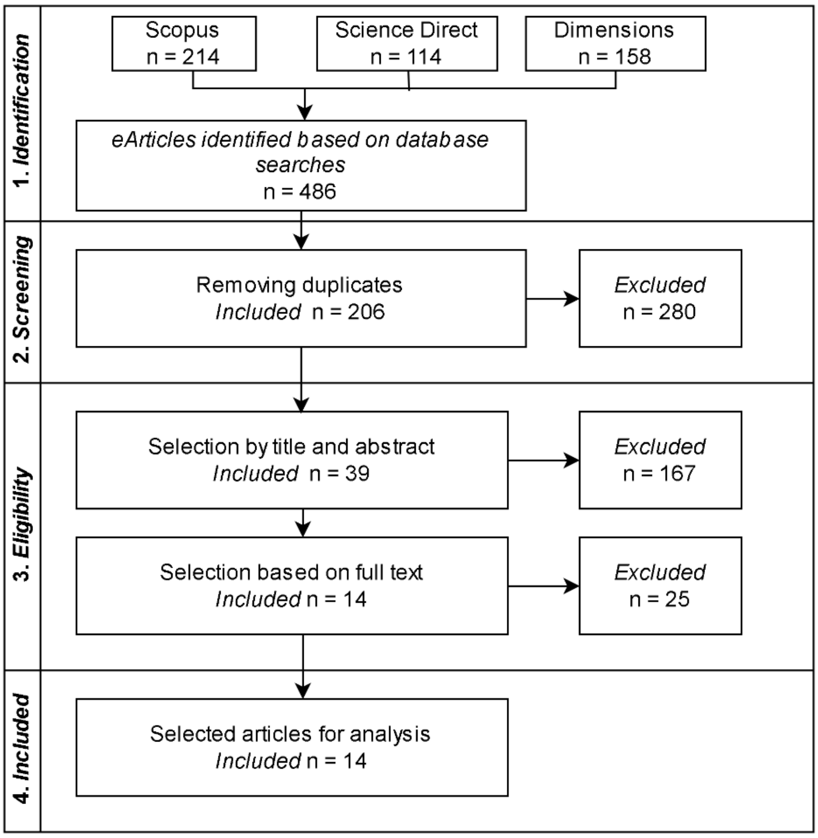

Figure 1. PRISMA Flow Diagram for Literature Search

Bibliometric Analysis

Bibliometric analysis often provides a bibliographic overview of high-citation scientific publications [18]. This study employs VOSviewer and RStudio for bibliometric analysis, each offering distinct advantages. VOSviewer excels in network visualization with superior graphical representation [18], RStudio enables more in-depth data exploration due to its wide range of statistical and graphical analysis options.

Data from 14 articles identified through the PRISMA method in the previous process were utilized to generate bibliometric mapping. The bibliometric mapping process follows three main steps as outlined in [11], [19], [20] : mining bibliometric data, analyzing bibliometric data, and mapping the results. These steps are implemented as illustrated in Figure 2.

In the first step, bibliometric data were mined from academic databases such as Scopus, ScienceDirect, and Dimensions to ensure the inclusion of relevant publications on robust MILP models using Benders decomposition in facility location problems. The extracted metadata, including titles, authors, abstracts, keywords, citations, and publication years, were carefully reviewed, cleaned, and prepared for analysis.

In the next stage, bibliometric data were analyzed using descriptive and statistical methods. Descriptive statistics, such as the total number of articles published per year, citation counts, and average citations per publication, were calculated to identify trends in scientific productivity and the influence of key works. Keyword frequency analysis was performed to uncover prominent research themes, while co-occurrence matrices of author keywords were generated to examine relationships between recurring terms. Citation analysis was conducted to determine the most influential authors, articles, and journals in the field by analyzing total citation counts and identifying high-impact publications.

The final step involved mapping the results through visualization and thematic analysis. VOSviewer was employed to construct co-occurrence networks of keywords, revealing clusters of interconnected research topics through graph-based clustering techniques, such as modularity optimization. RStudio was used to analyze thematic trends over time by applying time-series analysis and linear regression, enabling the identification of emerging research directions and the evolution of key themes within the field.

This integrated approach, combining statistical analysis and visualization tools, provides a comprehensive understanding of the intellectual structure, influential contributors, and evolving trends in research on robust MILP models with Benders decomposition for facility location problems.

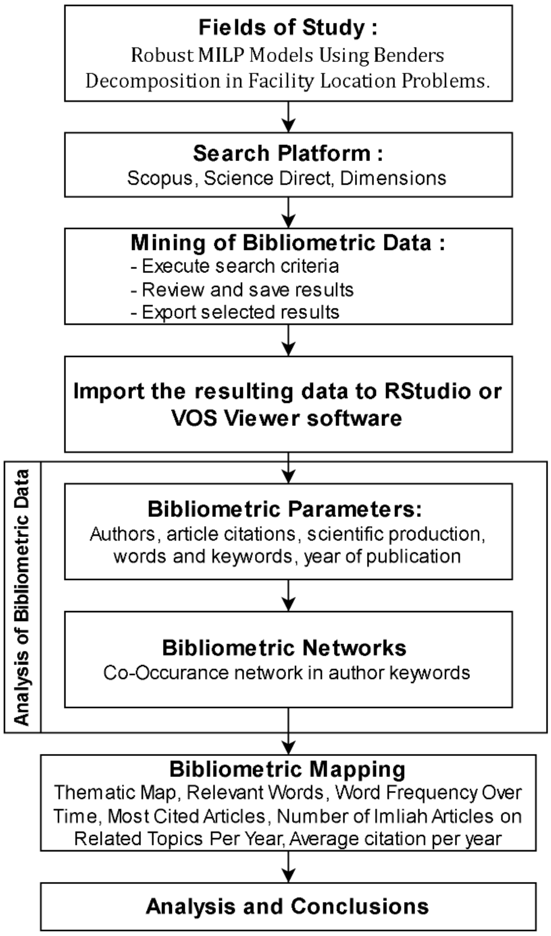

Figure 2. Bibliometric Mapping Process

RESULTS AND DISCUSSION

The State of the Art in Solving Robust Optimization Models for MILP Problems using Benders Decomposition and its Application to Facility Location Problems

Robust Optimization and Benders Decomposition have emerged as powerful tools for tackling mixed-integer linear programming (MILP) problems involving uncertainty. While Robust Optimization focuses on generating solutions resilient to variations within an uncertainty set, Benders Decomposition excels in handling MILP models with a combination of discrete and continuous decision variables by partitioning the problem into simpler subproblems.

The MILP model involves optimizing a linear objective function subject to linear constraints, with at least one decision variable required to be an integer [21]. The general form of a MILP model can be expressed as follows [22]:

$$\begin{aligned} &\min_{\mathbf{x},\mathbf{y}} f(\mathbf{x},\mathbf{v}) \\ &s.t \quad \mathbf{a}_i{}^T\mathbf{x} + \mathbf{b}_i{}^T\mathbf{y} \le f_i \quad (i = 1,2,3,\dots,m) \\ &\qquad \mathbf{x} \in \mathbb{R}^{n_x}, \mathbf{y} \in \mathbb{Z}^{n_y} \end{aligned} \tag{1}$$

where n_x and n_y represent the number of continuous and integer variables, respectively. The vector $\mathbf{x} \in \mathbb{R}^{n_x}$ represents continuous decision variables, and $\mathbf{y} \in \mathbb{Z}^{n_y}$ represents integer decision variables. The inequality $\mathbf{a}_i{}^T\mathbf{x} + \mathbf{b}_i{}^T\mathbf{y} \le f_i$ for $i = 1,2,3,\dots,m$ represents the i-th constraint, where $\mathbf{a}_i \in \mathbb{R}^{n_x}$, $\mathbf{b}_i \in \mathbb{R}^{n_y}$ are the technology coefficient vectors, and $f_i \in \mathbb{R}$ is the right-hand side value of the constraint function. If the coefficients $\mathbf{a}_i$ and $\mathbf{b}_i$ are assumed to be uncertain, the problem's uncertainty can be addressed using robust optimization by considering the uncertainty within box/interval, ellipsoidal, or polyhedral sets [23].

Benders [24] introduced the Benders Decomposition method to solve MILP problems. The method involves decomposing the problem into two subproblems: one involving the continuous variables and the other involving the discrete variables. By temporarily treating some variables as constants, these problems become easier to solve [25]. Consider the following minimization problem with the initial problem $P(\mathbf{x},\mathbf{y})$ [26]:

$$\begin{aligned} \min_{\mathbf{x},\mathbf{y}} \quad & \mathbf{c}^{\mathrm{T}}\mathbf{x} + f(\mathbf{y}) \\ s.t \quad & A\mathbf{x} + F(\mathbf{y}) = \mathbf{b} \\ & \mathbf{x} \ge \mathbf{0} \\ & \mathbf{y} \in Y \end{aligned} \tag{2}$$

where $A \in \mathbb{R}^{m\times n}, \mathbf{x}$ and $\mathbf{c} \in \mathbb{R}^n, \mathbf{b} \in \mathbb{R}^m$, and $\mathbf{y} \in Y \subset \mathbb{R}^p$. Here, $f(\mathbf{y})$ and $F(\mathbf{y})$ may be nonlinear and Y either discrete or continuous. For a fixed value of $\mathbf{y} \in Y$, this problem becomes a linear programming problem in $\mathbf{x}$, represented mathematically as $P(\mathbf{x}|\mathbf{y})$. Assume that $P(\mathbf{x}|\mathbf{y})$ has a bounded optimal solution $\mathbf{x}$, for each $\mathbf{y} \in Y$. The model (6) can be formulated as:

$$\min_{\mathbf{y}\in Y} \left\{ f(\mathbf{y}) + \min_{\mathbf{x}} \{\mathbf{c}^T\mathbf{x} | A\mathbf{x} = \mathbf{b} - F(\mathbf{y}), \mathbf{x} \ge \mathbf{0}\} \right\}. \tag{3}$$

The inner optimization problem of equation (3) is:

$$\min_{\mathbf{x}} \{\mathbf{c}^T\mathbf{x} | A\mathbf{x} = \mathbf{b} - F(\mathbf{y}), \mathbf{x} \ge \mathbf{0}\}. \tag{4}$$

The dual formulation of this inner problem is:

$$\max_{\mathbf{u}} \{[\mathbf{b} - F(y)]^T\mathbf{u} | A^T\mathbf{u} \le \mathbf{c}\}. \tag{5}$$

Substituting equation (5) into equation (3) yields:

$$\min_{\mathbf{y}\in Y} \left\{ f(\mathbf{y}) + \max_{\mathbf{u}} \{[\mathbf{b} - F(\mathbf{y})]^T\mathbf{u} | A^T\mathbf{u} \le \mathbf{c}\} \right\}. \tag{6}$$

This formulation simplifies the inner problem constraints, as they no longer depend on $\mathbf{y}$. The optimal solution of the inner problem is bounded because $P(\mathbf{x}|\mathbf{y})$ has a bounded optimal solution for each $\mathbf{y} \in Y$. The solution will always lie on the extreme point $u \in U$, so equation (6) can be expressed as:

$$\min_{\mathbf{y}\in Y} \left\{ f(\mathbf{y}) + \max_{u\in U} \{[\mathbf{b} - F(\mathbf{y})]^T\mathbf{u}\} \right\}. \tag{7}$$

The original model (2) can then be simplified into the Full Master Problem:

$$\begin{aligned} \min\ & f(\mathbf{y}) + m \\ s.t\ & [\mathbf{b} - F(\mathbf{y})]^{\mathrm{T}}\mathbf{u} \le m \\ & \mathbf{y} \in Y \\ & \mathbf{u} \in U \end{aligned} \tag{8}$$

The Relaxed Master Problem $M(\mathbf{y}, m)$ is defined as:

$$\begin{aligned} \min\ & f(\mathbf{y}) + m \\ s.t\ & [\mathbf{b} - F(\mathbf{y})]^{T}\mathbf{u} \le m, \qquad \mathbf{u} \in B \\ & \mathbf{y} \in Y \end{aligned} \tag{9}$$

where B is the empty set, and m is initially set to 0. The *Subroblem* $S(\mathbf{u}|\mathbf{y})$ is:

$$\begin{aligned} \max\ & [\mathbf{b} - F(\mathbf{y})]^{T}\mathbf{u} \\ s.t\ & A^{\mathrm{T}}\mathbf{u} \le \mathbf{c} \end{aligned} \tag{10}$$

with $\mathbf{u} \in \mathbb{R}^m$, $S(\mathbf{u}|\mathbf{y})$ has a bounded optimal solution, assuming that P(x|y) has a bounded optimal solution for each $\mathbf{y} \in Y$.

The iterative process of this method involves solving the subproblem to find $\mathbf{u}$, given a $\mathbf{y} \in Y$ from the *Master Problem*. The method then checks if a constraint involving $\mathbf{u}$ needs to be added to the Master Problem. If necessary, the Master Problem is resolved to generate a new $\mathbf{y}$ to be used as input for the subproblem, and the process continues until an optimal solution is reached with a specified tolerance. Further details of the Benders Decomposition algorithm are shown in Figure 3.

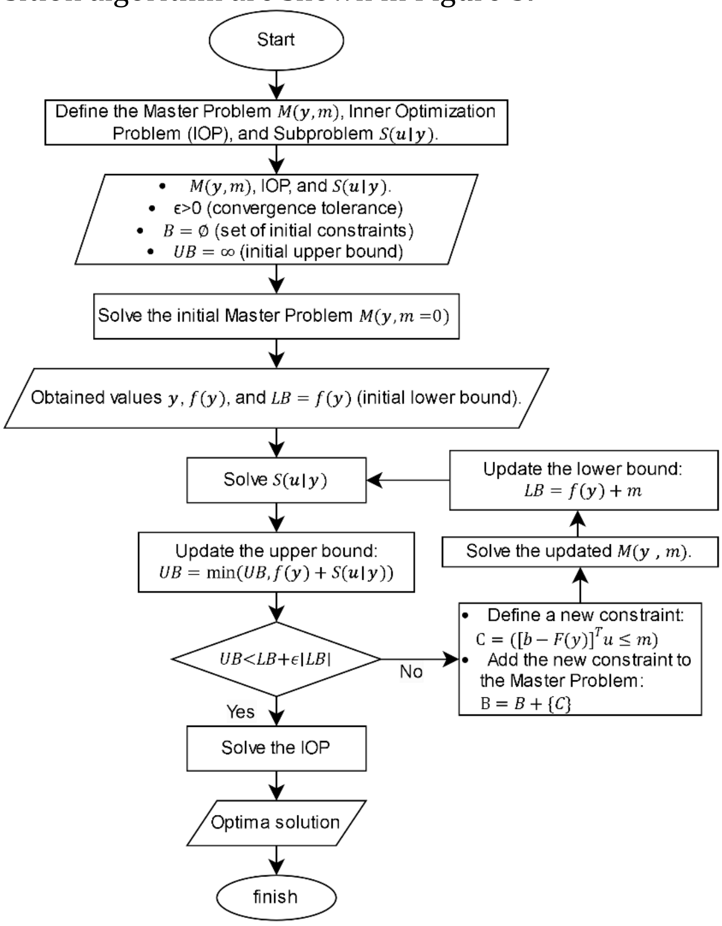

Figure 3. Benders Decomposition Algorithm

As discussed earlier, robust MILP models can be solved using the Benders Decomposition. Table 4 presents an overview of recent advancements in solving robust MILP problems using Benders Decomposition, particularly in the context of facility location problems. This information is derived from 14 articles identified through the PRISMA methodology. These articles, published in the past five years, explore the application of Benders Decomposition to robust optimization models for MILP problems involving various uncertainty sets.

Table 4. State of the art on solving robust optimization models for MILP problems using Benders Decomposition in facility location problems

Paper	MILP	Uncertainty		Benders Decomposition Method	Facility Location Problem
		Robust Optimization	Uncertainty set		
[27]	✓	✓	Multi-band	✓	-
[28]	✓	✓	Polyhedral	Combined with cutting plane	✓
[29]	✓	✓	Interval	✓	-
[30]	✓	✓	Polyhedral	✓	-
[31]	✓	✓	Polyhedral	✓	-
[32]	✓	✓	Interval	✓	-
[33]	✓	✓	Interval	✓	-
[34]	✓	✓	Interval	✓	✓
[35]	✓	✓	Interval	✓	-
[36]	✓	✓	Interval, polyhedral	Enhanced Benders' Decomposition (EBD)	-
[37]	✓	✓	Interval, polyhedral	✓	-
[38]	✓	✓	Interval	robustness based benders decomposition (RBBD)	-
[39]	✓	✓	Interval, polyhedral	Enhanced Benders' Decomposition (EBD)	-
[40]	✓	✓	Interval	✓	-

Table 4 shows that over the past five years, no studies have addressed solving robust optimization models for MILP with ellipsoidal uncertainty sets using the Benders Decomposition Method. Robust optimization models with ellipsoidal uncertainty can be applied to various real-world problems, such as scheduling operating rooms in hospitals [41], stock portfolio optimization [6], facility location problems [7], and others.

This gap highlights further research and innovation opportunities, particularly in applying Benders Decomposition to solve robust optimization models for MILP problems with ellipsoidal uncertainty sets, specifically focusing on facility location problems. The analysis suggests that there is a need for new approaches and methodologies to address this challenge, offering a clear avenue for novelty in this area of study.

Research Trends on Solving Robust MILP Models using the Benders Decomposition Method

Bibliometric analysis was conducted using a dataset of 14 articles filtered through the full-text selection stage to identify trends and developments in research related to this topic. VOSViewer software was used to visualize keyword occurrences and the relationships between them. The dataset includes 39 keywords, with 36 retained after synonym filtering. The bibliometric map for co-occurrence of keywords is shown in Figure 4.

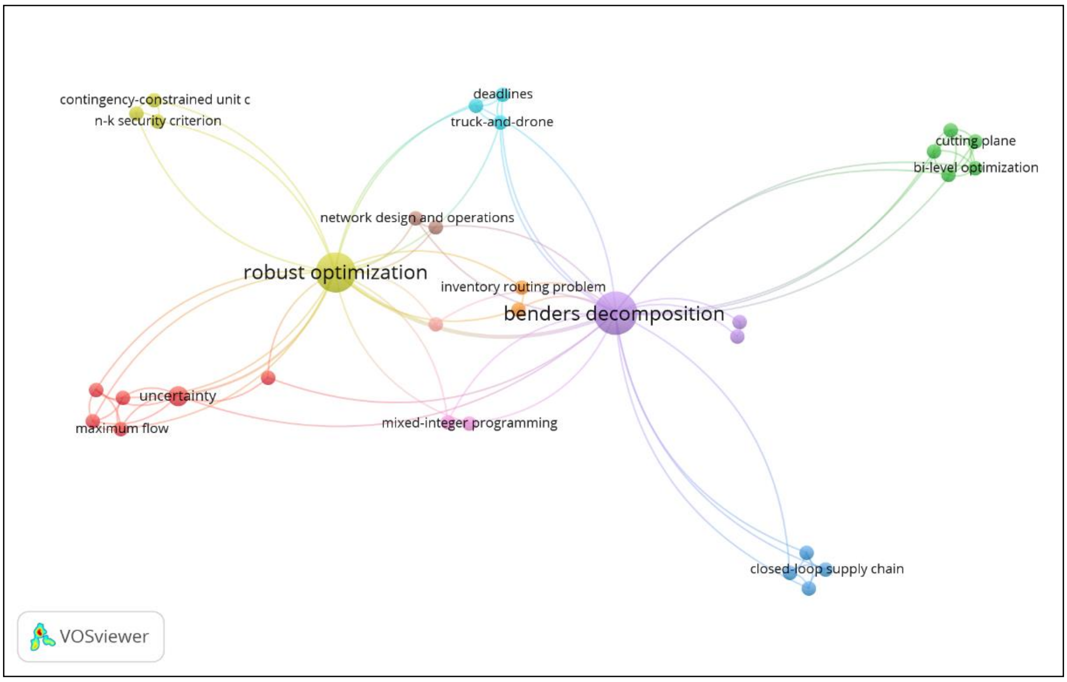

Figure 4. Co-Occurrence Bibliometric Map

The map in Figure 4 shows that the dataset's keywords are grouped into ten clusters, distinguished by different colors. The most frequent keywords are "Benders decomposition" (10 occurrences), "robust optimization" (8 occurrences), and "uncertainty" (2 occurrences), with others appearing once each. The keyword most frequently associated with others is "Benders decomposition," highlighting its widespread use in solving robust optimization problems.

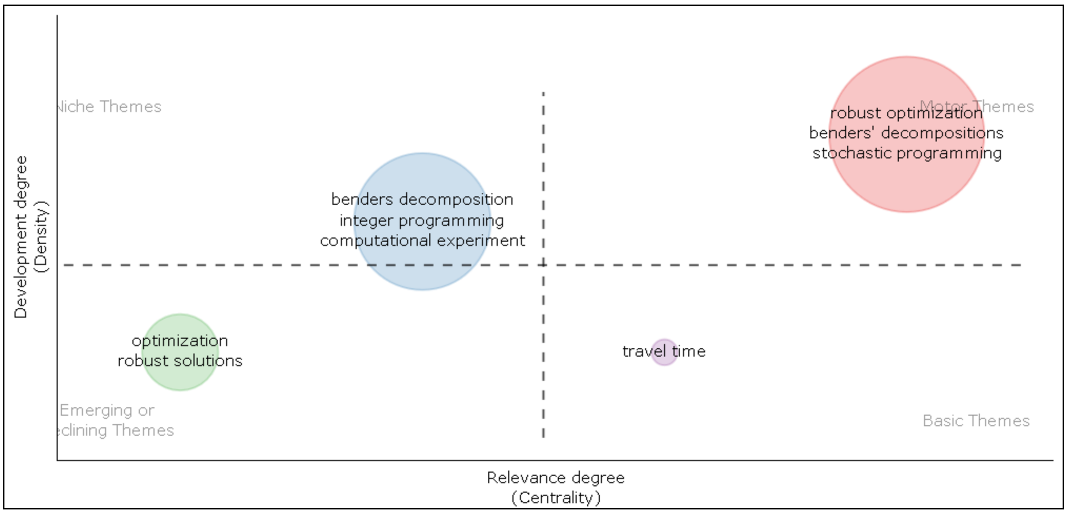

Figure 5. Thematic Map

Figure 5 displays the thematic map generated using RStudio. The vertical axis represents the density of publications on the topic, while the horizontal axis represents the centrality or impact of the study on the field. There are two main clusters in the top right and top left quadrants. Cluster I (top right) includes "robust optimization," "Benders decomposition," and "stochastic programming," which fall under the category of motor themes. This category is well-established, with numerous publications and significant influence. Cluster II (top left), with keywords like "Benders decomposition," "computational experiment," and "integer programming," falls under Niche Themes.

These topics are often studied but remain isolated in specific areas, offering potential for further exploration.

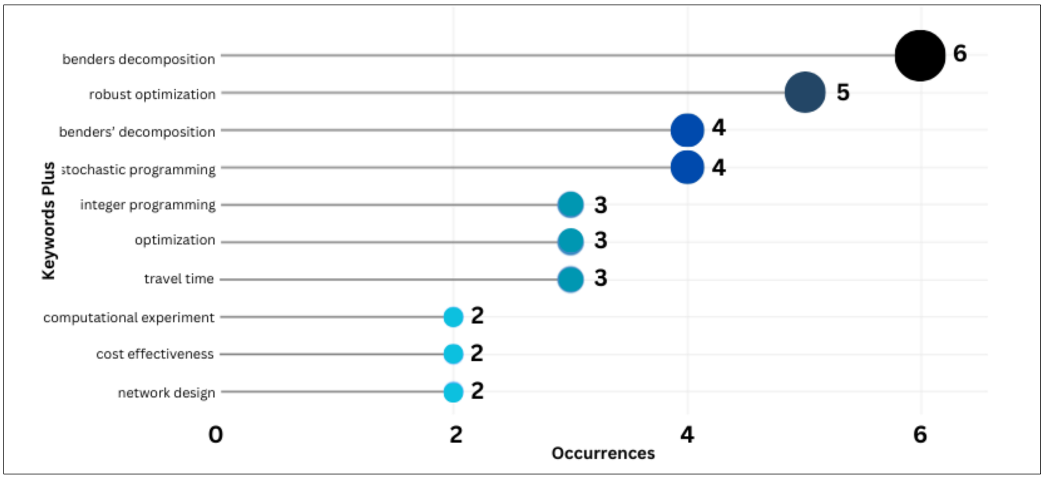

Figure 6. Relevant Keyword

Figure 6 shows the ten most common words, with "Benders decomposition" appearing the most (10 occurrences), followed by "robust optimization" (5 occurrences) and "stochastic programming" (4 occurrences). This distribution suggests that the research community focuses primarily on Benders Decomposition and robust optimization, indicating their centrality to the field. The prominence of "stochastic programming" also reflects the interest in addressing uncertainty in optimization models, which is often associated with Benders Decomposition.

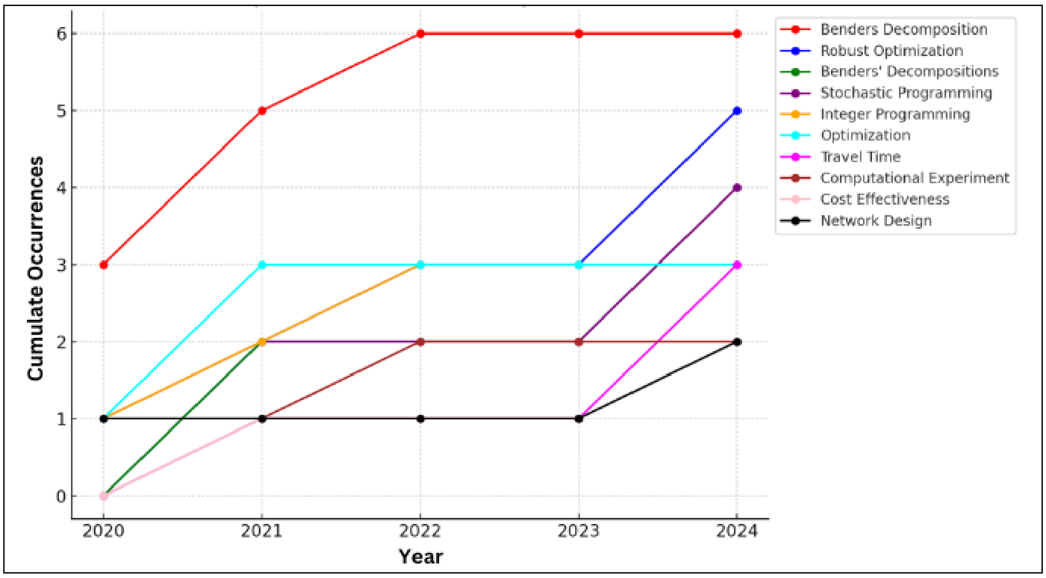

Figure 7. Word Frequency Over Time

Figure 7 depicts the frequency of the keyword "integer programming" between 2020 and 2024. The keyword appeared three times, with a steady but low frequency of occurrences. This suggests that while integer programming remains relevant, it is somewhat overshadowed by the more specialized focus on Benders Decomposition and robust optimization in recent years. Meanwhile, "robust optimization" has shown an increasing frequency of usage annually, reaching five occurrences in 2024. This trend reflects the growing importance of robust optimization as a critical method for handling uncertainty in optimization models. "Benders decomposition" showed consistent appearances from 2020 to 2022, then stabilized at six occurrences from 2023 to 2024. This stabilization could indicate that the foundational research on Benders Decomposition has plateaued, and the focus is shifting toward more specialized applications or variations of the method.

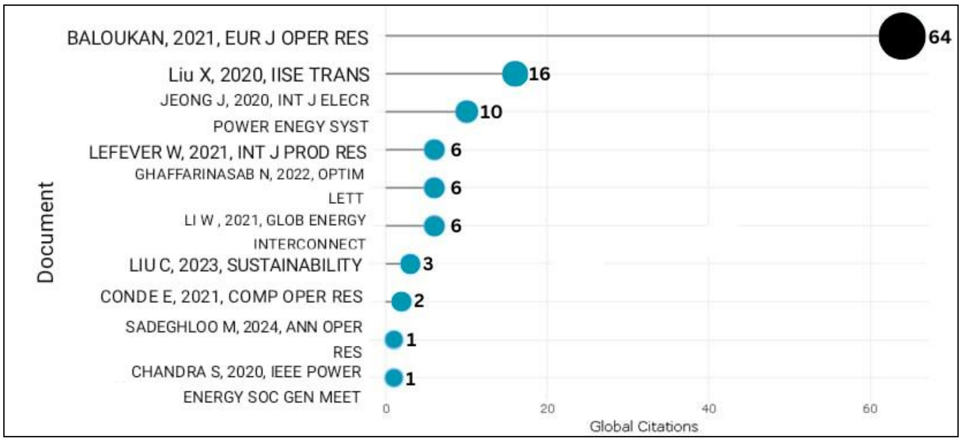

Figure 8. Most-Cited Article

Figure 8 lists the most-cited articles. The top citation goes to [30]. This article discusses robust MILP models for solving scheduling problems using the Benders Decomposition method. The high citation count signifies its influential role in shaping the field, highlighting its relevance and importance in advancing the application of Benders Decomposition to real-world problems.

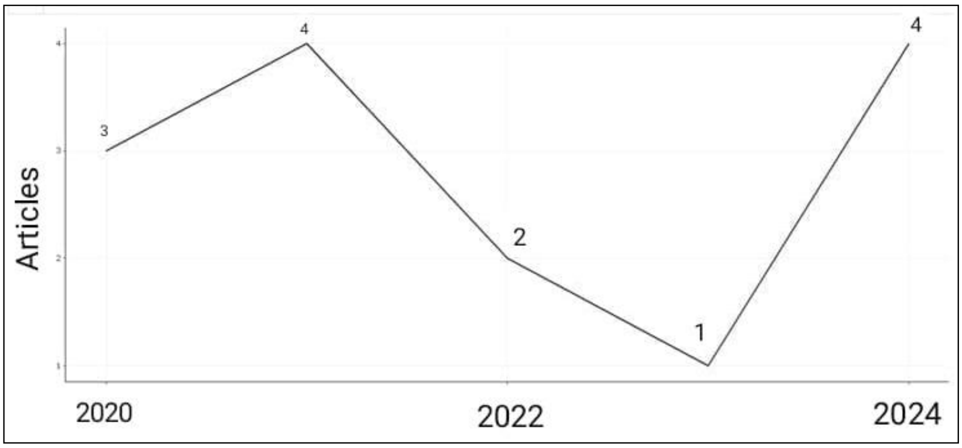

Figure 9. Number of Related Articles Published per Year

Figure 9 illustrates the trend in the number of articles published from 2020 to 2024. The data shows a decrease in publications in 2022 and 2023, followed by an increase in 2021 and 2024. This fluctuation in the publication count may indicate a temporary dip in research output or shifts in research focus during these years. However, the increase in 2024 suggests a resurgence of interest in the topic, possibly due to new advancements in methodology or the growing application of Benders Decomposition in robust optimization models.

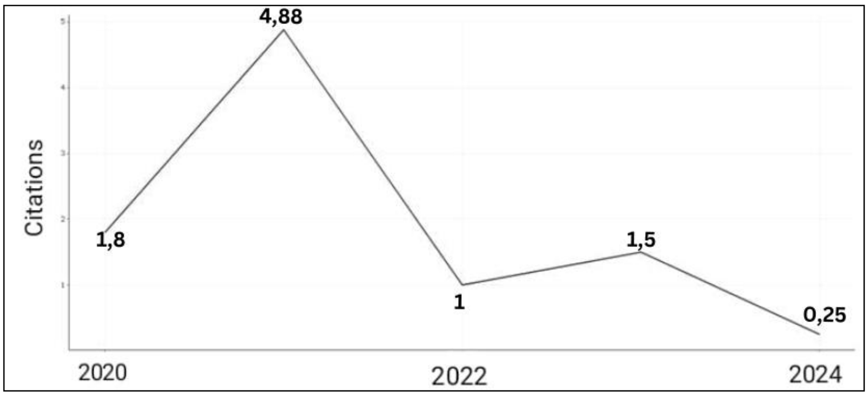

Figure 10. Average Citations per Year

Figure 10 presents the average number of citations per article per year. In 2021, articles received the highest average of five citations each. This suggests that publications in 2021 garnered significant attention from the academic community, likely due to the novelty or relevance of the research at that time. In the subsequent years, the average citation count per article declined, which could be attributed to an increase in the volume of publications, resulting in a wider distribution of citations across more articles. This decline might also suggest a shift in research priorities or the maturation of the field.

The bibliometric analysis shows that Benders Decomposition has become a widely applied method for solving robust MILP optimization problems. The observed trends in keyword frequency, publication numbers, and citation data indicate that this topic continues to be influential, with potential for further exploration and development, especially in the context of solving complex optimization problems under uncertainty.

The Most Commonly Used Uncertainty Sets in Solving Robust Optimization Models for MILP Problems Using Benders Decomposition

Mulvey et al. [42] first introduced robust optimization as an approach to handling parameter uncertainty in optimization processes. It aims to find optimal solutions despite variations or uncertainties in the input parameters, and generate "robust" solutions that perform well across various uncertainty scenarios. The general formulation for uncertain linear programming problems is as follows [20]:

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{c}^{\mathrm{T}}\mathbf{x} \\ s.t \quad & A\mathbf{x} \leq \mathbf{f} \\ & (\mathbf{c}, A, \mathbf{f}) \in \mathcal{U} \end{aligned} \tag{11}$$

where $\mathbf{c} \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, $\mathbf{f} \in \mathbb{R}^m$ represent uncertain coefficients, $\mathcal{U}$ denotes the uncertainty set, and $\mathbf{x} \in \mathbb{R}^n$ is the decision variable vector.

Based on the assumptions in robust optimization on [4], if $\mathbf{c} \in \mathbb{R}^n$ and $\mathbf{f} \in \mathbb{R}^m$ are deterministic, the robust reformulation of (11), known as the robust counterpart, is given by:

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{c}^{\mathrm{T}}\mathbf{x} \\ s.t \quad & A(\mathbf{w})\mathbf{x} \leq \mathbf{f} \qquad \forall \mathbf{w} \in \mathcal{Z} \end{aligned} \tag{12}$$

where $\mathcal{Z} \subseteq \mathbb{R}^L$ is the uncertainty set used. The matrix $A(\mathbf{w})$ represents the technology coefficient matrix expressed as a parameter $\mathbf{w} \in \mathcal{Z}$. A solution $\mathbf{x} \in \mathbb{R}^n$ is said to be robust and feasible if it satisfies the uncertainty constraint $[A(\mathbf{w})\mathbf{x} \leq \mathbf{f}]$ for all possible $\mathbf{w} \in \mathcal{Z}$. A constraint derived from (12) can be modeled as:

$$(\mathbf{a} + P\mathbf{w})^T \mathbf{x} \le f \qquad \forall \mathbf{w} \in \mathcal{Z} \tag{13}$$

when $\mathbf{a} \in \mathbb{R}^n$ represents the nominal value of the parameter and $P \in \mathbb{R}^{n \times L}$ represents the disturbance. The uncertainty set $\mathcal{U}$ is defined as:

$$\mathcal{U} = \{\mathbf{a}: \mathbf{a} = \mathbf{a} + P\mathbf{w}, \mathbf{w} \in \mathcal{Z}\} \tag{14}$$

Note that (12) involves multiple constraints due to the universal quantifier (∀) imposed by the worst-case formulation, which appears intractable in its current form. This means the solution to the problem cannot be computed practically [43]. One way to handle this is by applying robust reformulation techniques to eliminate the universal quantifier (∀). The tractability of the robust counterpart reformulation depends on the uncertainty set used. Tractability for box/interval, ellipsoidal, and polyhedral uncertainty sets is presented in Table 5.

Table 5. Tractable Reformulations for Various Uncertainty Sets [4]

Uncertainty Set	$\mathcal{Z}$	Robust Counterpart	Tractibility
Box	$\|\|\mathbf{w}\|\|_\infty$	$\mathbf{a}^T\mathbf{x} + \|\|P^T\mathbf{x}\|\|_1 \le f$	Linear Programming
Ellipsoidal	$\|\|\mathbf{w}\|\|_2$	$\mathbf{a}^T\mathbf{x} + \|\|P^T\mathbf{x}\|\|_2 \le f$	Conic quadratic programming
Polyhedral	$D\mathbf{w} + \mathbf{q} \ge \mathbf{0}$	$\begin{cases} \mathbf{a}^T\mathbf{x} + \mathbf{q}^T\mathbf{w} \le f \\ D^T\mathbf{x} = -\mathbf{P}^T\mathbf{x} \\ \mathbf{w} \ge \mathbf{0} \end{cases}$	Semidefinit Programming

The methodologies of all 14 articles were reviewed to identify the types of uncertainty sets employed. Table 4 summarizes these findings, showing that the interval uncertainty set is used 10 times, the polyhedral set 6 times, the multi-band set once, and the ellipsoidal set not at all. The frequency of uncertainty set usage is visualized in Figure 11.

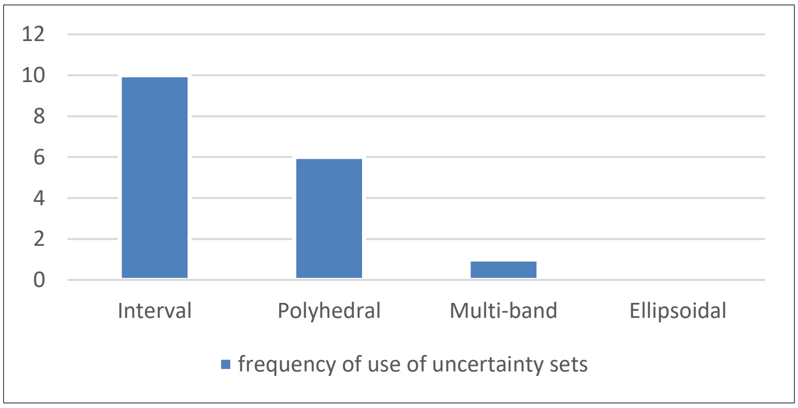

Figure 11. Frequency of Uncertainty Set Usage

The interval uncertainty set is the most frequently used, indicating a preference for interval-based approaches to handle uncertainty in optimization models. The polyhedral uncertainty set is also used frequently, suggesting its relevance, though it is less popular than the interval set. In contrast, the multi-band uncertainty set is less common in the reviewed literature.

MILP Problems Modeled in Robust Optimization and Solved Using Benders Decomposition

From the 14 articles previously obtained, we examined the MILP problems modeled in robust optimization and solved using Benders Decomposition. The results of this examination are presented in Table 6.

Table 6. MILP Problems Modeled

Paper	Problem
[27]	Power plant operation scheduling
[28]	Facility location and network design
[29]	Transmission switching
[30]	Project scheduling
[31]	Distribution network design
[32]	Inventory routing
[33]	Transmission expansion planning
[34]	Distribution center location
[35]	Robust optimization with interval uncertainty sets
[36]	Multi-project scheduling
[37]	Penjadwalan multi proyek
[38]	E-commerce distribution
[39]	Network protection
[40]	Truck and drone routing

Table 6 shows various MILP problems modeled in robust optimization and solved using Benders Decomposition. These articles cover a wide range of practical applications in energy, logistics, and supply chain, as well as planning and scheduling. This information highlights the extensive use of robust optimization in solving various MILP problems with Benders Decomposition. Facility location problems have also been formulated in robust optimization and solved using the Benders decomposition method in [28] and [34].

The facility location problem is crucial to logistics and supply chain management. It aims to determine the optimal facility locations to minimize costs and maximize distribution efficiency [15]. The facility location problem modeled in [34] exhibits a structure similar to the one described in [26]. The objective of the discussed model is to select p hubs and allocate traffic flows to minimize transportation costs, subject to several constraints, as follows:

$$\min \sum_{i,j\in A}\sum_{k\in H}\sum_{m\in H} w_{ij}c_{ijkm}x_{ijkm} \tag{15}$$

$$\text{s.t} \sum_{k\in H} z_k = p \tag{16}$$

$$\sum_{m\in H} x_{ijkm} = 1 \qquad \forall (i,j)\in A \tag{17}$$

$$\sum_{m\in H} x_{ijkm} + \sum_{m\in H, m\neq k} x_{ijmk} \leq z_k \quad \forall (i,j)\in A, k\in H \tag{18}$$

$$z_k \in \{0,1\} \qquad \forall k\in H \tag{19}$$

$$x_{ijmk} \geq 0 \tag{20}$$

where H r epresents the set of candidate nodes for hub facilities, with p nodes to be selected as hubs. A is the set of origin-destination (O/D) pairs, w_{ij} denotes the traffic volume from node $i$to node j, and c_{ijkm} is the transportation cost from node i to node j via hubs k and m. The decision variables include z_k, a binary variable that equals 1 if node k is selected as a hub and 0 otherwise, and x_{ijmk}, a continuous variable representing the fraction of flow w_{ij} routed from node i to j through the link between hubs k and m. This model is a mixed-integer linear programming (MILP) problem due to the combination of binary and continuous variables.

In the robust approach, it is assumed that traffic flows w_{ij} dmay vary within a specific interval, $(w_{ij} - \bar{w}_{ij})$ and $(w_{ij} + \bar{w}_{ij})$, where $\bar{w}_{ij}$ represents the uncertainty radius. The interval uncertainty set used, as shown in Table 2, is formulated as:

$$U = \{\widetilde{w} \in \mathbb{R}_+^{|A|} : w_{ij} - \bar{w}_{ij} \leq \widetilde{w}_{ij} \leq \left(w_{ij} + \bar{w}_{ij}\right) \in A\} \quad (21)$$

The Benders Decomposition method is employed to solve the robust counterpart of the problem. This approach divides the problem into a master problem, which contains only the location variables z_k and a subproblem, which includes the remaining variables. Experimental results demonstrate that the Benders Decomposition method can solve large-scale instances of the problem efficiently, achieving very short computational times.

As a future direction, this research could serve as a foundation for developing more complex facility location models that incorporate various sources of uncertainty or alternative forms of uncertainty sets. Moreover, the potential of Benders Decomposition to tackle large-scale optimization problems with complex structures opens up opportunities for applying this approach to a wide range of optimization challenges, including facility location problems.

Discussion

Bender decomposition has been widely applied to various optimization problems, showcasing its versatility. It has been extended to solve more complex models, such as mixed-integer nonlinear programming (MINLP) [44]–[47], and mixed-integer programming with conic constraints [22], demonstrating its potential across diverse fields. However, a key research gap exists in applying Benders Decomposition to robust Mixed-Integer Linear Programming (MILP) models, particularly for facility location problems. While robust optimization using interval and polyhedral uncertainty sets has been well-explored, the integration of ellipsoidal uncertainty sets with Benders Decomposition remains underexplored in this context. This study addresses this gap by identifying directions for future research.

Previous studies have demonstrated the effectiveness of Benders Decomposition in solving large-scale MILP problems across various fields, including energy, logistics, and supply chain management. These studies, however, predominantly rely on interval or polyhedral uncertainty sets. For instance, in the energy sector, research by [29] and [33] focuses on interval uncertainty sets to model uncertainties in power injections and wind power output. Similarly, studies in network design optimization, such as those [28] and [31], have extensively used interval uncertainty sets to model factors such as construction and operational costs, supply or demand requirements, and arc capacities. Although these models have proven beneficial, they are limited by the assumptions made about the uncertainty structure, which may not fully capture the complexity of real-world uncertainties.

In contrast, this study emphasizes the potential of ellipsoidal uncertainty sets, which offer greater flexibility in representing uncertainties, especially when dealing with correlated or multi-dimensional uncertainties. The use of ellipsoidal sets allows for a more accurate approximation of the uncertainty structure, leading to more robust and efficient optimization solutions. By integrating this advanced uncertainty modeling technique with Benders Decomposition, we could significantly improve both the robustness and computational efficiency of optimization models, particularly in complex facility location problems where multiple uncertainties are present, such as fluctuating demand and uncertain transportation costs.

Despite the growing interest in robust optimization, the application of ellipsoidal uncertainty sets with Benders Decomposition in facility location problems remains sparse in recent literature. A few works have explored this integration, but none in the past five years have specifically focused on the combination of ellipsoidal uncertainty sets with Benders Decomposition for MILP models in facility location. This gap represents a promising avenue for future research, where the unique strengths of ellipsoidal uncertainty sets could be leveraged to overcome current limitations in optimization models. However, challenges in terms of increased computational complexity and the need for efficient algorithmic approaches may arise when integrating these sets.

CONCLUSIONS

Based on the current state of the art, a significant gap exists in the literature concerning the solution of robust MILP models for facility location problems using Benders Decomposition. While robust optimization models with interval and polyhedral uncertainty sets have been widely applied, no studies in the last five years have explored integrating ellipsoidal uncertainty sets with Benders Decomposition for such problems. Addressing this gap could lead to substantial advancements in optimization methods, particularly by exploring the integration of ellipsoidal uncertainty sets with Benders Decomposition in facility location problems, offering improved robustness and computational efficiency.

The bibliometric analysis highlights the growing significance of Benders Decomposition in solving robust MILP problems across various domains, such as energy, logistics, supply chain management, and scheduling. However, the predominance of interval uncertainty sets indicates an opportunity to expand the application of alternative sets, such as ellipsoidal, to enhance the flexibility and applicability of optimization models.

This study contributes by systematically identifying the research gap and providing a comprehensive overview of existing methodologies. Future research should prioritize developing and testing frameworks integrating ellipsoidal uncertainty sets with Benders Decomposition in facility location problems. Such advancements are expected to bridge the identified gap and expand the practical relevance of robust optimization techniques in addressing real-world challenges.

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