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Energy and Topological Indices of Complete Bipartite Subgraphs

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ABSSTRACT

This paper investigates the complete bipartite subgraphs induced within the zero-divisor graph of a commutative ring formed by the direct product of three distinct modular integer rings. The set of nonzero zero-divisors is partitioned into six disjoint subsets based on the position of the zero component in each element. Six complete bipartite subgraphs are constructed and analysed by pairing subsets with zeros in different positions. For each subgraph, we compute the energy, Laplacian energy, and three degree-based multiplicative topological indices, namely the Narumi–Katayama index, and the first and second multiplicative Zagreb indices. The results are expressed in closed-form formulas and reveal consistent structural patterns, highlighting the relationship between the algebraic properties of the ring and the graph-theoretic characteristics of the induced subgraphs.

Keywords: complete bipartite graph; energy; Laplacian energy; topological indices; zero-divisor graph

INTRODUCTION

Graph theory is a branch of mathematics that is widely developed and applied in various fields. Its historical roots trace back to the Königsberg bridge problem in Prussia, which Euler in 1736 modeled using a graph by representing land areas as vertices and bridges as edges [1]. A graph G is defined as an ordered pair $(V(G), E(G))$, where $V(G)$ is a non-empty finite set of elements called vertices and $E(G)$ is a finite (possibly empty) set of pairs of vertices called edges. A subgraph of a graph can be formed by removing vertices or edges in the original graph. Graphs exhibit various unique structures based on the properties of their vertex and edge sets, including complete graphs and bipartite graphs. A complete graph is one in which every pair of distinct vertices is adjacent, whereas a bipartite graph is one whose vertex set can be partitioned into two disjoint sets V_1 and V_2 such that every edge connects a vertex in V_1 to a vertex in V_2. A bipartite graph is said to be complete bipartite if and only if every vertex in V_1 is connected to every vertex in V_2. If V_1 has m vertices and V_2 has n vertices, then the complete bipartite graph is denoted by $K_{m,n}$ [2].

One of the fundamental concepts in graph theory is the degree of a vertex, defined as the number of edges incident to that vertex. Various matrices can also be associated with

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a graph, such as the degree matrix, adjacency matrix, and Laplacian matrix. For a graph of order n, the degree matrix is an $n \times n$ diagonal matrix whose diagonal entries represent the degrees of the vertices. The adjacency matrix is an $n \times n$ matrix where the entries are 1 if two vertices are adjacent, and 0 otherwise [3]. The Laplacian matrix is derived by computing the difference between the degree matrix and the adjacency matrix [4].

Graph theory can be combined with algebraic structures, especially commutative rings. A commutative ring is a non-empty set R equipped with two binary operations, say addition and multiplication that satisfy the commutative group axioms under addition, are associative and commutative under multiplication, and follow the distributive laws. One graph construction based on a commutative ring is the zero-divisor graph. Many researchers are interested to investigate this graph for many cases. This concept was first introduced by Beck in 1988, where the vertices represent all elements of R and two vertices are adjacent if and only if their product is zero [5]. Later, Anderson and Livingston in 1999 modified Beck's definition. The resulting graph, denoted by $\Gamma(R)$, has as its vertices all the zero-divisors of the commutative ring R [6]. Moreover, the research of zero-divisor graphs can be seen in [7] and [8].

Dancheng and Tongsou studied bipartite graphs on zero-divisor graphs [9]. The result shows that a zero-divisor graph is bipartite if and only if it contains no triangles. Moreover, if such a graph is bipartite and contains no vertex of degree one, then it is a complete bipartite graph. Sharma et al. studied the adjacency matrix of a zero-divisor graph $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p)$, where p is a prime number [10]. Akgunes and Togan studied the degree of each vertex and the distance between pairs of vertices in $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q)$, where p and q are prime numbers [11]. Furthermore, Aykac and Akgunes studied the diameter, radius, girth, maximum and minimum degree, dominance number, chromatic number, and clique number of the zero-divisor graph $\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2})$ [12] .

Graph theory is closely related to energy which can be applied to molecules and chemical compounds, as well as topological indices as a tool to understand the properties of molecules based on their graphical structure. The energy of a graph is defined as the sum of the absolute values of all eigenvalues of its adjacency matrix. This concept was first introduced by Gutman in 1978 inspired by the total energy of electrons in molecules [13]. In 2006, Gutman and Zhou extended this idea by introducing Laplacian energy, calculated as the sum of the absolute values of the eigenvalues of the Laplacian matrix [14]. Ahmadi and Jahani-Nezhad studied the energy of the zero-divisor graph $\Gamma(\mathbb{Z}_{pq})$where p and q are prime numbers [15]. This graph is isomorphic to $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q)$ because it has the same structure, which both form a complete bipartite graph. Furthermore, in 2021 Singh and Bhat studied the Laplacian energy of the zero-divisor graph $\Gamma(\mathbb{Z}_{pq})$ [16] .

A topological index is a numerical parameter of a graph that describes its topological characteristics. Degree, distance, or eccentricity-based topological indices are widely used to characterise molecular graphs, establish relationships between molecular structure and properties, and predict the biological activity of chemical compounds. Topological indices are generally categorized as additive or multiplicative. Among the additive degree-based indices are the first and second Zagreb indices, introduced by Gutman and Trijnastic in 1972 [17]. The Zagreb indices were initially applied in chemistry to study total molecular energy, particularly the resonance energy of polyenes and aromatic hydrocarbons. This research focuses on the relationship between molecular graph structures, where vertices represent atoms and edges represent chemical bonds, and chemically relevant properties such as molecular stability. A higher vertex degree indicates more chemical bonds, suggesting that molecules with many high-degree atoms tend to be more stable due to more efficient electron distribution. The first Zagreb index is defined as the sum of the squares of the vertex degrees, while the second Zagreb index is defined as the sum of the products of the degrees of adjacent vertices.

In 1975, Gutman et al. developed the first and second Zagreb indices into multiplicative topological indices [18]. The first multiplicative Zagreb index is calculated as the product of the squares of the degrees of all vertices, while the second multiplicative Zagreb index is the product of the degree products of all adjacent vertex pairs. Furthermore, in 1984 Narumi and Katayama also introduced a simpler degree-based multiplicative topological index, namely the Narumi-Katayama index which focuses on multiplying the degree of each vertex [19].

Akgunes and Nacaroglu in 2019 studied the properties of the zero-divisor graph $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r)$, where p, q, r are prime numbers, including adjacency, degree, distance, diameter, radius, and girth [20]. They also analyzed the first and second Zagreb indices for this graph. Applicatively, Mondal et al. in 2021 studied degree-based multiplicative topological indices on molecular structures, namely nanostar dendrimers. The topological indices studied include the first and second multiplicative Zagreb indices, and the Narumi-Katayama index [21].

Theoretically, zero-divisor graphs derived from the ring $\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r$, where p, q, and r are distinct primes, exhibit complex structural properties. Although much of the existing literature focuses on analysing the entire zero-divisor graph, this study concentrates on specific complete bipartite subgraphs, which provide a more tractable and structured framework for investigation. These subgraphs offer a controlled setting in which both spectral and topological characteristics can be systematically examined, with meaningful implications for algebraic and combinatorial optimisation. Complete bipartite subgraphs have particular relevance in modelling real-world systems, such as communication networks and molecular structures, where strong interconnections occur between two distinct sets of components [22], [23]. In such settings, complete bipartite interactions serve as natural representations of highly organised and balanced interactions.

To the best of our knowledge, this research presents a novel approach by systematically forming complete bipartite subgraphs through two disjoint subsets of nonzero zero-divisors. We investigate their energy, Laplacian energy, and three degree-based multiplicative topological indices, namely the Narumi–Katayama index, the first multiplicative Zagreb index, and the second multiplicative Zagreb index. Previous studies have explored these invariants on full zero-divisor graphs, few have addressed them within the context of specifically induced bipartite substructures. Our contribution lies in deriving closed-form expressions that connect these graph invariants to the prime-based parameters of the underlying ring, offering both theoretical insight and applied potential.

We focus to study induced complete bipartite subgraphs, rather than the entire graph, since they provide the structural clarity. Such subgraphs, denoted by the union of distinct vertex classes, isolate specific interactions among zero-divisors and enable explicit computation of graph invariants. Moreover, in areas such as algebraic coding theory and cryptography, bipartite graphs offer natural models for structured data flows, such as those between encoding and decoding processes, or between public and private components in secure protocols. Investigating the spectral and topological properties of these well-defined subgraphs reveals algebraic and combinatorial patterns that may remain hidden in the global structure of the graph [24], [25].

Furthermore, constructing the zero-divisor graph from the product of three distinct primes introduces greater combinatorial richness compared to products involving only two primes. This enhanced complexity provides deeper insight into how the multiplicative structure of the ring influences the behaviour of the associated graph.

METHODS

This study applies an analytical approach based on concepts from graph theory and ring theory. To ensure conceptual clarity and provide a solid theoretical basis for the methodology, several key definitions are introduced first. These definitions guide the construction of graphs, matrix formulations, and the computation of the graph-theoretic invariants discussed in the following stages.

Definition 1 [5] Let R be a commutative ring, and let $Z^*(R)$ denote the set of nonzero zero-divisors. The zero-divisor graph of R, denoted $\Gamma(R)$, is a graph whose vertex set is $Z^*(R)$, and two vertices $u, v \in Z^*(R)$ are adjacent if and only if $u \cdot v = 0$.

Definition 2 [10] The energy of a graph G, denoted $E(G)$, is the sum of the absolute values of all the eigenvalues λ_i of the adjacency matrix $A(G)$.

$$E(G) = \sum_{i=1}^{n} |\lambda_i|.$$

Definition 3 [11] The Laplacian energy of a graph G, denoted $E_L(G)$, is the sum of the absolute values of all the eigenvalues μ_i of the Laplacian matrix $L(G)$.

$$E_L(G) = \sum_{i=1}^{n} |\mu_i|.$$

Definition 4 [16] The Narumi-Katayama index of a graph G, denoted $NK(G)$, is the product of the degrees of all vertices in G.

$$NK(G) = \prod_{v \in V(G)} \deg(v).$$

Definition 5 [15] The first multiplicative Zagreb index of the graph G, denoted $\Pi_1(G)$, is the product of the squares of the degrees of all vertices in G.

$$\Pi_1(G) = \prod_{v \in V(G)} (\deg(v))^2.$$

Definition 6 [15] The second multiplicative Zagreb index of the graph G, denoted $\Pi_2(G)$, is the product of the degree products of all adjacent vertex pairs in G.

$$\Pi_2(G) = \prod_{uv \in E(G)} \deg(u) \deg(v).$$

Based on the definitions presented above, the analytical procedure in this study proceeds through a series of structured steps. Each step is designed to construct and analyze complete bipartite subgraphs within the zero-divisor graph, incorporating both algebraic and spectral graph-theoretic techniques. The method is structured into five stages, as follows:

2.1 Subset Partition

Let R be the ring $\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r$, where p, q, r are distinct prime numbers. The set of nonzero zero-divisors $Z^*(R)$ is partitioned into six mutually disjoint subsets V_i, for $i = 1, 2, \cdots, 6$, based on the position of the zero component in each element. These subsets form the basis for constructing complete bipartite subgraphs by pairing sets $V_i \cup V_j$ such that the zero positions differ.

2.2 Graph Construction

For each valid pair (V_i, V_j), a complete bipartite subgraph is constructed where each vertex in V_i is adjacent to every vertex in V_j. According to **Definition 1**, two vertices are adjacent if their product yields the zero element in all components, which holds when the zero positions are distinct.

2.3 Matrix Assembly

For each induced bipartite graph isomorphic to $K_{m,n}$, the following matrices are constructed:

i. Adjacency matrix of the complete bipartite graph $K_{m,n}$, denoted $A(K_{m,n})$ is defined as a block matrix in which the off-diagonal blocks consist of all-ones matrices J, while the diagonal blocks are zero matrices O,

$$A(K_{m,n}) = \begin{bmatrix} O_{m\times m} & J_{m\times n} \\ J_{n\times m} & O_{n\times n} \end{bmatrix}.$$

ii. Degree matrix of the complete bipartite graph $K_{m,n}$, denoted $D(K_{m,n})$ is diagonal matrix where each vertex in V_i has degree n, and each vertex in V_j has degree m

iii. Laplacian matrix of the complete bipartite graph K_{mn}, denoted $L(K_{m,n})$ is constructed by subtracting the adjacency matrix from the degree matrix, that is,

$$L(K_{m,n}) = D(K_{m,n}) - A(K_{m,n}).$$

2.4 Eigenvalue Derivation

To compute the energy and Laplacian energy, the eigenvalues of the adjacency and Laplacian matrices are determined. Previously, the calculation of the characteristic polynomial of adjacency matrix $A(K_{m,n})$ using the Schur complement method will be given as follows:

$$\begin{gathered} \left|\lambda I_{m+n} - A(K_{m,n})\right| = 0 \\ \Leftrightarrow \left|\begin{bmatrix} \lambda I_{m\times m} & O_{m\times n} \\ O_{n\times m} & \lambda I_{n\times n} \end{bmatrix} - \begin{bmatrix} O_{m\times m} & J_{m\times n} \\ J_{n\times m} & O_{n\times n} \end{bmatrix}\right| = 0 \\ \Leftrightarrow \begin{vmatrix} \lambda I_{m\times m} & -J_{m\times n} \\ -J_{n\times m} & \lambda I_{n\times n} \end{vmatrix} = 0 \\ \Leftrightarrow |\lambda I_{m\times m}||\lambda I_{n\times n} - (-J_{n\times m})(\lambda I_{m\times m}^{-1})(-J_{m\times n})| = 0 \\ \Leftrightarrow \lambda^m \left(\frac{\lambda^2 - mn}{\lambda} \cdot \lambda^{n-1}\right) = 0 \\ \Leftrightarrow \lambda^{m+n-2}(\lambda^2 - mn) = 0. \end{gathered}$$

The same method is applied to derive the characteristic polynomial of the Laplacian matrix $L(K_{m,n})$. Using **Definitions 2** and **3**, the energy and Laplacian energy are computed from the eigenvalues of the respective matrices.

2.5 Degree-Based Multiplicative Topological Indices

Once the vertex degrees are established, the Narumi–Katayama index, the first multiplicative Zagreb index, and the second multiplicative Zagreb index are calculated based on **Definitions 4**, **5**, and **6**.

RESULTS AND DISCUSSION

Based on the work of Akgunes and Nacaroglu, which explores various properties of the zero-divisor graph over the ring $\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r$, the structure of $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r)$ exhibits several important characteristics, including its vertex set and the degrees of its vertices [20]. We provide a summary of the fundamental concepts as a basis for deriving the results.

Let $R = \mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r$, where p, q, and r are distinct prime numbers. Based on **Definition 1** we obtain the set of nonzero zero-divisors in R, can be partitioned into six disjoint subsets as follows:

1. $V_1 = \{(0, v_2, v_3) | v_2 \in \mathbb{Z}_q \backslash \{0\}, v_3 \in \mathbb{Z}_r \backslash \{0\}\}$,
2. $V_2 = \{(v_1, 0, v_3) | v_1 \in \mathbb{Z}_p \backslash \{0\}, v_3 \in \mathbb{Z}_r \backslash \{0\}\}$,
3. $V_3 = \{(v_1, v_2, 0) | v_1 \in \mathbb{Z}_p \backslash \{0\}, v_2 \in \mathbb{Z}_q \backslash \{0\}\}$,
4. $V_4 = \{(0, 0, v_3) | v_3 \in \mathbb{Z}_r \backslash \{0\}\}$,
5. $V_5 = \{(0, v_2, 0) | v_2 \in \mathbb{Z}_q \backslash \{0\}\}$,
6. $V_6 = \{(v_1, 0, 0) | v_1 \in \mathbb{Z}_p \backslash \{0\}\}$.

The cardinalities of these sets are given by:

- $|V_1| = (q-1)(r-1)$
- $|V_2| = (p-1)(r-1)$
- $|V_3| = (p-1)(q-1)$
- $|V_4| = r-1$
- $|V_5| = q-1$
- $|V_6| = p-1$

Next, we obtain the degree of the vertices of graph $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r)$ as stated in **Lemma 1**.

Lemma 1 The degrees of the vertices of graph $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r)$ are given by:

$$\deg(v) = \begin{cases} p-1, & v \in V_1, \\ q-1, & v \in V_2, \\ r-1, & v \in V_3, \\ pq-1, & v \in V_4, \\ pr-1, & v \in V_5, \\ qr-1, & v \in V_6. \end{cases}$$

Proof.

Let $u, v \in \Gamma(\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r)$ and define the degree of vertex v as the number of vertices u such that $u \cdot v = (0,0,0)$. We obtain the vertex degrees as follows:

1. For $v \in V_1$, the vertex v is only adjacent to all elements in V_6. Therefore, the degree of v is $\deg(v) = p-1$.
2. For $v \in V_2$, the vertex v is only adjacent to all elements in V_5. Hence, the degree of v is $\deg(v) = q-1$.
3. For $v \in V_3$, the vertex v is only adjacent to all elements in V_4. Thus, the degree of v is $\deg(v) = r-1$.
4. For $v \in V_4$, the vertex v is adjacent to all elements in V_3, V_5, and V_6. Therefore, the degree of v is $\deg(v) = (p-1)(q-1) + (q-1) + (p-1) = pq - p - q + 1 + q - 1 + p - 1 = pq - 1$.
5. For $v \in V_5$, the vertex v is adjacent to all elements in V_2, V_4, and V_6. Thus, the degree of v is $\deg(v) = (p-1)(r-1) + (r-1) + (p-1) = pr - p - r + 1 + r - 1 + p - 1 = pr - 1$.

   For $v \in V_6$, the vertex v is adjacent to all elements in V_1, V_4, and V_5. Hence, the degree of v is $\deg(v) = (q-1)(r-1) + (r-1) + (q-1) = qr - q - r + 1 + r - 1 + q - 1 = qr - 1$. ■

Example 1 If $p = 2, q = 3$, and $r = 5$, the set of zero-divisors in $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5$ as follows:
1. $V_1 = \{(0,1,1), (0,1,2), (0,1,3), (0,1,4), (0,2,1), (0,2,2), (0,2,3), (0,2,4)\}$,
2. $V_2 = \{(1,0,1), (1,0,2), (1,0,3), (1,0,4)\}$,
3. $V_3 = \{(1,1,0), (1,2,0)\}$,
4. $V_4 = \{(0,0,1), (0,0,2), (0,0,3), (0,0,4)\}$,
5. $V_5 = \{(0,1,0), (0,2,0)\}, \}$,
6. $V_6 = \{(1,0,0)\}$.

Figure 1 visualizes the structure of the zero-divisor graph induced by the ring $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5$, where each vertex represents a nonzero zero-divisor, and edges indicate products resulting in zero. The partitioning of vertices based on the zero component is clearly illustrated.

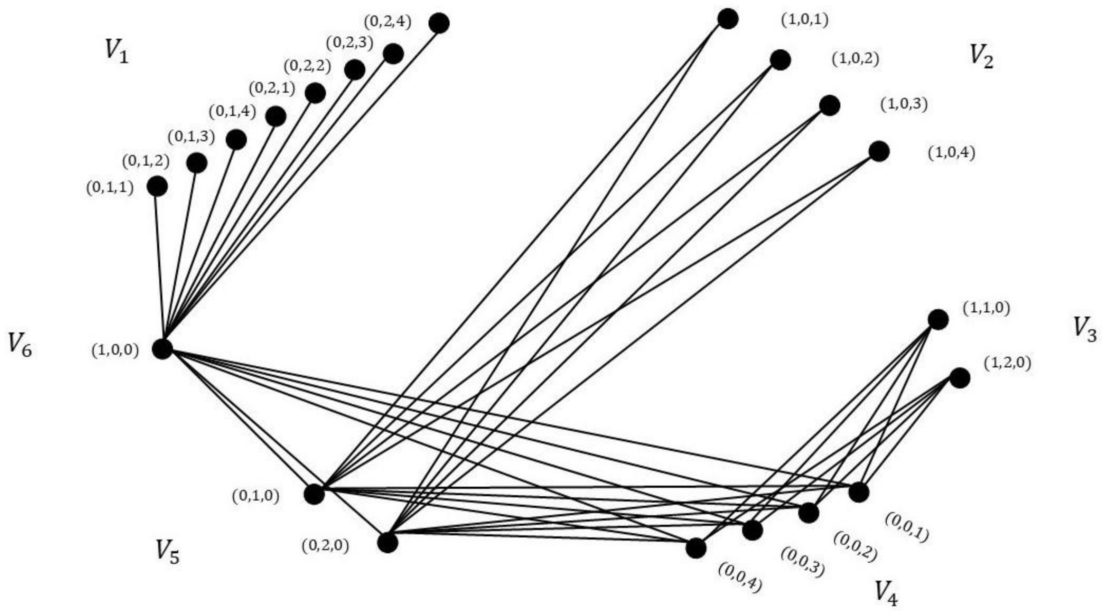

Figure 1. The zero-divisor graph $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5)$

To analyze the structure of the zero-divisor graph $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r)$, we focus on specific pairs of subsets of nonzero zero-divisors. When these subsets consist of elements with zero components in different positions, their union induces a subgraph with a bipartite structure, as described below.

Proposition 1 Let V_i and V_j be two disjoint subsets of nonzero zero-divisors in $\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r$, where each element has exactly one zero component, and the position of the zero component in elements of V_i is different from that in elements of V_j. Then, the induced subgraph $\Gamma(V_i \cup V_j)$ is isomorphic to the complete bipartite graph $K_{|V_i|,|V_j|}$.

Proof.
Let $u = (u_1, u_2, u_3) \in V_i$ and $v = (v_1, v_2, v_3) \in V_j$, where the zero components in u and v are located in different positions. As a result, their product is $u \cdot v = (0,0,0)$, implying that u and v are adjacent in the zero-divisor graph. This holds for every pair $(u, v) \in V_i \times V_j$, meaning that $\Gamma(V_i \cup V_j)$ forms a complete bipartite graph. On the other hand, all elements within the same subset have their zero component in the same position. Consequently,

the product of any two vertices from the same subset will have at least one nonzero entry and thus is not equal to zero. Therefore, no edges exist within V_i or V_j. Thus, the induced subgraph $\Gamma(V_i \cup V_j)$ is a complete bipartite graph $K_{|V_i|,|V_j|}$, where each vertex in V_i has degree $|V_j|$ and each vertex in V_j has degree $|V_i|$. ■

The **Table 1** describes the characteristics of the complete bipartite subgraph formed from $V_i \cup V_j$ which consists of the order and degree of each set.

Table 1. Degree distribution and structure of complete bipartite subgraphs $\Gamma(V_i \cup V_j)$

$V_i \cup V_j$	Order of $\lvert V_i\rvert$	Order of $\lvert V_j\rvert$	Degree of each $u \in V_i$	Degree of each $v \in V_j$	Complete bipartite graph
$V_1 \cup V_6$	$(q-1)(r-1)$	$p-1$	$p-1$	$(q-1)(r-1)$	$K_{(q-1)(r-1),(p-1)}$
$V_2 \cup V_5$	$(p-1)(r-1)$	$q-1$	$q-1$	$(p-1)(r-1)$	$K_{(p-1)(r-1),(q-1)}$
$V_3 \cup V_4$	$(p-1)(q-1)$	$r-1$	$r-1$	$(p-1)(q-1)$	$K_{(p-1)(q-1),(r-1)}$
$V_4 \cup V_5$	$r-1$	$q-1$	$q-1$	$r-1$	$K_{(r-1)(q-1)}$
$V_4 \cup V_6$	$r-1$	$p-1$	$p-1$	$r-1$	$K_{(r-1),(p-1)}$
$V_5 \cup V_6$	$q-1$	$p-1$	$p-1$	$q-1$	$K_{(q-1),(p-1)}$

The complete bipartite subgraphs that can be formed from the zero-divisor graph $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r)$ are $\Gamma(V_1 \cup V_6), \Gamma(V_2 \cup V_5), \Gamma(V_3 \cup V_4), \Gamma(V_4 \cup V_5), \Gamma(V_4 \cup V_6)$, and $\Gamma(V_5 \cup V_6)$. If $p = 2, q = 3$, and $r = 5$, the complete bipartite subgraph of $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5)$ is shown in **Figure 2**.

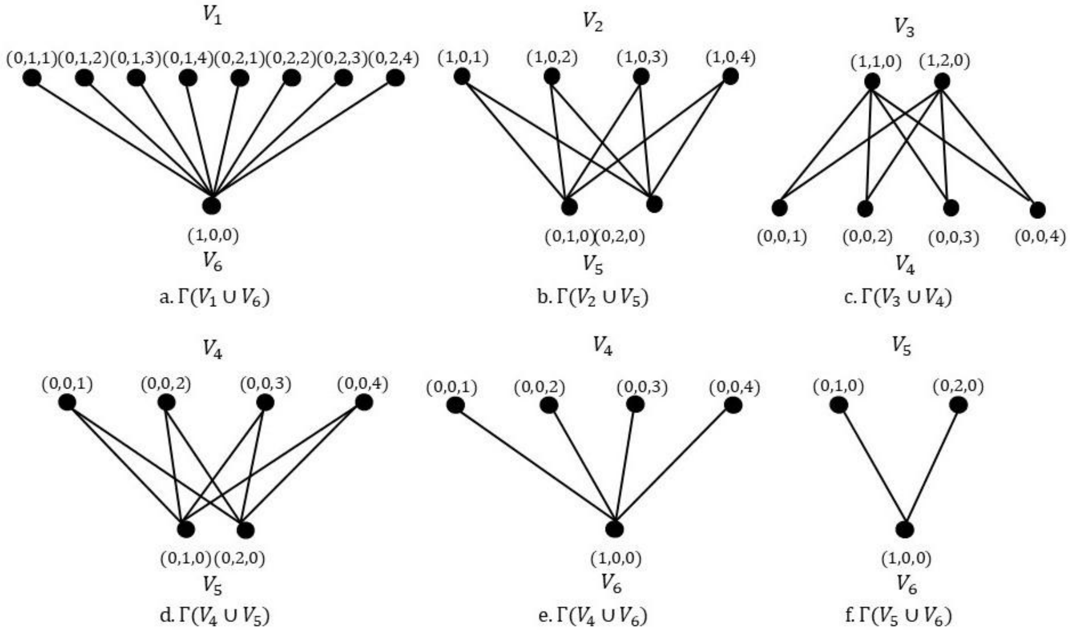

Figure 2. The complete bipartite subgraph of $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5)$

Figure 2 shows a complete bipartite subgraph formed by the union of two disjoint subsets of zero-divisors, each subset having its zero component in different positions. This configuration satisfies the criteria for inducing $K_{m,n}$ within the zero-divisor graph. Next, the energy, Laplacian energy, and degree-based multiplicative topological index on the complete bipartite subgraph of the zero-divisor graph $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r)$ are determined as follows:

Theorem 1 If $\Gamma(V_i \cup V_j)$ is a complete bipartite subgraph of the zero-divisor graph $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r)$, then the energy of $\Gamma(V_i \cup V_j)$ is given by:

$$E\left(\Gamma(V_i \cup V_j)\right) = 2\sqrt{|V_i||V_j|}.$$

Proof.
The adjacency matrix from subgraph $\Gamma(V_i \cup V_j)$, as follows:

$$A\left(\Gamma(V_i \cup V_j)\right) = \begin{bmatrix} O_{|V_i|\times|V_i|} & J_{|V_i|\times|V_j|} \\ J_{|V_j|\times|V_i|} & O_{|V_j|\times|V_j|} \end{bmatrix},$$

where O is a zero matrix and J is a matrix of all ones.
The characteristic polynomial of $A\left(\Gamma(V_i \cup V_j)\right)$ is

$$\left|\lambda I_{|V_i|+|V_j|} - A\left(\Gamma(V_i \cup V_j)\right)\right| = 0$$
$$\Leftrightarrow \lambda^{|V_i|+|V_j|-2}\left(\lambda^2 - |V_i||V_j|\right) = 0,$$

which gives the eigenvalues of $\sqrt{|V_i||V_j|}$ and $-\sqrt{|V_i||V_j|}$ with multiplicity 1, and 0 otherwise.
Therefore, according to **Definition 2** the energy of $\Gamma(V_i \cup V_j)$ is:

$$E\left(\Gamma(V_i \cup V_j)\right) = \sqrt{|V_i||V_j|} + \sqrt{|V_i||V_j|} = 2\sqrt{|V_i||V_j|}. \quad \blacksquare$$

We begin by evaluating the energy of the subgraph induced by the union $V_1 \cup V_6$. This case serves as a representative example for the spectral analysis of the induced bipartite subgraphs. The resulting eigenvalues are used to compute the energy explicitly, illustrating the procedure that applies similarly to other cases.

Example 2 The energy of the subgraph $\Gamma(V_1 \cup V_6)$ in the zero-divisor graph $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r)$ is given by:

$$E\left(\Gamma(V_1 \cup V_6)\right) = 2\sqrt{(p-1)(q-1)(r-1)}.$$

Proof.
The adjacency matrix from subgraph $\Gamma(V_1 \cup V_6)$, as follows:

$$A\left(\Gamma(V_1 \cup V_6)\right) = \begin{bmatrix} O_{(q-1)(r-1)\times(q-1)(r-1)} & J_{(q-1)(r-1)\times(p-1)} \\ J_{(p-1)\times(q-1)(r-1)} & O_{(p-1)\times(p-1)} \end{bmatrix}.$$

The characteristic polynomial of $A\left(\Gamma(V_1 \cup V_6)\right)$ is

$$\left|\lambda I_{(q-1)(r-1)+(p-1)} - A\left(\Gamma(V_1 \cup V_6)\right)\right| = 0$$
$$\Leftrightarrow \lambda^{(q-1)(r-1)+(p-1)-2}\left(\lambda^2 - (p-1)(q-1)(r-1)\right) = 0,$$

which gives the eigenvalues of $\sqrt{(p-1)(q-1)(r-1)}$ and $-\sqrt{(p-1)(q-1)(r-1)}$ with multiplicity 1, and 0 otherwise.

Therefore, the energy of the zero-divisor graph $\Gamma(V_1 \cup V_6)$ is:

$$\begin{aligned} E\big(\Gamma(V_1 \cup V_6)\big) &= \sqrt{(p-1)(q-1)(r-1)} + \sqrt{(p-1)(q-1)(r-1)} \\ &= 2\sqrt{(p-1)(q-1)(r-1)}. \\ &= 2\sqrt{pqr - pq - pr - qr + p + q + r - 1}. \end{aligned} \qquad \blacksquare$$

Similarly, the energy of the subgraphs $\Gamma(V_2 \cup V_5), \Gamma(V_3 \cup V_4), \Gamma(V_4 \cup V_5), \Gamma(V_4 \cup V_6)$, and $\Gamma(V_5 \cup V_6)$ can be determined using the same approach. By applying **Theorem 1** to each pair of disjoint subsets V_i and V_j, the energy of the corresponding complete bipartite subgraphs $\Gamma(V_i \cup V_j)$ is obtained. The results of these computations are summarized in **Table 2** below.

Table 2. The energy of complete bipartite subgraphs $\Gamma(V_i \cup V_j)$

$V_i \cup V_j$	$\lvert V_i\rvert$	$\lvert V_j\rvert$	$E\left(\Gamma(V_i \cup V_j)\right)$
$V_1 \cup V_6$	$(q-1)(r-1)$	$p-1$	$2\sqrt{pqr-pq-pr-qr+p+q+r-1}$
$V_2 \cup V_5$	$(p-1)(r-1)$	$q-1$	$2\sqrt{pqr-pq-pr-qr+p+q+r-1}$
$V_3 \cup V_4$	$(p-1)(q-1)$	$r-1$	$2\sqrt{pqr-pq-pr-qr+p+q+r-1}$
$V_4 \cup V_5$	$r-1$	$q-1$	$2\sqrt{qr-q-r+1}$
$V_4 \cup V_6$	$r-1$	$p-1$	$2\sqrt{pr-p-r+1}$
$V_5 \cup V_6$	$q-1$	$p-1$	$2\sqrt{pq-p-q+1}$

Theorem 2 If $\Gamma(V_i \cup V_j)$ is a complete bipartite subgraph of the zero-divisor graph $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r)$, then the Laplacian energy of $\Gamma(V_i \cup V_j)$ is given by:

$$E_L\left(\Gamma(V_i \cup V_j)\right) = 2|V_i||V_j|.$$

Proof.

The Laplacian matrix $L(\Gamma(V_i \cup V_j)) = D(\Gamma(V_i \cup V_j)) - A(\Gamma(V_i \cup V_j))$ is constructed based on the degree matrix $D(\Gamma(V_i \cup V_j))$ and adjacency matrix $A(\Gamma(V_i \cup V_j))$, as follows:

$$D\left(\Gamma(V_i \cup V_j)\right) = \begin{bmatrix} P_{|V_i|\times|V_i|} & O_{|V_i|\times|V_j|} \\ O_{|V_j|\times|V_i|} & S_{|V_j|\times|V_j|} \end{bmatrix},$$

$$A\left(\Gamma(V_i \cup V_j)\right) = \begin{bmatrix} O_{|V_i|\times|V_i|} & J_{|V_i|\times|V_j|} \\ J_{|V_j|\times|V_i|} & O_{|V_j|\times|V_j|} \end{bmatrix},$$

where P is a diagonal matrix with all entries $|V_j|$ and S is a diagonal matrix with all entries $|V_i|$.

Therefore, the Laplacian matrix is

$$\begin{aligned} L\left(\Gamma(V_i \cup V_j)\right) &= D\left(\Gamma(V_i \cup V_j)\right) - A\left(\Gamma(V_i \cup V_j)\right) \\ &= \begin{bmatrix} P_{|V_i|\times|V_i|} & Q_{|V_i|\times|V_j|} \\ R_{|V_j|\times|V_i|} & S_{|V_j|\times|V_j|} \end{bmatrix}, \end{aligned}$$

where Q and R are matrices whose entries all -1.

The characteristic polynomial of $L\left(\Gamma(V_i \cup V_j)\right)$ is

$$\left|\mu I_{|V_i|+|V_j|} - L\left(\Gamma(V_i \cup V_j)\right)\right| = 0$$

$$\Leftrightarrow (|V_j| - \mu)^{|V_i|}\left(\frac{(-\mu)[|V_i| - \mu]^{|V_j|-1}[|V_j| + |V_i| - \mu]}{|V_j| - \mu}\right) = 0$$
$$\Leftrightarrow \mu\left[\mu - [|V_i| + |V_j|]\right][\mu - |V_j|]^{|V_i|-1}[\mu - |V_i|]^{|V_j|-1} = 0,$$
which gives the eigenvalues of $0, |V_i| + |V_j|, |V_j|$ with multiplicity $|V_i| - 1$, and $|V_i|$ with multiplicity $|V_j| - 1$.
Thus, based on **Definition 3**, the Laplacian energy of $\Gamma(V_i \cup V_j)$ can be expressed as:

$$\begin{aligned} E_L\left(\Gamma(V_i \cup V_j)\right) &= |V_i| + |V_j| + |V_j|(|V_i| - 1) + |V_i|(|V_j| - 1) \\ &= 2|V_i||V_j|. \end{aligned}$$
■

To complement the Laplacian energy analysis, we also determine the Laplacian energy for the same subgraph $\Gamma(V_1 \cup V_6)$. The Laplacian matrix is constructed from the degree and adjacency matrices, and its spectrum is used to compute the Laplacian energy. This example reflects the general pattern observed in other bipartite subgraphs formed from disjoint zero-divisor subsets.

Example 3 The Laplacian energy of the subgraph $\Gamma(V_1 \cup V_6)$ in the zero-divisor graph $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r)$ is given by:
$$E_L(\Gamma(V_1 \cup V_6)) = 2(pqr - pq - pr - qr + p + q + r - 1).$$
Proof.
The Laplacian matrix $L(\Gamma(V_1 \cup V_6)) = D(\Gamma(V_1 \cup V_6)) - A(\Gamma(V_1 \cup V_6))$ is constructed based on the degree matrix $D(\Gamma(V_1 \cup V_6))$ and adjacency matrix $A(\Gamma(V_1 \cup V_6))$, as follows:

$$D(\Gamma(V_1 \cup V_6)) = \begin{bmatrix} P_{(q-1)(r-1)\times(q-1)(r-1)} & O_{(q-1)(r-1)\times(p-1)} \\ O_{(p-1)\times(q-1)(r-1)} & S_{(p-1)\times(p-1)} \end{bmatrix},$$

$$A(\Gamma(V_1 \cup V_6)) = \begin{bmatrix} O_{(q-1)(r-1)\times(q-1)(r-1)} & J_{(q-1)(r-1)\times(p-1)} \\ J_{(p-1)\times(q-1)(r-1)} & O_{(p-1)\times(p-1)} \end{bmatrix},$$

where P is a diagonal matrix with all entries $p - 1$ and S is a diagonal matrix with all entries $(q - 1)(r - 1)$.
Therefore, the Laplacian matrix is

$$\begin{aligned} L(\Gamma(V_1 \cup V_6)) &= D(\Gamma(V_1 \cup V_6)) - A(\Gamma(V_1 \cup V_6)) \\ &= \begin{bmatrix} P_{(q-1)(r-1)\times(q-1)(r-1)} & Q_{(q-1)(r-1)\times(p-1)} \\ R_{(p-1)\times(q-1)(r-1)} & S_{(p-1)\times(p-1)} \end{bmatrix}, \end{aligned}$$

where Q and R are matrices whose entries all -1.
The characteristic polynomial of $L(\Gamma(V_1 \cup V_6))$ is

$$|\mu I_{(q-1)(r-1)+(p-1)} - L(\Gamma(V_1 \cup V_6))| = 0$$
$$\Leftrightarrow \mu[\mu - [(p - 1) + (q - 1)(r - 1)]][\mu - (p - 1)]^{(q-1)(r-1)-1}[\mu - (q - 1)(r - 1)]^{p-2} = 0,$$

which gives the eigenvalues of $0, (p - 1) + (q - 1)(r - 1), (p - 1)$ with multiplicity $(q - 1)(r - 1) - 1$, and $(q - 1)(r - 1)$ with multiplicity $p - 2$.
Therefore, the Laplacian energy of subgraph $\Gamma(V_1 \cup V_6)$ is:

$$E_L(\Gamma(V_1 \cup V_6)) = (p-1) + (q-1)(r-1) + (p-1)[(q-1)(r-1) - 1] + (q-1)(r-1)(p-2)$$
$$= 2(pqr - pq - pr - qr + p + q + r - 1). \qquad \blacksquare$$

Similarly, the Laplacian energy of the subgraphs $\Gamma(V_2 \cup V_5), \Gamma(V_3 \cup V_4), \Gamma(V_4 \cup V_5), \Gamma(V_4 \cup V_6)$, and $\Gamma(V_5 \cup V_6)$ can be determined using the same approach. Applying **Theorem 2** to each subset V_i and V_j, we can compute the Laplacian energy for several pairs of subgraphs $\Gamma(V_i \cup V_j)$. The results of these computations are summarized in **Table 3** below.

Table 3. The Laplacian energy of complete bipartite subgraphs $\Gamma(V_i \cup V_j)$

$\boldsymbol{V_i \cup V_j}$	$\boldsymbol{\lvert V_i \rvert}$	$\boldsymbol{\lvert V_j \rvert}$	$\boldsymbol{E_L\left(\Gamma(V_i \cup V_j)\right)}$
$V_1 \cup V_6$	$(q-1)(r-1)$	$p-1$	$2(pqr - pq - pr - qr + p + q + r - 1)$
$V_2 \cup V_5$	$(p-1)(r-1)$	$q-1$	$2(pqr - pq - pr - qr + p + q + r - 1)$
$V_3 \cup V_4$	$(p-1)(q-1)$	$r-1$	$2(pqr - pq - pr - qr + p + q + r - 1)$
$V_4 \cup V_5$	$r-1$	$q-1$	$2(qr - q - r + 1)$
$V_4 \cup V_6$	$r-1$	$p-1$	$2(pr - p - r + 1)$
$V_5 \cup V_6$	$q-1$	$p-1$	$2(pq - p - q + 1)$

There are three degree-based multiplicative topological indices discussed in this paper, namely the Narumi-Katayama index, the first multiplicative Zagreb index, and the second multiplicative Zagreb index.

Theorem 3 Let $V_i \cup V_j \subseteq \mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r$ be such that $\Gamma(V_i \cup V_j)$ forms a complete bipartite subgraph of the zero-divisor graph $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r)$. Then the Narumi–Katayama index of $\Gamma(V_i \cup V_j)$ is given by:

$$NK\left(\Gamma(V_i \cup V_j)\right) = |V_i|^{|V_j|} \cdot |V_j|^{|V_i|}.$$

Proof.
From **Proposition 1**, it is known that if $v \in V_i, \deg(v) = |V_j|$ and if $v \in V_j, \deg(v) = |V_i|$. Consequently, according to **Definition 4**, the Narumi-Katayama index of $\Gamma(V_i \cup V_j)$ is given by:

$$NK\left(\Gamma(V_i \cup V_j)\right) = \prod_{v \in V(\Gamma(V_i \cup V_j))} \deg(v)$$
$$= \prod_{v \in V_i} \deg(v) \cdot \prod_{v \in V_j} \deg(v)$$
$$= |V_j|^{|V_i|} \cdot |V_i|^{|V_j|}.$$
$$= |V_i|^{|V_j|} \cdot |V_j|^{|V_i|}. \qquad \blacksquare$$

We begin the degree-based topological analysis by computing the Narumi–Katayama index for the subgraph induced by $V_1 \cup V_6$. This index is obtained as the product of the degrees of all vertices and serves as a measure of overall connectivity in the graph structure.

Example 4 The Narumi–Katayama index of the subgraph $\Gamma(V_1 \cup V_6)$ in the zero-divisor graph $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r)$ is given by:

$$NK\big(\Gamma(V_1 \cup V_6)\big) = ((q-1)(r-1))^{p-1} \cdot (p-1)^{(q-1)(r-1)}.$$

Proof.
From **Table 1**, it is known if $v \in V_1, \deg(v) = p-1$ and if $v \in V_6, \deg(v) = (q-1)(r-1)$. Thus, the Narumi-Katayama index is obtained as follows:

$$\begin{aligned} NK\big(\Gamma(V_1 \cup V_6)\big) &= \prod_{v \in V(\Gamma(V_1 \cup V_6))} \deg(v) \\ &= \prod_{v \in V_1} \deg(v) \cdot \prod_{v \in V_6} \deg(v) \\ &= (p-1)^{(q-1)(r-1)} \cdot ((q-1)(r-1))^{p-1} \\ &= ((q-1)(r-1))^{p-1} \cdot (p-1)^{(q-1)(r-1)}. \quad \blacksquare \end{aligned}$$

Similarly, the Narumi-Katayama index of the subgraphs $\Gamma(V_2 \cup V_5), \Gamma(V_3 \cup V_4), \Gamma(V_4 \cup V_5), \Gamma(V_4 \cup V_6)$, and $\Gamma(V_5 \cup V_6)$ can be determined using the same approach. Applying **Theorem 3** to each subset V_i and V_j, we can compute the Narumi–Katayama index for several pairs of subgraphs $\Gamma(V_i \cup V_j)$. The results of these computations are summarized in **Table 4** below.

Table 4. The Narumi-Katayama index of complete bipartite subgraphs $\Gamma(V_i \cup V_j)$

$V_i \cup V_j$	$\lvert V_i \rvert$	$\lvert V_j \rvert$	$NK\left(\Gamma(V_i \cup V_j)\right)$
$V_1 \cup V_6$	$(q-1)(r-1)$	$p-1$	$((q-1)(r-1))^{p-1} \cdot (p-1)^{(q-1)(r-1)}$
$V_2 \cup V_5$	$(p-1)(r-1)$	$q-1$	$((p-1)(r-1))^{q-1} \cdot (q-1)^{(p-1)(r-1)}$
$V_3 \cup V_4$	$(p-1)(q-1)$	$r-1$	$((p-1)(q-1))^{r-1} \cdot (r-1)^{(p-1)(q-1)}$
$V_4 \cup V_5$	$r-1$	$q-1$	$(r-1)^{(q-1)} \cdot (q-1)^{(r-1)}$
$V_4 \cup V_6$	$r-1$	$p-1$	$(r-1)^{(p-1)} \cdot (p-1)^{(r-1)}$
$V_5 \cup V_6$	$q-1$	$p-1$	$(q-1)^{(p-1)} \cdot (p-1)^{(q-1)}$

Theorem 4 Let $V_i \cup V_j \subseteq \mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r$ be such that $\Gamma(V_i \cup V_j)$ forms a complete bipartite subgraph of the zero-divisor graph $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r)$. Then the first multiplicative Zagreb index of $\Gamma(V_i \cup V_j)$ is given by:

$$\Pi_1\left(\Gamma(V_i \cup V_j)\right) = |V_i|^{2|V_j|} \cdot |V_j|^{2|V_i|} = \left(|V_i|^{|V_j|} \cdot |V_j|^{|V_i|}\right)^2 = NK\left(\Gamma(V_i \cup V_j)\right)^2.$$

Proof.
According to **Proposition 1**, the degree of each vertex $v \in V_i, \deg(v) = |V_j|$, while the degree of each vertex $v \in V_j, \deg(v) = |V_i|$. Thus, by **Definition 5**, the first multiplicative Zagreb index of $\Gamma(V_i \cup V_j)$ is given by:

$$\begin{aligned} \Pi_1\left(\Gamma(V_i \cup V_j)\right) &= \prod_{v \in V(\Gamma(V_i \cup V_j))} (\deg(v))^2 \\ &= \prod_{v \in V_i} (\deg(v))^2 \cdot \prod_{v \in V_j} (\deg(v))^2 \\ &= \left(|V_j|\right)^{2|V_i|} \cdot |V_i|^{2|V_j|} \end{aligned}$$

$$= \left(|V_i|^{|V_j|} \cdot |V_j|^{|V_i|}\right)^2$$
$$= NK\left(\Gamma(V_i \cup V_j)\right)^2. \qquad \blacksquare$$

Next, we calculate the first multiplicative Zagreb index, which involves squaring the degrees of vertices and taking their product. For the subgraph $\Gamma(V_1 \cup V_6)$, this index provides further insight into how vertex degrees are distributed across both partitions.

Example 5 The first multiplicative Zagreb index of the subgraph $\Gamma(V_1 \cup V_6)$ in the zero-divisor graph $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r)$ is given by:

$$\Pi_1\big(\Gamma(V_1 \cup V_6)\big) = ((q-1)(r-1))^{2(p-1)} \cdot (p-1)^{2((q-1)(r-1))}.$$

Proof.
From **Table 1**, it can be seen that the degree of each vertex $v \in V_1, \deg(v) = p - 1$, while the degree of each vertex $v \in V_6, \deg(v) = (q-1)(r-1)$. Thus, the first multiplicative Zagreb index is obtained as follows:

$$\Pi_1\big(\Gamma(V_1 \cup V_6)\big) = \prod_{v \in V(\Gamma(V_1 \cup V_6))} (\deg(v))^2$$
$$= \prod_{v \in V_1} (\deg(v))^2 \cdot \prod_{v \in V_6} (\deg(v))^2$$
$$= (p-1)^{2((q-1)(r-1))} \cdot ((q-1)(r-1))^{2(p-1)}$$
$$= ((q-1)(r-1))^{2(p-1)} \cdot (p-1)^{2((q-1)(r-1))}. \qquad \blacksquare$$

Similarly, the first multiplicative Zagreb index of the subgraphs $\Gamma(V_2 \cup V_5), \Gamma(V_3 \cup V_4), \Gamma(V_4 \cup V_5), \Gamma(V_4 \cup V_6)$, and $\Gamma(V_5 \cup V_6)$ can be determined using the same approach. Applying **Theorem 4** to each subset V_i and V_j, we can compute the first multiplicative Zagreb index for several pairs of subgraphs $\Gamma(V_i \cup V_j)$. The results of these computations are summarized in **Table 5** below.

Table 5. The first multiplicative Zagreb index of complete bipartite subgraphs $\Gamma(V_i \cup V_j)$

$V_i \cup V_j$	$\lvert V_i\rvert$	$\lvert V_j\rvert$	$\Pi_1\left(\Gamma(V_i \cup V_j)\right)$
$V_1 \cup V_6$	$(q-1)(r-1)$	$p-1$	$((q-1)(r-1))^{2(p-1)} \cdot (p-1)^{2((q-1)(r-1))}$
$V_2 \cup V_5$	$(p-1)(r-1)$	$q-1$	$((p-1)(r-1))^{2(q-1)} \cdot (q-1)^{2((p-1)(r-1))}$
$V_3 \cup V_4$	$(p-1)(q-1)$	$r-1$	$((p-1)(q-1))^{2(r-1)} \cdot (r-1)^{2((p-1)(q-1))}$
$V_4 \cup V_5$	$r-1$	$q-1$	$(r-1)^{2(q-1)} \cdot (q-1)^{2(r-1)}$
$V_4 \cup V_6$	$r-1$	$p-1$	$(r-1)^{2(p-1)} \cdot (p-1)^{2(r-1)}$
$V_5 \cup V_6$	$q-1$	$p-1$	$(q-1)^{2(p-1)} \cdot (p-1)^{2(q-1)}$

Theorem 5 Let $V_i \cup V_j \subseteq \mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r$ be such that $\Gamma(V_i \cup V_j)$ forms a complete bipartite subgraph of the zero-divisor graph $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r)$. Then the second multiplicative Zagreb index of $\Gamma(V_i \cup V_j)$ is given by:

$$\Pi_2\left(\Gamma(V_i \cup V_j)\right) = \left(|V_i||V_j|\right)^{|V_i||V_j|}.$$

Proof.
For any $uv \in E(\Gamma(V_i \cup V_j))$, we have $\deg(u) = |V_j|$ and $\deg(v) = |V_i|$. Therefore, according to **Definition 6**, the second multiplicative Zagreb index of $\Gamma(V_i \cup V_j)$ is:

$$\Pi_2\left(\Gamma(V_i \cup V_j)\right) = \prod_{uv\in E(\Gamma(V_i\cup V_j))} \deg(u)\cdot\deg(v)$$
$$= |V_j||V_i|^{|V_i||V_j|}$$
$$= \left[|V_i||V_j|\right]^{|V_i||V_j|}. \qquad \blacksquare$$

Finally, the second multiplicative Zagreb index is determined by multiplying the degrees at both ends of each edge and taking the product over all edges. In the case of $V_1 \cup V_6$, this index captures how interactions between vertex pairs contribute to the graph's topological complexity.

Example 6 The second multiplicative Zagreb index of the subgraph $\Gamma(V_1 \cup V_6)$ in the zero-divisor graph $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r)$ is given by:

$$\Pi_2\left(\Gamma(V_1 \cup V_6)\right) = [(p-1)(q-1)(r-1)]^{(p-1)(q-1)(r-1)}.$$

Proof.
Given that $uv \in V(\Gamma(V_1 \cup V_6))$, with $\deg(u) = p-1$ and $\deg(v) = (q-1)(r-1)$, the second multiplicative Zagreb index can be expressed as follows:

$$\Pi_2\left(\Gamma(V_1 \cup V_6)\right) = \prod_{uv\in V(\Gamma(V_1\cup V_6))} \deg(u)\cdot\deg(v)$$
$$= [(p-1)(q-1)(r-1)]^{(q-1)(r-1)(p-1)}$$
$$= [(p-1)(q-1)(r-1)]^{(p-1)(q-1)(r-1)}. \qquad \blacksquare$$

Similarly, the second multiplicative Zagreb index of the subgraphs $\Gamma(V_2 \cup V_5), \Gamma(V_3 \cup V_4), \Gamma(V_4 \cup V_5), \Gamma(V_4 \cup V_6)$, and $\Gamma(V_5 \cup V_6)$ can be determined using the same approach. Applying **Theorem 5** to each subset V_i and V_j, we can compute the second multiplicative Zagreb index for several pairs of subgraphs $\Gamma(V_i \cup V_j)$. The results of these computations are summarized in **Table 6** below.

Table 6. The second multiplicative Zagreb index of complete bipartite subgraphs $\Gamma(V_i \cup V_j)$

$V_i \cup V_j$	$\lvert V_i\rvert$	$\lvert V_j\rvert$	$\Pi_2\left(\Gamma(V_i \cup V_j)\right)$
$V_1 \cup V_6$	$(q-1)(r-1)$	$p-1$	$[(p-1)(q-1)(r-1)]^{(p-1)(q-1)(r-1)}$
$V_2 \cup V_5$	$(p-1)(r-1)$	$q-1$	$[(p-1)(q-1)(r-1)]^{(p-1)(q-1)(r-1)}$
$V_3 \cup V_4$	$(p-1)(q-1)$	$r-1$	$[(p-1)(q-1)(r-1)]^{(p-1)(q-1)(r-1)}$
$V_4 \cup V_5$	$r-1$	$q-1$	$[(q-1)(r-1)]^{(q-1)(r-1)}$
$V_4 \cup V_6$	$r-1$	$p-1$	$[(p-1)(r-1)]^{(p-1)(r-1)}$
$V_5 \cup V_6$	$q-1$	$p-1$	$[(q-1)(r-1)]^{(q-1)(r-1)}$

CONCLUSIONS

This paper investigates the structural and topological properties of the zero-divisor graph $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r)$, where p, q, and r are distinct prime numbers. The graph is analysed based on the structure of its nonzero zero-divisors, with particular emphasis on induced complete bipartite subgraphs $\Gamma(V_i \cup V_j)$, formed by combining disjoint subsets whose zero components occupy different positions. For each such subgraph, the energy, Laplacian energy, and three degree-based multiplicative topological indices are computed: the Narumi–Katayama index, and the first and second multiplicative Zagreb indices. The results reveal consistent patterns in the degree distributions and energy expressions, which depend on the cardinalities of the corresponding sets V_i and V_j.

This study is limited to commutative rings of the form $\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r$, where p, q, and r are distinct prime numbers, and considers only induced complete bipartite subgraphs. A full analysis of the entire zero-divisor graph and other potential subgraph types is beyond the scope of this work. Future research directions may include extending the framework to rings of the form $\mathbb{Z}_{p^a} \times \mathbb{Z}_{q^b} \times \mathbb{Z}_{r^c}$, where $a, b, c \in \mathbb{Z}^+$, to examine the influence of higher powers of primes on the resulting graph structures. Other possibilities involve generalisations to noncommutative rings or semirings, and investigating similar spectral and topological characteristics in alternative algebraic graphs, such as unit graphs and total graphs. Moreover, the application of the topological indices in fields such as algebraic coding theory and cryptography could be a practical significance, particularly since bipartite structures can serve as natural models for secure data transmission. In summary, the findings of this study provide deeper insights into how the algebraic properties of commutative rings influence the topological and spectral features of their associated graphs.

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