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Some Properties of Prime Graph of Cartesian Product of The Rings and It's Line Graph

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Abstract

The prime graph of the ring $\mathcal{R}$, $PG(\mathcal{R})$, is a graph which set of vertices consists of elements of R and two different vertices are adjacent if their product in the ring is zero. We study the prime graph of cartesian product of the rings $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}$ for distinct prime numbers p_1 and p_2. We find that some properties of $PG(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2})$ such as order, size, the number of triangles, and Wiener index. Further, we construct the line graph of $PG(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2})$ and calculate the order, size, and Wiener index of $L(PG(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}))$.

Keywords: Prime Graph; Cartesian Product of Rings; Line Graph; Wiener Index.

1. Introduction

A graph $G = (V(G), E(G))$ is defined as an ordered pair composed of two distinct sets that is a non empty set of vertices, $V(G)$ and a set of edges, $E(G)$. The order (size) of G refers to the number of vertices (edges) respectively. When $u, v \in V(G)$ are adjacent, we write $u \sim v$; otherwise, if they are non-adjacent, we write $u \nsim v$. The length of the shortest path between the two distinct vertices u and v in G is called distance and is denoted by $d(u, v)$. In a graph G, the neighborhood of a vertex u refers to the set of vertices that are adjacent to u. Two graphs G_1 and G_2 are isomorphic, written $G_1 \cong G_2$, if there is a bijective function $\varphi : V(G_1) \to V(G_2)$ such that u and v are adjacent in G_1 if only if $\varphi(u)$ and $\varphi(v)$ are adjacent in G_2. Any other undefined notations and terms related to graph theory can be referred to in [1].

One of the important topics in graph theory is the study of graph invariants. Among them, the Wiener index is one of the most widely studied distance-based invariants. For a connected graph G, the Wiener index, denoted by $W(G)$, is defined as the sum of distances between all unordered pairs of vertices in G. This index was originally introduced by Wiener in 1947 in mathematical chemistry to study the relationship between molecular structure and the physicochemical properties of organic compounds [2]. Since then, the Wiener index has been further developed in graph theory and has been applied in the study of molecular structures and nanomaterials [3], network analysis [4], and educational systems [5].

In addition to graph invariants, line graph have long been studied in graph theory. The concept of a line graph was first introduced by Whitney in 1992 in his study on graph isomorphism. For a simple graph G, the line graph of G, denoted by $L(G)$, is defined as the graph whose

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vertex set corresponds to the edge set of G. Two vertices in $L(G)$ are adjacent if and only if the corresponding edges in the original graph G share a common endpoint [6].

This paper explores the connection between graph theory and algebra, focusing particularly on an algebraic structure called a prime ring, which is associated with a graph called the prime graph of a ring. In 2010, Bhavanari et al. [7] proposed the concept of the prime graph of a ring $\mathcal{R}$, $PG(\mathcal{R})$. They defined the vertex set of this graph as the set of all elements of $\mathcal{R}$, where two different vertices x and y are adjacent if only in $x\mathcal{R}y = 0$ or $y\mathcal{R}x = 0$. Moreover, they also examined several properties of $PG(\mathbb{Z}_p)$ for the prime number p. In 2013, Kalita and Patra [8] extended this notion to the ring $\mathbb{Z}_m \times \mathbb{Z}_n$ and determined the chromatic number of its prime graph, but their study was limited to graph coloring and did not address fundamental structural properties. In 2015, Bhavanari and Srinivasulu [9] investigated the graphs $PG(\mathbb{Z}_n) \times PG(\mathbb{Z}_n)$ and $PG(\mathbb{Z}_n \times \mathbb{Z}_n)$, and in 2019 Rajendra et al. [10] refined earlier results that $PG(\mathbb{Z}_n)$ is a star graph only when n is a prime or $n = 4$. In the same year, Pawar and Joshi [11] investigated degree, girth, and the number of triangles of $PG(\mathbb{Z}_n)$ and its complement, while in 2018 Joshi and Pawar [12] studied the line graph and Wiener index of $PG(\mathbb{Z}_n)$ for $n \in \{p, p^2, p^3\}$. For more literature on the prime graph of the ring [13–17].

Motivated by the work of Kalita and Patra [8], which focused only on the chromatic number of prime graphs, this paper aims to provide a more detailed investigation of the structural properties of the prime graph $PG(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2})$, where p_1 and p_2 are prime numbers. In particular, we study fundamental graph parameters such as the order, size, number of triangles, and the Wiener index of $PG(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2})$. Furthermore, we construct the line graph of $PG(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2})$ and discuss its order, size, and Wiener index.

2. Methods

This study employs a literature-based approach to investigate the properties of $PG(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2})$, where p_1 and p_2 are distinct prime numbers. The research methodology consists of the following steps:

(i) Identity the set of vertices of $PG(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2})$ by determining all element of ring $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}$ and analyze the set of edges to determine the size of $PG(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2})$.

(ii) Determine the number of triangles in $PG(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2})$ by identifying the number of subgraph of $PG(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2})$ which forms triangle.

(iii) Determine the Wiener index in $PG(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2})$ by calculating the sum of the distances between all pairs of vertices in the graph.

(iv) Construct the line graph of $PG(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2})$ by forming a graph in which each vertex represents an edge of $PG(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2})$ and two vertices in the line graph are adjacent if and only if their corresponding edges in $PG(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2})$ share a common vertex.

(v) Consequently of step *iv*, identify the order, size, and Wiener index of line graph of $PG(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2})$.

Before presenting the result and discussion of the prime graph $PG(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2})$, we first review some formal definitions of the prime graph, its line graph, and its Wiener index.

Definition 1. Let $\mathcal{R} = \mathcal{R}_1 \times \mathcal{R}_2 \times \ldots \times \mathcal{R}_n$ be a ring and let $(x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n) \in \mathcal{R}$. Prime graph of a ring $\mathcal{R}$ (in short $PG(\mathcal{R}_1 \times \ldots \times \mathcal{R}_n)$) is a graph with vertex set is $V(PG(\mathcal{R})) = \mathcal{R}$. Two distinct vertices $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ are adjacent if only if [8]:

$$(x_1, x_2, \ldots, x_n)\, \mathcal{R}\, (y_1, y_2, \ldots, y_n) = \{(0, 0, \ldots, 0)\}$$

Definition 2. The line graph of the prime graph $PG(\mathcal{R})$ (in short $L(PG(\mathcal{R}))$ is a graph which set of vertices consists of the edges of $PG(\mathcal{R})$ and two vertices being adjacent if their corresponding edges share a common vertex in in$PG(\mathcal{R})$ [12].

Definition 3. The Wiener index of $PG(\mathcal{R})$ is defined as the sum of the distances between all pair of vertices of $PG(\mathcal{R})$, i.e. [12]:

$$\mathcal{W}(PG(\mathcal{R})) = \sum_{u,v \in V(G)} d(u,v)$$

3. Results and Discussion

This section discusses several properties of the prime graph over the ring $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}$ and its line graph, for prime numbers p_1 and p_2. The properties examined for the prime graph include order, size, the number of triangles, and Wiener index, while for the line graph, the discussion covers order, size, and the Wiener index.

3.1. Prime Graph of Ring

In this subsection, we show that some properties about order, size, number of triangles, and Wiener index of $PG(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2})$. By Definition 1, for $PG(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2})$, where $p_1, p_2 \in \mathbb{Z}^+$ a vertex $(\overline{0}, \overline{0})$ is adjacent to every other vertex. Now, we show the order and size of $PG(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2})$.

Lemma 1. *Let $PG(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2})$ be a prime graph of ring $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}$, where p_1, p_2 the prime numbers. Then, $PG(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2})$ has order p_1p_2 and size $2p_1p_2 - p_1 - p_2$.*

Proof. By Definition 1, it is clear that the order of $PG(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2})$ is p_1p_2. Then, neighbourhood of vertex in $PG(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2})$ we consider in two cases as follows:

(i) The neighbourhood of vertex containing $(\overline{0}, \overline{0})$. It is clear that the number of those neighbourhood are $(p_1p_2 - 1)$.

(ii) The neighbourhood of vertex not containing $(\overline{0}, \overline{0})$. Since $(\overline{x}, \overline{0})$ and $(\overline{0}, \overline{y})$, where $x = \overline{1}, \overline{2}, \ldots, \overline{p_1 - 1}$ and $y = \overline{1}, \overline{2}, \ldots, \overline{p_2 - 1}$ is adjacent, then the number of those neighbourhood is $(p_1 - 1)(p_2 - 1)$.

Therefore, by (i) and (ii), we obtain that the size of $PG(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2})$ is

$$(p_1p_2 - 1) + (p_1 - 1)(p_2 - 1) = 2p_1p_2 - p_1 - p_2.$$

□

Example 1. Let us contruct the graph $PG(\mathbb{Z}_3 \times \mathbb{Z}_5)$. We know that $V(PG(\mathbb{Z}_3 \times \mathbb{Z}_5)) = \mathbb{Z}_3 \times \mathbb{Z}_5$ and the edges satisfy $(x_1, y_1)(\mathbb{Z}_3 \times \mathbb{Z}_5)(x_2, y_2) = (\overline{0}, \overline{0})$. Then, $PG(\mathbb{Z}_3 \times \mathbb{Z}_5)$ is shown in Fig. 1.

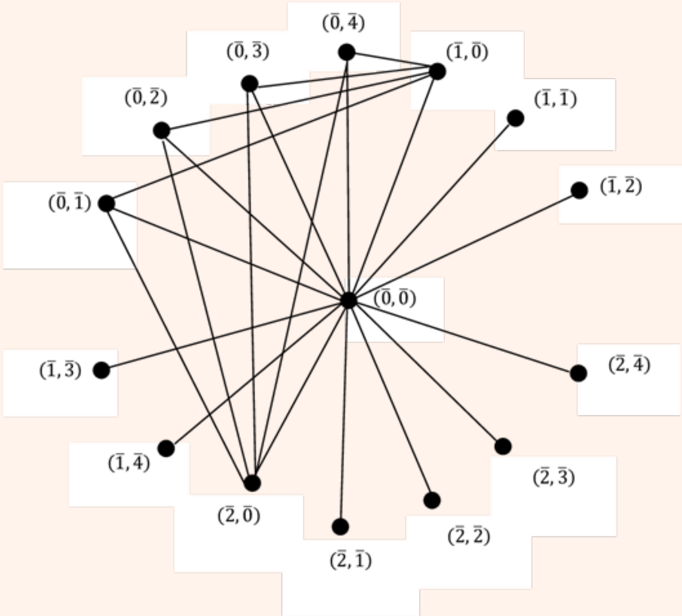

Fig. 1: $PG\left(\mathbb{Z}_3 \times \mathbb{Z}_5\right)$

Now, we construct $PG\left(\mathbb{Z}_5 \times \mathbb{Z}_3\right)$ in Fig. 2.

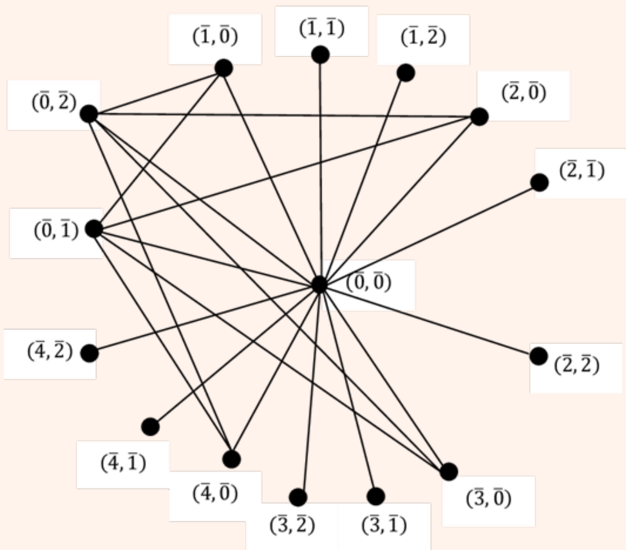

Fig. 2: $PG\left(\mathbb{Z}_5 \times \mathbb{Z}_3\right)$

We can see that, based on Fig. 1 and Figure Fig. 2, two graphs $PG\left(\mathbb{Z}_3 \times \mathbb{Z}_5\right)$ and $PG\left(\mathbb{Z}_5 \times \mathbb{Z}_3\right)$ are isomorphic, we write $PG\left(\mathbb{Z}_3 \times \mathbb{Z}_5\right) \cong PG\left(\mathbb{Z}_5 \times \mathbb{Z}_3\right)$. Both of order 15 and size 22. An explanation will be given shortly.
There are exists a bijective function

$$\varphi : V\left(PG\left(\mathbb{Z}_3 \times \mathbb{Z}_5\right)\right) \to V\left(PG\left(\mathbb{Z}_5 \times \mathbb{Z}_3\right)\right)$$

defined by

$$\begin{aligned}
&\varphi(\bar{0},\bar{0}) = (\bar{0},\bar{0}), \quad \varphi(\bar{1},\bar{0}) = (\bar{0},\bar{1}), \quad \varphi(\bar{2},\bar{0}) = (\bar{0},\bar{2}),\\
&\varphi(\bar{0},\bar{1}) = (\bar{1},\bar{0}), \quad \varphi(\bar{0},\bar{2}) = (\bar{2},\bar{0}), \quad \varphi(\bar{0},\bar{3}) = (\bar{3},\bar{0}),\\
&\varphi(\bar{0},\bar{4}) = (\bar{4},\bar{0}), \quad \varphi(\bar{1},\bar{1}) = (\bar{1},\bar{1}), \quad \varphi(\bar{1},\bar{2}) = (\bar{1},\bar{2}),\\
&\varphi(\bar{1},\bar{3}) = (\bar{3},\bar{1}), \quad \varphi(\bar{1},\bar{4}) = (\bar{4},\bar{1}), \quad \varphi(\bar{2},\bar{1}) = (\bar{2},\bar{1}),\\
&\varphi(\bar{2},\bar{2}) = (\bar{2},\bar{2}), \quad \varphi(\bar{2},\bar{3}) = (\bar{3},\bar{2}), \quad \varphi(\bar{2},\bar{4}) = (\bar{4},\bar{2}).
\end{aligned}$$

In Fig. 1, there are triangles containing vertex $(\overline{0},\overline{0})$. Therefore, we can calculate the number of triangles in $PG\left(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}\right)$.

Lemma 2. *If p_1 and p_2 are distinct prime numbers, then $PG\left(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}\right)$ has the number of triangles $(p_1 - 1)(p_2 - 1)$.*

Proof. It is known that the vertex $(\overline{0},\overline{0})$ is adjacent to every other vertex in $PG\left(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}\right)$. So, it is clear that one of vertex in triangles is $(\overline{0},\overline{0})$. Hence $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}$ not containing zero divisors, then the other vertices we get:
$(\overline{0},\overline{1})$ adjacent to $(\overline{1},\overline{0}), (\overline{2},\overline{0}), \ldots, (\overline{p_1 - 1},\overline{0})$
$(\overline{0},\overline{2})$ adjacent to $(\overline{1},\overline{0}), (\overline{2},\overline{0}), \ldots, (\overline{p_1 - 1},\overline{0})$
$\vdots$
$(\overline{0},\overline{p_2 - 1})$ adjacent to $(\overline{1},\overline{0}), (\overline{2},\overline{0}), \ldots, (\overline{p_1 - 1},\overline{0})$.
Therefore, these all together forms $(p_1 - 1)(p_2 - 1)$ number of triangles. $\square$

Next, we calculate the Wiener index of prime graph of ring $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}$.

Theorem 1. *If graph $G \cong PG\left(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}\right)$, for prime numbers p_1,p_2, then the Wiener index of G is*

$$\mathcal{W}(G) = p_1^2 p_2^2 - 3p_1p_2 + p_1 + p_2.$$

Proof. The distances between all pair vertices of $PG\left(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}\right)$ divided into two cases; the distance between two adjacent vertices and non-adjacent vertices.

(i) The distances between two adjacent vertices. Based on Lemma 1 the number of adjacent vertex in $PG\left(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}\right)$ is $2p_1p_2 - p_1 - p_2$, Since $d(u,v) = 1$, for two vertices $u = (x_1, y_1)$ and $v = (x_2, y_2)$ are adjacent, we get the sum of the distance between two adjacent vertex is

$$\sum_{\substack{v, v \in V(G) \\ u \sim v}} d(u,v) = (2p_1p_2 - p_1 - p_2)[1] = 2p_1p_2 - p_1 - p_2.$$

(ii) The distances between two non-adjacent vertices. Suppose the number of edges in complete graph, K_n where $n = p_1p_2$ vertex is

$$\frac{n(n-1)}{2} = \frac{p_1p_2(p_1p_2 - 1)}{2} = \frac{1}{2}\left(p_1^2p_2^2 - p_1p_2\right).$$

Therefore, by (i), the number of non-adjacent vertex is given by

$$\frac{1}{2}\left(p_1^2p_2^2 - p_1p_2\right) - (2p_1p_2 - p_1 - p_2) = \frac{1}{2}\left(p_1^2p_2^2 - 5p_1p_2 + 2p_1 + 2p_2\right).$$

Since, the vertex $(\overline{0},\overline{0})$ is adjacent to all other vertices in $PG\left(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}\right)$, the distance between two non-adjacent vertices $u = (x_1, y_1)$ and $v = (x_2, y_2)$ is always 2.

$$\sum_{\substack{v,v \in V(G) \\ u \nsim v}} d(u,v) = \frac{1}{2}\left(p_1^2p_2^2 - 5p_1p_2 + 2p_1 + 2p_2\right)[2] = p_1^2p_2^2 - 5p_1p_2 + 2p_1 + 2p_2.$$

Thus, by (i) and (ii), we obtain that the Wiener index of $PG\left(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}\right)$ is sum of distances between two adjacent vertices and distances between two non-adjacent vertices, that is

$$\begin{aligned}\mathcal{W}(G) &= \sum_{\substack{v,v \in V(L(G)) \\ u \sim v}} d(u,v) + \sum_{\substack{v,v \in V(L(G)) \\ u \nsim v}} d(u,v) \\ &= 2p_1p_2 - p_1 - p_2 + p_1^2p_2^2 - 5p_1p_2 + 2p_1 + 2p_2 \\ &= p_1^2p_2^2 - 3p_1p_2 + p_1 + p_2.\end{aligned}$$

□

Example 2. Based on Theorem 1, we present the Wiener index of $PG\left(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}\right)$, for various prime number p_1 and p_2 as shown in Table 1.

Table 1: Wiener index of $PG(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2})$ for various prime numbers

p_1	p_2	$W(PG(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}))$
2	3	23
2	5	77
2	7	163
3	5	188
3	7	388
5	7	1132

3.2. Line Graph of Ring

In this subsection, we establish a line graph of a ring $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}$. This is a theorem that show the order and size in $L\left(PG\left(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}\right)\right)$.

Theorem 2. *Let $PG\left(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}\right)$ be a prime graph of $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}$. Then, $L\left(PG\left(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}\right)\right)$ has order $2p_1p_2 - p_1 - p_2$ and size*

$$\frac{1}{2}\left(2p_1^2p_2^2 - 2p_1^2p_2 - 2p_1p_2^2 + p_1^2 + p_2^2 + 4p_1p_2 - 5p_1 - 5p_2 + 6\right).$$

Proof. According to the Lemma 1, size of $PG\left(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}\right)$ is $2p_1p_2 - p_1 - p_2$. Hence, we get the order of $L\left(PG\left(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}\right)\right)$ is $2p_1p_2 - p_1 - p_2$. Then, the number of neighbourhood of vertex in $L\left(PG\left(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}\right)\right)$, where $x = \overline{1}, \overline{2}, \ldots, \overline{p_1 - 1}$ and $y = \overline{1}, \overline{2}, \ldots, \overline{p_2 - 1}$, is determined by considering the following nine cases:

(i) For $\left(\left(\overline{0},\overline{0}\right),\left(\overline{x},\overline{0}\right)\right)$ adjacent to $\left(\left(\overline{0},\overline{0}\right),\left(\overline{x},\overline{0}\right)\right)$, we obtain that the number of neighborhood is $\binom{p_1-1}{2}=\frac{1}{2}\left(p_1-1\right)\left(p_1-2\right)$ vertices.

(ii) For $\left(\left(\overline{0},\overline{0}\right),\left(\overline{x},\overline{0}\right)\right)$ adjacent to $\left(\left(\overline{0},\overline{0}\right),\left(\overline{0},\overline{y}\right)\right)$, the number of neighborhood is $\left(p_1-1\right)\left(p_2-1\right)$ vertices.

(iii) For $\left(\left(\overline{0},\overline{0}\right),\left(\overline{x},\overline{0}\right)\right)$ adjacent to $\left(\left(\overline{0},\overline{0}\right),\left(\overline{x},\overline{y}\right)\right)$, the number of neighborhood is $\left(p_1-1\right)^2\left(p_2-1\right)$ vertices.

(iv) For $\left(\left(\overline{0},\overline{0}\right),\left(\overline{x},\overline{0}\right)\right)$ adjacent to $\left(\left(\overline{x},\overline{0}\right),\left(\overline{0},\overline{y}\right)\right)$, the number of neighborhood is $\left(p_1-1\right)\left(p_2-1\right)$ vertices.

(v) For $\left(\left(\overline{0},\overline{0}\right),\left(\overline{0},\overline{y}\right)\right)$ adjacent to $\left(\left(\overline{0},\overline{0}\right),\left(\overline{0},\overline{y}\right)\right)$, the number of neighborhood is $\binom{p_2-1}{2}=\frac{1}{2}(p_2-1)(p_2-2)$ vertices.

(vi) For $\left(\left(\overline{0},\overline{0}\right),\left(\overline{0},\overline{y}\right)\right)$ adjacent to $\left(\left(\overline{0},\overline{0}\right),\left(\overline{x},\overline{y}\right)\right)$, the number of neighborhood is $\left(p_1-1\right)\left(p_2-1\right)^2$ vertices.

(vii) For $\left(\left(\overline{0},\overline{0}\right),\left(\overline{0},\overline{y}\right)\right)$ adjacent to $\left(\left(\overline{x},\overline{0}\right),\left(\overline{0},\overline{y}\right)\right)$, the number of neighborhood is $(p_1-1)(p_2-1)$ vertices.

(viii) For $\left(\left(\overline{x},\overline{0}\right),\left(\overline{0},\overline{y}\right)\right)$ adjacent to $\left(\left(\overline{x},\overline{0}\right),\left(\overline{0},\overline{y}\right)\right)$, the number of neighborhood is $\binom{\left(p_1-1\right)\left(p_2-1\right)}{2}=\frac{1}{2}\left(p_1p_2-p_1-p_2+1\right)\left(p_1p_2-p_1-p_2\right)$ vertices.

(ix) For $\left(\left(\overline{0},\overline{0}\right),\left(\overline{x},\overline{y}\right)\right)$ adjacent to $\left(\left(\overline{0},\overline{0}\right),\left(\overline{x},\overline{y}\right)\right)$, the number of neighborhood is $\binom{\left(p_1-1\right)\left(p_2-1\right)}{2}=\frac{1}{2}\left(p_1p_2-p_1-p_2+1\right)\left(p_1p_2-p_1-p_2\right)$ vertices.

Therefore, the size in $L\left(PG\left(\mathbb{Z}_{p_1}\times\mathbb{Z}_{p_2}\right)\right)$ is the sum of the number of neighborhood (i) - (ix), i.e

$$\frac{1}{2}\left(2p_1^2p_2^2-2p_1^2p_2-2p_1p_2^2+p_1^2+p_2^2+4p_1p_2-5p_1-5p_2+6\right)$$

$\square$

Example 3. $PG\left(\mathbb{Z}_2\times\mathbb{Z}_3\right)$ is shown in Fig. 3.

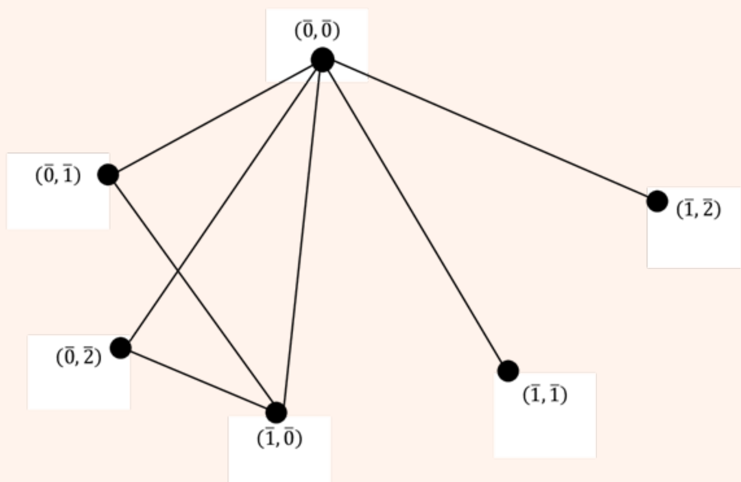

Fig. 3: $PG(\mathbb{Z}_2\times\mathbb{Z}_3)$

Now, let us construct $L(PG(\mathbb{Z}_2 \times \mathbb{Z}_3))$. As we know, the set vertices of $L(PG(\mathbb{Z}_2 \times \mathbb{Z}_3))$ constitutes of the edges of $PG(\mathbb{Z}_2 \times \mathbb{Z}_3)$. Therefore, we have a graph $L(PG(\mathbb{Z}_2 \times \mathbb{Z}_3))$ in Fig. 4.

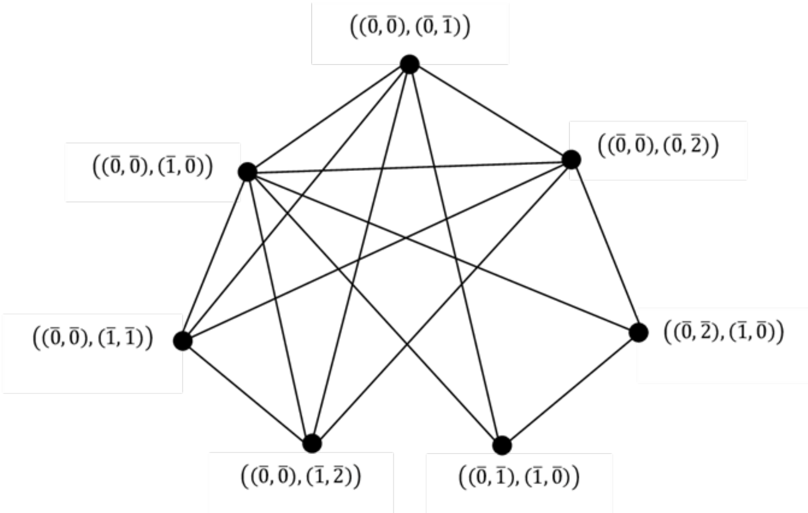

Fig. 4: $L(PG(\mathbb{Z}_2 \times \mathbb{Z}_3))$

We can see that in Fig. 4, $L(PG(\mathbb{Z}_2 \times \mathbb{Z}_3))$ has order 7 and size 15.
Next, we calculate the Wiener index of line graph of prime graph of ring $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}$.

Theorem 3. *If graph $G \cong PG(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2})$, for prime numbers p_1 and p_2, then the Wiener index of line graph G is*

$$\mathcal{W}(L(G)) = \frac{1}{2}\left(6p_1^2p_2^2 - 6p_1^2p_2 - 6p_1p_2^2 + p_1^2 + p_2^2 - 4p_1p_2 + 7p_1 + 7p_2 - 6\right).$$

Proof. Let graph $G \cong PG(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2})$. The distances between all pair vertices of $L(G)$ divided into two cases, the distance between two adjacent vertices and non-adjacent vertices.

(i) The distances between two adjacent vertices. Based on Theorem 2, the number of adjacent vertex in G is

$$\frac{1}{2}\left(2p_1^2p_2^2 - 2p_1^2p_2 - 2p_1p_2^2 + p_1^2 + p_2^2 + 4p_1p_2 - 5p_1 - 5p_2 + 6\right).$$

Since the distance for any two different vertices, $u = ((x_1, y_1), (x_2, y_2))$ and $v = ((x_3, y_3), (x_4, y_4))$ are adjacent is 1. Then we get the sum of distance between two adjacent vertex is

$$\begin{aligned}\sum_{\substack{v, v \in V(L(G)) \\ u \sim v}} d(u, v) &= \frac{1}{2}\left(2p_1^2p_2^2 - 2p_1^2p_2 - 2p_1p_2^2 + p_1^2 + p_2^2 + 4p_1p_2 - 5p_1 - 5p_2 + 6\right) \\ &= \frac{1}{2}\left(2p_1^2p_2^2 - 2p_1^2p_2 - 2p_1p_2^2 + p_1^2 + p_2^2 + 4p_1p_2 - 5p_1 - 5p_2 + 6\right)\end{aligned}$$

(ii) The distances between two non-adjacent vertices. Suppose the number of edges in complete graph K_n, for $n = 2p_1p_2 - p_1 - p_2$ is

$$\begin{aligned}\frac{n(n-1)}{2} &= \frac{(2p_1p_2 - p_1 - p_2)(2p_1p_2 - p_1 - p_2 - 1)}{2} \\ &= \frac{1}{2}\left(4p_1^2p_2^2 - 4p_1^2p_2 - 4p_1p_2^2 + p_1^2 + p_2^2 + p_1 + p_2\right).\end{aligned}$$

Therefore, by (i), the number of non-adjacent vertex is given by

$$\begin{aligned}&\frac{1}{2}\left(4p_1^2p_2^2-4p_1^2p_2-4p_1p_2^2+p_1^2+p_2^2+p_1+p_2\right)\\&\quad-\frac{1}{2}\left(2p_1^2p_2^2-2p_1^2p_2-2p_1p_2^2+p_1^2+p_2^2+4p_1p_2-5p_1-5p_2+6\right)\\&\quad=p_1^2p_2^2-p_1^2p_2-p_1p_2^2-2p_1p_2+3p_1+3p_2-3.\end{aligned}$$

Based on the proof of Theorem 2, every vertex in $L(G)$ have adjacent to $\left(\left(\overline{0},\overline{0}\right),\left(\overline{x},\overline{0}\right)\right)$ or $\left(\left(\overline{0},\overline{0}\right),\left(\overline{0},\overline{y}\right)\right)$, then the distance for any two different vertices, $u=\left(\left(x_1,y_1\right),\left(x_2,y_2\right)\right)$ and $v=\left(\left(x_3,y_3\right),\left(x_4,y_4\right)\right)$ are non-adjacent is 2. Additionaly, we conclude that sum of the distance between two non-adjacent vertices is

$$\begin{aligned}\sum_{\substack{v,v\in V(L(G))\\ u\nsim v}} d\left(u,v\right)&=\left(p_1^2p_2^2-p_1^2p_2-p_1p_2^2-2p_1p_2+3p_1+3p_2-3\right)[2]\\&=2p_1^2p_2^2-2p_1^2p_2-2p_1p_2^2-4p_1p_2+6p_1+6p_2-6.\end{aligned}$$

Thus, by (i) and (ii), we obtain that the Wiener *index* of $G\cong L\left(PG\left(\mathbb{Z}_{p_1}\times\mathbb{Z}_{p_2}\right)\right)$ is sum of distances between two adjacent vertices and distances between two non-adjacent vertices, that is

$$\begin{aligned}\mathcal{W}\left(G\right)&=\sum_{\substack{v,v\in V(L(G))\\ u\sim v}} d\left(u,v\right)+\sum_{\substack{v,v\in V(L(G))\\ u\nsim v}} d\left(u,v\right)\\&=\frac{1}{2}\left(2p_1^2p_2^2-2p_1^2p_2-2p_1p_2^2+p_1^2+p_2^2+4p_1p_2-5p_1-5p_2+6\right)\\&\quad+2p_1^2p_2^2-2p_1^2p_2-2p_1p_2^2-4p_1p_2+6p_1+6p_2-6\\&=\frac{1}{2}\left(6p_1^2p_2^2-6p_1^2p_2-6p_1p_2^2+p_1^2+p_2^2-4p_1p_2+7p_1+7p_2-6\right).\end{aligned}$$

□

Example 4. Based on Theorem 3, we present the Wiener index of $L\left(PG\left(\mathbb{Z}_{p_1}\times\mathbb{Z}_{p_2}\right)\right)$, for various prime numbers p_1 and p_2 as shown in Table 2.

Table 2: Wiener index of $L(PG(\mathbb{Z}_{p_1}\times\mathbb{Z}_{p_2}))$ for various prime numbers

p_1	p_2	$W(L(PG(\mathbb{Z}_{p_1}\times\mathbb{Z}_{p_2})))$
2	3	27
2	5	106
2	7	237
3	5	327
3	7	712
5	7	2421

4. Conclusion

This paper discuss some properties of prime graph of ring $\mathbb{Z}_{p_1}\times\mathbb{Z}_{p_2}$, denoted as $PG(\mathbb{Z}_{p_1}\times\mathbb{Z}_{p_2})$ for prime numbers p_1, p_2 and its line graph. The properties examined for the prime graph include order, size, the number of triangles, and Wiener index, while for the line graph, the discussion covers order, size, and the Wiener index. For further research, other properties of $PG(\mathbb{Z}_m \times \mathbb{Z}_n)$ can be explored, such as the energy and other topological indices, such as the Zagreb index, sombor index and Randic index.

CRediT Authorship Contribution Statement

Vira Hari Krisnawati: Conceptualization, Methodology, Formal analysis, Writing–Original Draft. **Ayunda Faizatul Musyarrofah:** Methodology, Visualization, Writing–Review & Editing. **Noor Hidayat:** Supervision, Validation, Writing–Review & Rditing. **Farah Maulidya Fatimah:** Investigation, Formal analysis, Writing–Original Draft. All authors have read and approved the final version of the manuscript.

Declaration of Generative AI and AI-assisted technologies

No generative AI or AI-assisted technologies were used during the preparation of this manuscript.

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The authors declare no competing interests

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