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b-coloring Analysis on Tree Graph Families

Rafiantika Megahnia Prihandini[1], Arika Indah Kristiana[1*], Lusita Risma Dana[1], Edy Wihardjo[1], Robiatul Adawiyah[1], and Hutkemri Zulnaidi[2]

[1] *Department of Mathematics Education, Faculty of Teacher Training and Education, Jember University, Indonesia*
[2] *Department of Mathematics and Sciences Education, Universiti Malaya, Kuala Lumpur, Malaysia*

Abstract

A tree graph is a connected graph and has no circuits. Tree graphs used in this study include: broom graph, centipede graph, and Banana Tree graph. Graph coloring is the process of giving color to graph elements with the rule that neighboring elements must not have the same color, and the number of colors used must be as minimal as possible. b-coloring of a graph G is a coloring of the vertices of G such that each color class has at least one vertex adjacent to all other color classes. The b-chromatic number of a graph G is denoted by $\varphi(G)$, is the largest integer k such that G has a b-coloring with k colors. The limit of b-coloring of graph G with maximum degree $\Delta(G)$ is as follows, $\chi(G) \leq \varphi(G) \leq \Delta(G) + 1.\chi(G)$ is the chromatic number of a graph G where $\chi(G)$ is the minimum value of the color required for proper coloring of graph G. While $\Delta(G)$ is the maximum degree of the vertices in graph G. This study uses an exploratory research type with an axiomatic deductive method and a pattern detection method. Based on this study, the results of the b-coloring analysis on the tree graph family are known. The results of this study are expected to be used as study material and the development of scientific knowledge related to b-coloring analysis on other graphs.

Keywords: b-Coloring; b-Chromatic number; broom graph; banana tree graph; centipede graph; Tree graph.

1 Introduction

The 21st century has witnessed rapid technological advancements that have simultaneously driven progress in scientific fields, including mathematics. As a fundamental discipline, mathematics forms the foundation for technological development and plays a crucial role in enhancing cognitive abilities. It is a branch of science that offers numerous practical applications for problem-solving. Mathematics continues to evolve in response to emerging challenges, reflecting its dynamic nature. One of the key areas of development in applied mathematics is graph theory, which is increasingly used to model and solve complex problems, particularly those related to networks, transportation, and the flow of information [1]. In the context of traffic systems, graph theory serves as a tool to understand and optimize vehicle flow, aiming to alleviate congestion and other related issues [2].

Graph theory, as a broader mathematical framework, is defined by Slamin as an ordered set of (V, E), where V represents a non-empty set of elements called vertices, and E is a finite set of

*Corresponding author. E-mail: arika.fkip@unej.ac.id

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edges, which may be empty, connecting pairs of distinct vertices in $V(G)$ [3]. A graph G is a finite set of vertices and edges, denoted by $G = (V, E)$, where V is a set of vertices and E is a set of edges. A graph may not have edges but must have at least one vertex [4]. $V(G)$ is called the vertex set of G and $E(G)$ is called the edge set of G. Two vertices in a graph connected by a common edge are called adjacent vertices, while a vertex that is an end of an edge is said to be adjacent [5]. The degree of a vertex v in graph G is denoted by $d(v)$, defined as the number of edges attached to the vertex [6]. Graph coloring is the process of assigning colors to the elements of the graph in question [7]. From this definition, graph coloring can be interpreted as the process of assigning colors to the elements of the graph with the rule that adjacent elements must not have the same color and the number of colors used must be as minimal as possible.

A concept closely related to graph coloring is b-coloring. The concept of b-coloring was first introduced by Irving and Manlove in 1999 as a development of the classical graph coloring problem [8]. A tree graph, which is a type of graph that does not contain cycles [9], is one of the types of graphs where b-coloring can be applied. As cited in [10], the calculation of the b-chromatic number is more challenging in graphs with complex structures, such as branched tree graphs. Kornelia also conducted related research [11], which discusses b-coloring in cross-section graphs, origami graphs, and tadpole graphs. Researchers are interested in developing b-coloring in other graphs, one of which is the tree graph family. Previous research was conducted by [12], who studied the tree graph family entitled *On the b-coloring of cographs and P4-sparse.* In addition, another study by [13] discusses b-coloring of unicyclic and bicyclic graphs.

The novelty of this research lies in its exploration of the b-chromatic number for various tree graph families, specifically focusing on how their structural properties influence the b-coloring process. This study extends the current understanding by developing new b-coloring results and providing insights into how these results can contribute to broader applications in optimization and network theory. The objective of this research is to fill this gap by analyzing b-coloring for tree graphs and formulating a set of patterns applicable to these structures, thereby advancing the theoretical foundation of graph coloring. The following definitions and lemma outline the fundamental concepts of graph coloring and b-coloring, providing the theoretical foundation for the analysis of graph coloring in this study.

Definition 1. *[14] A coloring of all vertices of a graph $G(V, E)$ is a mapping $F : V \to N$ with the condition that adjacent vertices have different colors in N, meaning that $v_1v_2 \in E$ then $F(v_1) \neq F(v_2)$.*

Definition 2. *[11] The chromatic number of a graph G, denoted $\chi(G)$, is the minimum number of colors required to produce a proper coloring of the graph G*

Definition 3. *[10] A b-coloring of a graph G is a coloring of the vertices of G such that each color class has at least one vertex adjacent to all other color classes.*

A color class in b-coloring is a set of vertices that have the same color. The color class in question is the set of vertices that have color i and $1 \leq i \leq k$. Based on this definition, the letter "b" in "b-coloring" stands for "bounded", which refers to the limit on the number of colors used in the coloring process.

Lemma 1. *[15] For every graph G, it holds: $\chi(G) \leq \varphi(G) \leq \Delta(G) + 1$.*

$\chi(G)$ is the chromatic number of a graph G, where $\chi(G)$ is the minimum value of the color required for a correct coloring of the graph G.

2 Methods

There are two research methods in this study, namely the axiomatic deductive method and the pattern detection method. The axiomatic deductive method is one of the research methods that uses the principles of deductive proof in mathematical logic, applying existing theorems, axioms, and lemmas to b-coloring on a family of tree graphs. Then, pattern detection is used to formulate patterns and determine the b-chromatic number on the family of tree graphs studied. The research procedure is a systematic series of steps used to gather, analyze, and interpret data to answer research questions or test hypotheses. This procedure ensures that the research is structured, consistent, and replicable. The procedure for determining the b-coloring in tree graph families is as follows:

1. Selection of Graphs for b-coloring Study: The first step is to select the tree graphs to be studied for b-coloring. These graphs include the broom graph, banana tree graph, and centipede graph. These specific graphs are chosen due to their distinct structural properties, which will be analyzed under the b-coloring concept.
2. Determining Cardinality for Each Graph: The next step is to determine the cardinality (the number of vertices and edges) for each selected graph: the broom graph, banana tree graph, and centipede graph. This step is crucial because the graph's size and structure directly influence the coloring process.
3. Assigning Colors to the Graph Vertices: In this step, colors are assigned to the vertices of each graph according to the definition of b-coloring. The b-coloring is applied so that adjacent vertices receive different colors, with the condition that each color class must have at least one vertex adjacent to all other color classes.
4. Verification of b-coloring Definition Compliance: After assigning colors, the next step is to verify whether all vertices in the graphs (broom graph, banana tree graph, and centipede graph) comply with the b-coloring definition. This means ensuring that adjacent vertices do not share the same color and that each color class meets the b-coloring criteria.
5. Determining the b-chromatic Number: Once b-coloring is verified, the b-chromatic number for each graph is determined. The b-chromatic number refers to the maximum number of color classes used in a valid b-coloring of the graph. This number is essential for understanding the graph's coloring complexity.
6. Formulating Theorems Based on b-coloring Results: After determining the b-chromatic number, the next step is to formulate theorems based on the results of the b-coloring analysis. These theorems describe the relationships between the graph's structure and its b-chromatic number, thereby formalizing the findings.
7. Proving the Theorems: The formulated theorems are then proven using logical reasoning and the axiomatic deductive method. This involves demonstrating that the theorems hold true for the selected tree graphs, relying on mathematical proofs that confirm the validity of the relationships discovered in the b-coloring process.

3 Results and Discussion

This study presents three theorems on the b-chromatic number for tree graph families, specifically the broom graph, banana tree graph, and centipede graph, as follows:

Theorem 1. *b-Chromatic number of on the $B_{n,3}$ for $9 \leq n \leq 13$ is $\varphi(B_{n,3}) = 5$.*

Proof. The broom graph ($B_{n,3}$) has a vertex set, namely $V(B_{n,3}) = \{x_i; 1 \leq i \leq n\} \cup \{y_i; 1 \leq i \leq 3\}$ and an edge set $E(B_{n,3}) = \{x_i x_{i+1}; 1 \leq i \leq n-1\} \cup \{x_1 y_i; 1 \leq i \leq 3\}$. The cardinalities of the vertex set and edge set of the broom graph ($B_{n,3}$) are $|V(B_{n,3}| = n + 3$ and $|E(B_{n,3})| = n + 2$, respectively. Furthermore, to determine the b-chromatic number on the broom graph $B_{n,3}$ with $9 \leq n \leq 13$, the upper bound on the broom graph $B_{n,3}$. Based on Definition 3 and Lemma 1, we obtain $\chi(B_{n,3}) \leq \varphi(B_{n,3}) \leq \Delta(B_{n,3}) + 1$. Next, determine the coloring function $f : V(G) \rightarrow \{1, 2, 3, 4, 5\}$ for each vertex on the broom graph $B_{n,3}$ for $n = 9$, the coloring function is defined as follows:

$$f(v) = \begin{cases} 1, & v = x_1, \\ 2, & v = x_5,\ x_9,\ y_1, \\ 3, & v = x_4,\ x_7,\ y_2, \\ 4, & v = x_3,\ x_6,\ y_3, \\ 5, & v = x_2,\ x_8. \end{cases}$$

Based on this function, the color class obtained in the broom graph for $n = 9$ is five with the following set of color classes :

$$C_1 = \{x_1\}, \quad C_2 = \{x_5, x_9, y_1\}, \quad C_3 = \{x_4, x_7, y_2\}, \quad C_4 = \{x_3, x_6, y_3\}, \quad C_5 = \{x_2, x_8\}$$

So that

$$\begin{aligned}
&\exists\, x_1 \in C_1 \text{ such that } x_1 \text{ is adjacent to } y_1,\ y_1 \in C_2,\\
&\exists\, x_1 \in C_1 \text{ such that } x_1 \text{ is adjacent to } y_2,\ y_2 \in C_3,\\
&\exists\, x_1 \in C_1 \text{ such that } x_1 \text{ is adjacent to } y_3,\ y_3 \in C_4,\\
&\exists\, x_1 \in C_1 \text{ such that } x_1 \text{ is adjacent to } x_2,\ x_2 \in C_5,\\
&\exists\, x_2 \in C_5 \text{ such that } x_2 \text{ is adjacent to } x_3,\ x_3 \in C_4,\\
&\exists\, x_3 \in C_4 \text{ such that } x_3 \text{ is adjacent to } x_4,\ x_4 \in C_3,\\
&\exists\, x_4 \in C_3 \text{ such that } x_4 \text{ is adjacent to } x_5,\ x_5 \in C_2,\\
&\exists\, x_5 \in C_2 \text{ such that } x_5 \text{ is adjacent to } x_6,\ x_6 \in C_4,\\
&\exists\, x_8 \in C_5 \text{ such that } x_8 \text{ is adjacent to } x_7,\ x_7 \in C_3,\\
&\exists\, x_9 \in C_2 \text{ such that } x_9 \text{ is adjacent to } x_8,\ x_8 \in C_5.
\end{aligned}$$

for $n = 10$

$$f(v) = \begin{cases} 1, & v \in \{x_1, x_{10}\}, \\ 2, & v \in \{x_5, x_9, y_1\}, \\ 3, & v \in \{x_4, x_7, y_2\}, \\ 4, & v \in \{x_3, x_6, y_3\}, \\ 5, & v \in \{x_2, x_8\}. \end{cases}$$

Based on this function, the broom graph $B_{n,3}$ for $n = 10$ admits a b-coloring using five colors with the following color classes:

$$C_1 = \{x_1, x_{10}\}, \quad C_2 = \{x_5, x_9, y_1\}, \quad C_3 = \{x_4, x_7, y_2\}, \quad C_4 = \{x_3, x_6, y_3\}, \quad C_5 = \{x_2, x_8\}.$$

So that

$$\begin{aligned}
&\exists\, x_1 \in C_1 \text{ such that } x_1 \text{ is adjacent to } y_1,\ y_1 \in C_2,\\
&\exists\, x_1 \in C_1 \text{ such that } x_1 \text{ is adjacent to } y_2,\ y_2 \in C_3,\\
&\exists\, x_1 \in C_1 \text{ such that } x_1 \text{ is adjacent to } y_3,\ y_3 \in C_4,\\
&\exists\, x_1 \in C_1 \text{ such that } x_1 \text{ is adjacent to } x_2,\ x_2 \in C_5,\\
&\exists\, x_2 \in C_5 \text{ such that } x_2 \text{ is adjacent to } x_3,\ x_3 \in C_4,\\
&\exists\, x_3 \in C_4 \text{ such that } x_3 \text{ is adjacent to } x_4,\ x_4 \in C_3,\\
&\exists\, x_4 \in C_3 \text{ such that } x_4 \text{ is adjacent to } x_5,\ x_5 \in C_2,\\
&\exists\, x_5 \in C_2 \text{ such that } x_5 \text{ is adjacent to } x_6,\ x_6 \in C_4,\\
&\exists\, x_8 \in C_5 \text{ such that } x_8 \text{ is adjacent to } x_7,\ x_7 \in C_3,\\
&\exists\, x_9 \in C_2 \text{ such that } x_9 \text{ is adjacent to } x_8,\ x_8 \in C_5,\\
&\exists\, x_9 \in C_2 \text{ such that } x_9 \text{ is adjacent to } x_{10},\ x_{10} \in C_1.
\end{aligned}$$

for $n = 11$

$$f(v) = \begin{cases} 1, & v \in \{x_1, x_{10}\}, \\ 2, & v \in \{x_5, x_9, y_1\}, \\ 3, & v \in \{x_4, x_7, y_2\}, \\ 4, & v \in \{x_3, x_6, y_3\}, \\ 5, & v \in \{x_2, x_8, x_{11}\}. \end{cases}$$

Based on this function, the broom graph $B_{n,3}$ for $n = 11$ admits a b-coloring using five colors with the following color classes:

$$C_1 = \{x_1, x_{10}\}, \quad C_2 = \{x_5, x_9, y_1\}, \quad C_3 = \{x_4, x_7, y_2\}, C_4 = \{x_3, x_6, y_3\}, \quad C_5 = \{x_2, x_8, x_{11}\}.$$

So that

$$\begin{aligned}
&\exists\, x_1 \in C_1 \text{ such that } x_1 \text{ is adjacent to } y_1,\ y_1 \in C_2,\\
&\exists\, x_1 \in C_1 \text{ such that } x_1 \text{ is adjacent to } y_2,\ y_2 \in C_3,\\
&\exists\, x_1 \in C_1 \text{ such that } x_1 \text{ is adjacent to } y_3,\ y_3 \in C_4,\\
&\exists\, x_1 \in C_1 \text{ such that } x_1 \text{ is adjacent to } x_2,\ x_2 \in C_5,\\
&\exists\, x_2 \in C_5 \text{ such that } x_2 \text{ is adjacent to } x_3,\ x_3 \in C_4,\\
&\exists\, x_3 \in C_4 \text{ such that } x_3 \text{ is adjacent to } x_4,\ x_4 \in C_3,\\
&\exists\, x_4 \in C_3 \text{ such that } x_4 \text{ is adjacent to } x_5,\ x_5 \in C_2,\\
&\exists\, x_5 \in C_2 \text{ such that } x_5 \text{ is adjacent to } x_6,\ x_6 \in C_4,\\
&\exists\, x_8 \in C_5 \text{ such that } x_8 \text{ is adjacent to } x_7,\ x_7 \in C_3,\\
&\exists\, x_9 \in C_2 \text{ such that } x_9 \text{ is adjacent to } x_8,\ x_8 \in C_5,\\
&\exists\, x_9 \in C_2 \text{ such that } x_9 \text{ is adjacent to } x_{10},\ x_{10} \in C_1,\\
&\exists\, x_{10} \in C_1 \text{ such that } x_{10} \text{ is adjacent to } x_{11},\ x_{11} \in C_5.
\end{aligned}$$

for $n = 12$

$$f(v) = \begin{cases} 1, & v \in \{x_1, x_{10}\}, \\ 2, & v \in \{x_5, x_9, y_1\}, \\ 3, & v \in \{x_4, x_7, x_{12}, y_2\}, \\ 4, & v \in \{x_3, x_6, y_3\}, \\ 5, & v \in \{x_2, x_8, x_{11}\}. \end{cases}$$

Based on this function, the graph $B_{n,3}$ for $n = 12$ admits a b-coloring using 5 colors with the following set of color classes:

$$C_1 = \{x_1, x_{10}\}, \quad C_2 = \{x_5, x_9, y_1\}, C_3 = \{x_4, x_7, x_{12}, y_2\}, C_4 = \{x_3, x_6, y_3\}, C_5 = \{x_2, x_8, x_{11}\}.$$

So that

$$\begin{aligned}
&\exists\, x_1 \in C_1 \text{ such that } x_1 \text{ is adjacent to } y_1,\ y_1 \in C_2,\\
&\exists\, x_1 \in C_1 \text{ such that } x_1 \text{ is adjacent to } y_2,\ y_2 \in C_3,\\
&\exists\, x_1 \in C_1 \text{ such that } x_1 \text{ is adjacent to } y_3,\ y_3 \in C_4,\\
&\exists\, x_1 \in C_1 \text{ such that } x_1 \text{ is adjacent to } x_2,\ x_2 \in C_5,\\
&\exists\, x_2 \in C_5 \text{ such that } x_2 \text{ is adjacent to } x_3,\ x_3 \in C_4,\\
&\exists\, x_3 \in C_4 \text{ such that } x_3 \text{ is adjacent to } x_4,\ x_4 \in C_3,\\
&\exists\, x_4 \in C_3 \text{ such that } x_4 \text{ is adjacent to } x_5,\ x_5 \in C_2,\\
&\exists\, x_5 \in C_2 \text{ such that } x_5 \text{ is adjacent to } x_6,\ x_6 \in C_4,
\end{aligned}$$

$\exists\, x_8 \in C_5$ such that x_8 is adjacent to $x_7,\ x_7 \in C_3,$
$\exists\, x_9 \in C_2$ such that x_9 is adjacent to $x_8,\ x_8 \in C_5,$
$\exists\, x_9 \in C_2$ such that x_9 is adjacent to $x_{10},\ x_{10} \in C_1,$
$\exists\, x_{10} \in C_1$ such that x_{10} is adjacent to $x_{11},\ x_{11} \in C_5,$
$\exists\, x_{11} \in C_5$ such that x_{11} is adjacent to $x_{12},\ x_{12} \in C_3.$

for $n = 13$

$$f(v) = \begin{cases} 1, & v \in \{x_1, x_{12}\}, \\ 2, & v \in \{x_5, x_8, x_{11}, y_1\}, \\ 3, & v \in \{x_4, x_{10}, y_2\}, \\ 4, & v \in \{x_3, x_6, y_3\}, \\ 5, & v \in \{x_2, x_7, x_9, x_{13}\}. \end{cases}$$

Based on this function, the color class obtained in the broom graph $B_{n,3}$ for $n = 13$ is five with the following set of color classes:

$C_1 = \{x_1, x_{12}\}$, $C_2 = \{x_5, x_8, x_{11}, y_1\}$, $C_3 = \{x_4, x_{10}, y_2\}$, $C_4 = \{x_3, x_6, y_3\}$, $C_5 = \{x_2, x_7, x_9, x_{13}\}$.

So that

$\exists\, x_1 \in C_1$ such that x_1 is adjacent to $y_1,\ y_1 \in C_2,$
$\exists\, x_1 \in C_1$ such that x_1 is adjacent to $y_2,\ y_2 \in C_3,$
$\exists\, x_1 \in C_1$ such that x_1 is adjacent to $y_3,\ y_3 \in C_4,$
$\exists\, x_1 \in C_1$ such that x_1 is adjacent to $x_2,\ x_2 \in C_5,$
$\exists\, x_2 \in C_5$ such that x_2 is adjacent to $x_3,\ x_3 \in C_4,$
$\exists\, x_3 \in C_4$ such that x_3 is adjacent to $x_4,\ x_4 \in C_3,$
$\exists\, x_4 \in C_3$ such that x_4 is adjacent to $x_5,\ x_5 \in C_2,$
$\exists\, x_5 \in C_2$ such that x_5 is adjacent to $x_6,\ x_6 \in C_4,$
$\exists\, x_8 \in C_2$ such that x_8 is adjacent to $x_7,\ x_7 \in C_5,$
$\exists\, x_9 \in C_5$ such that x_9 is adjacent to $x_8,\ x_8 \in C_2,$
$\exists\, x_{10} \in C_3$ such that x_{10} is adjacent to $x_{11},\ x_{11} \in C_2,$
$\exists\, x_{12} \in C_1$ such that x_{12} is adjacent to $x_{13},\ x_{13} \in C_5.$

It will be proven that $\chi(B_{n,3}) = 2$ is a lower bound of the broom graph $B_{n,3}$ if $\varphi(B_{n,3}) = 5$ for $9 \leq n \leq 13$. Based on Lemma 1, the following is obtained.

$$5 = \varphi(B_{n,3}) \geq 2. \tag{1}$$

Based on inequality 1, it is proven that $\chi(B_{n,3}) = 2$ is the lower bound of the broom graph $B_{n,3}$ with $9 \leq n \leq 13$. It will be proven that $\varphi(B_{n,3}) = 5$ for $9 \leq n \leq 13$ and $\Delta(B_{n,3}) + 1$ is the upper bound of the $B_{n,3}$. Based on Lemma 1, the following is obtained.

$$\varphi(B_{n,3}) \leq \Delta(B_{n,3}) + 1 = 5 \tag{2}$$

Based on inequality 2, it is proven that $\varphi(B_{n,3}) = 5$ for $9 \leq n \leq 13$. Based on the color class above, the broom graph $(B_{n,3})$ for $9 \leq n \leq 13$ satisfies the definition of b-coloring with the upper bound obtained $\varphi(B_{n,3}) \leq 5$ for $9 \leq n \leq 13$. According to the definition of b-chromatic number, the b-chromatic number value is obtained from the maximum value in the graph color class $(B_{n,3})$ for $9 \leq n \leq 13$ namely $\varphi(B_{n,3}) = 5$. Below is an illustration of b-coloring applied to a broom graph.

□

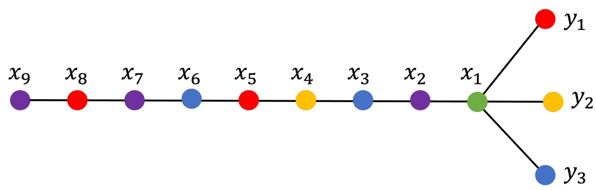

Figure 1: b-coloring on Broom Graph $(B_{9,3})$

Theorem 2. *The b-chromatic number of the Banana Tree graph* $(Bt_{2,n})$ *for* $n \geq 4$ *is* $\varphi(Bt_{2,n}) = 4$

Proof. The banana tree graph $(Bt_{2,n})$ has a set of points, namely $V(Bt_{2,n}) = \{y\} \cup \{x_i; 1 \leq i \leq 2\} \cup \{x_{i,j}; 1 \leq i \leq 2, 1 \leq j \leq n\}$ and a set of edges $E(Bt_{2,n}) = \{yx_{i,1}; 1 \leq i \leq 2\} \cup \{x_i x_{i,j}; 1 \leq i \leq 2, 1 \leq j \leq n\}$. The cardinalities of the vertex set and edge set of the banana tree graph $(Bt_{2,n})$ are respectively $|V(Bt_{2,n})| = 2n+3$ and $|E(Bt_{2,n})| = 2n+2$. To determine the b-chromatic number of the banana tree graph $(Bt_{2,n})$ with $n \geq 4$, the upper bound of the banana tree graph $(Bt_{2,n})$ will be analyzed. Based on Definition 3 and Lemma 1, we obtain $\chi(Bt_{2,n}) \leq \varphi(Bt_{2,n}) \leq \Delta(Bt_{2,n})+1$. Next, determine the coloring function $f : V(G) \to \{1, 2, 3, 4\}$ for each vertex in the banana tree graph $(Bt_{2,n})$.

For $n = 4$

$$f(v) = \begin{cases} 1, & v = x_1, \\ 2, & v \in \{x_{1,1},\ x_{1,2},\ x_2\}, \\ 3, & v \in \{x_{1,3},\ y,\ x_{2,2},\ x_{2,4}\}, \\ 4, & v \in \{x_{1,4},\ x_{2,1},\ x_{2,3}\}. \end{cases}$$

Based on this function, the color class obtained in the banana tree graph for $n = 4$ is four with the following color class sets:

$$C_1 = \{x_1\}, \quad C_2 = \{x_{1,1}, x_{1,2}, x_2\}, \quad C_3 = \{x_{1,3}, y, x_{2,2}, x_{2,4}\}, \quad C_4 = \{x_{1,4}, x_{2,1}, x_{2,3}\}.$$

Thus,

$$\begin{aligned} &\exists\, x_1 \in C_1 \text{ such that } x_1 \text{ is adjacent to } x_{1,1} \text{ and } x_{1,2},\ x_{1,1}, x_{1,2} \in C_2, \\ &\exists\, x_1 \in C_1 \text{ such that } x_1 \text{ is adjacent to } x_{1,3},\ x_{1,3} \in C_3, \\ &\exists\, x_1 \in C_1 \text{ such that } x_1 \text{ is adjacent to } x_{1,4},\ x_{1,4} \in C_4, \\ &\exists\, x_2 \in C_2 \text{ such that } x_2 \text{ is adjacent to } x_{2,2} \text{ and } x_{2,4},\ x_{2,2}, x_{2,4} \in C_3, \\ &\exists\, x_2 \in C_2 \text{ such that } x_2 \text{ is adjacent to } x_{2,1} \text{ and } x_{2,3},\ x_{2,1}, x_{2,3} \in C_4, \\ &\exists\, y \in C_3 \text{ such that } y \text{ is adjacent to } x_{1,1},\ x_{1,1} \in C_2. \end{aligned}$$

for $n = 5$

$$f(v) = \begin{cases} 1, & v \in \{x_1\}, \\ 2, & v \in \{x_{1,1}, x_{1,2}, x_2, x_{1,5}\}, \\ 3, & v \in \{x_{1,3}, y, x_{2,2}, x_{2,4}\}, \\ 4, & v \in \{x_{1,4}, x_{2,1}, x_{2,3}\}. \end{cases}$$

Based on this function, the color class obtained in the banana tree graph for $n = 5$ is 4 with the following color class sets:

$$C_1 = \{x_1\}, \quad C_2 = \{x_{1,1}, x_{1,2}, x_2, x_{1,5}\}, \quad C_3 = \{x_{1,3}, y, x_{2,2}, x_{2,4}\}, \quad C_4 = \{x_{1,4}, x_{2,1}, x_{2,3}, x_{2,5}\}.$$

Thus,

$$\begin{aligned}
&\exists\, x_1 \in C_1 \text{ such that } x_1 \text{ is adjacent to } x_{1,1},\ x_{1,2},\ \text{and } x_{1,5},\ x_{1,1}, x_{1,2}, x_{1,5} \in C_2,\\
&\exists\, x_1 \in C_1 \text{ such that } x_1 \text{ is adjacent to } x_{1,3},\ x_{1,3} \in C_3,\\
&\exists\, x_1 \in C_1 \text{ such that } x_1 \text{ is adjacent to } x_{1,4},\ x_{1,4} \in C_4,\\
&\exists\, x_2 \in C_2 \text{ such that } x_2 \text{ is adjacent to } x_{2,2} \text{ and } x_{2,4},\ x_{2,2}, x_{2,4} \in C_3,\\
&\exists\, x_2 \in C_2 \text{ such that } x_2 \text{ is adjacent to } x_{2,1},\ x_{2,3},\ \text{and } x_{2,5},\ x_{2,1}, x_{2,3}, x_{2,5} \in C_4,\\
&\exists\, y \in C_3 \text{ such that } y \text{ is adjacent to } x_{1,1},\ x_{1,1} \in C_2.
\end{aligned}$$

for $n = 6$

$$f(v) = \begin{cases} 1, & v \in \{x_1\},\\ 2, & v \in \{x_{1,2}, x_2, x_{1,5}\},\\ 3, & v \in \{x_{1,3}, y, x_{2,2}, x_{2,4}, x_{1,6}\},\\ 4, & v \in \{x_{1,1}, x_{1,4}, x_{2,1}, x_{2,3}, x_{2,5}\}. \end{cases}$$

Based on this function, the color class obtained in the banana tree graph for $n = 6$ is 4 with the following color class sets:

$$C_1 = \{x_1\}, \quad C_2 = \{x_{1,2}, x_2, x_{1,5}\}, C_3 = \{x_{1,3}, y, x_{2,2}, x_{2,4}, x_{1,6}\}, \quad C_4 = \{x_{1,1}, x_{1,4}, x_{2,1}, x_{2,3}, x_{2,5}\}.$$

Thus,

$$\begin{aligned}
&\exists\, x_1 \in C_1 \text{ such that } x_1 \text{ is adjacent to } x_{1,1} \text{ and } x_{1,4},\ x_{1,1}, x_{1,4} \in C_4,\\
&\exists\, x_1 \in C_1 \text{ such that } x_1 \text{ is adjacent to } x_{1,2} \text{ and } x_{1,5},\ x_{1,2}, x_{1,5} \in C_2,\\
&\exists\, x_1 \in C_1 \text{ such that } x_1 \text{ is adjacent to } x_{1,3} \text{ and } x_{1,6},\ x_{1,3}, x_{1,6} \in C_3,\\
&\exists\, x_2 \in C_2 \text{ such that } x_2 \text{ is adjacent to } x_{2,1} \text{ and } x_{2,3},\ x_{2,1}, x_{2,3} \in C_4,\\
&\exists\, x_2 \in C_2 \text{ such that } x_2 \text{ is adjacent to } x_{2,2} \text{ and } x_{2,4},\ x_{2,2}, x_{2,4} \in C_3,\\
&\exists\, y \in C_3 \text{ such that } y \text{ is adjacent to } x_{1,1} \text{ and } x_{2,1},\ x_{1,1}, x_{2,1} \in C_4.
\end{aligned}$$

It will be proven that $\chi(Bt_{2,n}) = 2$ is the lower bound of the banana tree graph $(Bt_{2,n})$ if $\varphi(Bt_{2,n}) = 4$ for $n \geq 4$. Based on Lemma 1, the following is obtained.

$$4 = \varphi(Bt_{2,n}) \geq 2 \tag{3}$$

Based on inequality 3, it is proven that $\chi(Bt_{2,n}) = 2$ is the lower bound of the banana tree graph $(Bt_{2,n})$ with $n \geq 4$. It will be proven that $\varphi(Bt_{2,n}) = 4$ for $n \geq 4$ and $\Delta(Bt_{2,n}) + 1$ is the upper bound of the banana tree graph. Based on Lemma 1, the following is obtained.

$$\varphi(Bt_{2,n}) \leq \Delta(Bt_{2,n}) + 1 = 5 \tag{4}$$

Based on inequality 4, this contradicts the color class of the banana tree graph $(Bt_{2,n})$ for $n \geq 4$. Meanwhile, according to the definition of b-chromatic number, the *b*-chromatic number value is obtained from the maximum value in the color class of the banana tree graph $(Bt_{2,n})$ for $n \geq 4$, namely $\varphi(Bt_{2,n}) = 5$. When viewed from the color class, it will fail at point y, because there are two pairs of non adjacent vertex, which does not comply with the definition of *b*-coloring. Thus, the *b*-chromatic number of the banana tree graph $(Bt_{2,n})$ for $n \geq 4$ is $\varphi(Bt_{2,n}) = 4$. □

Figure 2 shows an illustration of b-coloring on the banana tree graph $(Bt_{2,4})$ with the *b*-chromatic number of the Banana Tree graph $(Bt_{2,4})$ is 4.

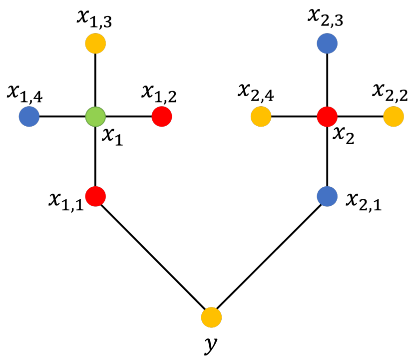

Figure 2: b-coloring on Banana Tree Graph ($B_{2,4}$)

Theorem 3. *The b-Chromatic number of a Centipede graph* (Cp_n) *for* $4 \leq n \leq 5$ *is* $\varphi(Cp_n) = 4$.

Proof. The centipede graph (Cp_n) has a vertex set, namely $V(Cp_n) = \{x_i; 1 \leq i \leq n\} \cup \{y_i; 1 \leq i \leq n\}$ and an edge set $E(Cp_n) = \{x_i x_{i+1}; 1 \leq i \leq n-1\} \cup \{x_i y_i; 1 \leq i \leq n\}$.The cardinalities of the vertex set and edge set of the centipede graph (Cp_n) are $|V(Cp_n)| = 2n$ and $|E(Cp_n)| = 2n-1$, respectively. Next, to determine the b-chromatic number of the centipede graph (Cp_n) with $4 \leq n \leq 11$, the upper bound of the centipede graph (Cp_n). Based on Definition 3 and Lemma 1, we obtain $\chi(Cp_n) \leq \varphi(Cp_n) \leq \Delta(Cp_n) + 1$. Next, determine the coloring function $f : V(G) \to \{1, 2, 3, 4\}$ for each vertex of the centipede graph (Cp_n) for $4 \leq n \leq 5$, the coloring function is defined as follows:

for $n = 4$

$$f(v) = \begin{cases} 1, & v = x_2, \\ 2, & v \in \{x_1, y_3, y_4\}, \\ 3, & v \in \{x_4, y_1, y_2\}, \\ 4, & v = x_3. \end{cases}$$

Based on this function, the color class obtained in the centipede graph for $n = 4$ uses 4 colors with the following set of color classes:

$$C_1 = \{x_2\}, \quad C_2 = \{x_1, y_3, y_4\}, \quad C_3 = \{x_4, y_1, y_2\}, \quad C_4 = \{x_3\}.$$

So that

$$\begin{aligned} &\exists x_2 \in C_1 \text{ such that } x_2 \text{ is adjacent to } x_1,\ x_1 \in C_2, \\ &\exists x_2 \in C_1 \text{ such that } x_2 \text{ is adjacent to } y_2,\ y_2 \in C_3, \\ &\exists x_2 \in C_1 \text{ such that } x_2 \text{ is adjacent to } x_3,\ x_3 \in C_4, \\ &\exists x_3 \in C_4 \text{ such that } x_3 \text{ is adjacent to } y_3,\ y_3 \in C_2, \\ &\exists x_3 \in C_4 \text{ such that } x_3 \text{ is adjacent to } x_4,\ x_4 \in C_3, \\ &\exists x_4 \in C_3 \text{ such that } x_4 \text{ is adjacent to } y_4,\ y_4 \in C_2. \end{aligned}$$

for $n = 5$

$$f(v) = \begin{cases} 1, & v \in \{x_2, x_5\}, \\ 2, & v \in \{x_1, y_3, y_4\}, \\ 3, & v \in \{x_4, y_1, y_2, y_5\}, \\ 4, & v = x_3. \end{cases}$$

Based on this function, the color class obtained in the centipede graph for $n = 5$ uses 4 colors with the following set of color classes:

$$C_1 = \{x_2, x_5\}, \quad C_2 = \{x_1, y_3, y_4\}, \quad C_3 = \{x_4, y_1, y_2, y_5\}, \quad C_4 = \{x_3\}.$$

So that

$$\begin{aligned}
&\exists x_2 \in C_1 \text{ such that } x_2 \text{ is adjacent to } x_1,\ x_1 \in C_2,\\
&\exists x_2 \in C_1 \text{ such that } x_2 \text{ is adjacent to } y_2,\ y_2 \in C_3,\\
&\exists x_2 \in C_1 \text{ such that } x_2 \text{ is adjacent to } x_3,\ x_3 \in C_4,\\
&\exists x_3 \in C_4 \text{ such that } x_3 \text{ is adjacent to } y_3,\ y_3 \in C_2,\\
&\exists x_3 \in C_4 \text{ such that } x_3 \text{ is adjacent to } x_4,\ x_4 \in C_3,\\
&\exists x_4 \in C_3 \text{ such that } x_4 \text{ is adjacent to } y_4,\ y_4 \in C_2.
\end{aligned}$$

It will be proven that $\chi(Cp_n) = 2$ is a lower bound of the centipede graph (Cp_n) if $\varphi(Cp_n) = 4$ for $4 \leq n \leq 5$. Based on Lemma 1, the following is obtained.

$$4 = \varphi(Cp_n) \geq 2 \tag{5}$$

Based on inequality 5, it is proven that $\chi(Cp_n) = 2$ is the lower b-chromatic bound of the centipede graph (Cp_n) for $4 \leq n \leq 5$. It will be proven that $\varphi(Cp_n) = 4$ for $4 \leq n \leq 5$ and $\Delta(Cp_n) + 1$ is the upper bound of the centipede graph. Based on Lemma 1, the following is obtained.

$$\varphi(Cp_n) \leq \Delta(Cp_n) + 1 = 4 \tag{6}$$

Based on inequality 6, it is proven that $\varphi(Cp_n) = 4$ for $4 \leq n \leq 5$. Based on the color class above, for the centipede graph (Cp_n) for $4 \leq n \leq 5$ it satisfies the definition of b-coloring with the upper bound obtained $\varphi(Cp_n) = 4$ for $4 \leq n \leq 5$. According to the definition of b-chromatic number, the b-chromatic number value is obtained from the maximum value in the color class of the centipede graph (Cp_n) for $4 \leq n \leq 5$, namely $\varphi(Cp_n) = 4$. Figure 3 illustrates the application of b-coloring to the Centipede Graph (Cp_4) where each vertex is assigned colors according to the b-coloring rules. □

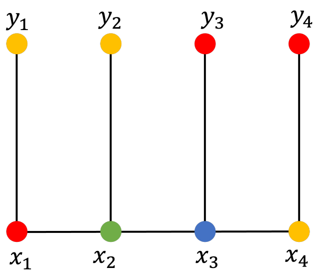

Figure 3: b-coloring on Centipede Graph (Cp_4)

4 Conclusion

Based on the results, three new theorems were derived on the b-coloring of tree graph families, specifically the broom graph $(B_{n,3})$, the banana tree graph $(Bt_{2,n})$, and the centipede graph (Cp_n). These theorems provide exact formulas for the b-chromatic numbers of these graphs, offering valuable insights into how their structure influences b-coloring. These results advance theoretical understanding and can be applied in practical fields such as network design, where minimizing color classes is essential for efficiency. Future research should explore b-coloring in other graph families, including hybrid and non-tree graphs, to further expand its applicability. Additionally, computational approaches can validate these theoretical findings in larger, more complex networks, and further investigate the relationship between b-coloring and other graph properties.

CRediT Authorship Contribution Statement

Rafiantika Megahnia Prihandini: Conceptualization, Methodology, Formal analysis, Investigation, Writing–original draft.
Arika Indah Kristiana: Conceptualization, Supervision, Validation, Writing–review & editing.
Lusita Risma Dana: Investigation, Visualization, Writing–review & editing.
Edy Wihardjo: Formal analysis, Validation, Visualization.
Robiatul Adawiyah: Methodology, Validation, Writing–review & editing.
Hutkemri Zulnaidi: Supervision, Validation, Writing–review & editing.
All authors have read and approved the final version of the manuscript.

Declaration of Generative AI and AI-assisted technologies

The authors declare that no generative AI and AI-assisted technologies were used in the writing process, the development of the mathematical results, or the preparation of this manuscript.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data and Code Availability

No dataset was generated or analyzed in this study. The results are derived analytically using deductive proofs and pattern detection on the studied graph families. Therefore, data and code sharing are not applicable. If any supplementary materials are prepared (e.g., additional illustrations or extended proofs), they will be made available from the corresponding author upon reasonable request.

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