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Pricing Modified Barrier Options Using the Bino-Trinomial Tree Model: A Strategy for Loss Minimization

Rima Aulia Rahayu[1*], Fitriani Agustina[2], and Kuntjoro Adji Sidarto[3]

[1]*Mathematics Education Study Program, Faculty of Mathematics and Science Education, Universitas Pendidikan Indonesia, Bandung, Indonesia*
[2]*Mathematics Study Program, Faculty of Mathematics and Science Education, Universitas Pendidikan Indonesia, Bandung, Indonesia*
[3]*Department of Mathematics, Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Indonesia*

Abstract

Barrier options represent a particular type of exotic option that has gained popularity in global financial markets. These options are characterized by their ability to mitigate risk through an activation mechanism contingent upon specific price limits. However, determining the price of options with complex structures requires an efficient and accurate numerical approach. The objective of this study is to develop and analyze a bino-trinomial tree model to determine the price of modified barrier options. These options serve as an alternative hedging strategy to minimize potential investor losses. The model is constructed by combining a trinomial tree scheme in the initial step to enhance the flexibility of asset price movements, followed by a binomial tree in subsequent steps to simplify computations. The novelty of this research lies in the application of the model to European down-and-out call Options with multi-step moving barriers and European up-and-out call window barrier options, which have not been extensively discussed in previous studies. The present study is grounded in the principles of numerical accuracy, computational efficiency, and practical relevance in the context of financial decision-making. The findings suggest that the bino-trinomial tree model possesses the capability to produce stable and adaptive price estimates for complex barrier options, thereby demonstrating its potential to serve as an effective alternative approach in pricing exotic options.

Keywords: barrier options; bino-trinomial tree; exotic derivatives; lattice method; option pricing.

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1 Introduction

Since the early twentieth century, mathematical models in modern finance have undergone rapid development, marked by the integration of probability and optimization in derivative pricing. This evolution has been driven by the need for realistic models that can adapt to the dynamic fluctuations of market dynamics.

Derivatives are financial instruments whose value is dependent on the value of the underlying asset [1]. A derivative instrument that is frequently utilized in risk management is options [2]. In Indonesia, options trading was first introduced in 2024, following its global inception through the Chicago Board Options Exchange (CBOE) in 1973. In the context of the capital market, options are classified into

*Corresponding author. E-mail: rimaauliarr@gmail.com

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two categories: vanilla options and exotic options [3]. Vanilla options are options whose payoff value depends on the stock price at the time of exercise. In contrast, exotic options are options whose payoff value depends not only on the stock price at the time of exercise but also on stock prices during the option's lifespan.

A particular type of exotic option is the barrier option [4]. This option is most popular because it offers lower prices than vanilla options [5] and has an activation mechanism based on certain price limits that serve as risk and return controls for investors and option issuers [6].

The pricing of barrier options represents a critical aspect of derivative modeling, particularly in cases where these options exhibit intricate characteristics. There are two general approaches to option pricing: the analytical method and the numerical method [7]. The analytical approach is frequently selected for its ability to produce explicit solutions. One of the most well-known analytical models is the Black-Scholes-Merton (BSM) model, which serves as the foundation for the development of barrier option pricing theory. The BSM model is capable of providing closed-form solutions for vanilla options and certain simple barrier options, under the assumption of an ideal market and constant volatility [8]. However, the efficacy of this model is significantly reduced when applied to options with additional features, such as dynamic barriers, partial barriers, or path-dependent barriers, as the mathematical formulation becomes too complex or even unattainable in closed form [9].

In contrast, numerical methods have been demonstrated to exhibit superiority due to their ability to manage modified option structures with high flexibility, despite their elevated computational costs [10]. Various numerical methods are frequently used in the context of option pricing, including the Monte Carlo simulation method [11], the finite difference method [12], and the lattice method [13].

In the context of modified barrier options, a numerical approach constitutes an appropriate alternative, as it is capable of capturing the complexity of market characteristics with greater realism. Among the various numerical methods, this study decided to utilize the lattice approach because it is regarded as more efficient and flexible. The finite difference method, while precise in certain circumstances, necessitates an explicit form of the option differential equation, which is not universally available, particularly when the option incorporates features such as dynamic barriers. In contrast, the Monte Carlo method requires a substantial number of simulations to achieve convergent estimates, resulting in higher computational costs [13].

The binomial tree model, which was introduced by Cox, Ross, and Rubinstein (CRR) [14], is a frequently employed lattice method. This is primarily due to the fact that it is both straightforward and simple to implement. Despite its application in the pricing of barrier options, this model exhibits suboptimal efficiency in managing the intricacies inherent in such options. The primary deficiency of the CRR model stems from distribution error, which stems from the approximation of the lognormal distribution with a discrete distribution, and nonlinear error, which emanates from the nonlinearity of option values, particularly in barrier options, leading to substantial fluctuations in value over an extensive number of steps [15].

In order to overcome the limitations of the binomial model, the Kamrad–Ritchken (K-R) trinomial method has been introduced for barrier option pricing [16]. Asnawi [17] applied the K-R trinomial model in the context of barrier options, producing results that were more accurate than those produced by the binomial method. However, the application of this model to complex options or those with large step sizes can result in a significant increase in complexity and computational expense.

Binomial and trinomial models have limitations in handling barrier options with dynamic structures. To address this challenge, Dai and Lyuu [18] proposed the bino-trinomial tree model (BTT), which integrates the strengths of both approaches. Specifically, the BTT model leverages the flexibility of price movements characteristic of the trinomial model in the early stages, while leveraging the computational efficiency of the binomial model in the later stages. This approach has been demonstrated to enhance accuracy without substantial augmentation of the computational load.

A relevant study by Agustina [19] modified the BTT model for European call options by incorporating volatility uncertainty using triangular fuzzy numbers. This modification resulted in a price range that reflects market fluctuations and risk preferences. However, the study's primary focus was on parameter uncertainty rather than on the design of options.

In contrast to the aforementioned approach, this study builds on Dai and Lyuu's BTT framework by expanding its application to more complex barrier options, specifically the shifting barrier type, also known as moving barrier options [20]. Moving barrier options enable investors to manage risk and adjust their hedging strategies to market changes gradually, without the need to establish a single, overly restrictive barrier level. In order to enhance the relevance of the model to the dynamic nature of capital markets, the researchers will also propose the utilization of this model to address multi-step single moving barrier cases. This approach is undertaken to furnish a substitute solution for the management of risk exposure that is more optimal for companies and investors who necessitate more flexible and effective hedging instruments in complex and dynamic markets [21].

However, researchers also recognize that companies or investors sometimes face challenges in determining the price of barrier options where the barrier does not apply throughout the option's lifespan but only during specific periods deemed most relevant. In such situations, partial barrier options, also known as window barrier options [22], offer an equally attractive solution. These options allow investors to project according to their specific needs to obtain more appropriate protection and returns. This option type is a highly relevant, measurable, and efficient hedging tool in dealing with market dynamics.

To the best of the author's knowledge, there has been no research on the use of the binomial-trinomial tree model to determine the price of partial barrier options or multi-step single moving barrier options, which is also the basis for conducting this research.

2 Methods

This study implements the bino-trinomial tree (BTT) method to valuation exotic barrier options. The following discussion will analyze two types of options: European call options with multi-step moving down-and-out barriers and European call options with window up-and-out barriers. Subsection 2.1 describes the construction of the model, Subsection 2.2 describes the barrier option framework, followed by the forward stock price construction in Subsection 2.3 and the determination of option value through backward induction in Subsection 2.4.

2.1 Model Construction

2.1.1 Multi-Step Moving Barrier

Referring to Figure 1, it can be posited that three monitoring time periods may be considered, namely $[0,T_1], (T_1,T_1+T_2]$, and $(T_1+T_2,T_1+T_2+T_3]$. For each of these times, there exist lower barriers designated L_1,L_2, and L_3, respectively.

The branching BTT that originates from node S is constructed through the integration of multiple basic bino-trinomial trees (bBTT) that emanate from node $S,X,D,E,F,Z,L,M,N,O,P,$ and Q. bBTT is a combination of the CRR binomial tree model and the K-R trinomial tree model. In this model, the initial step is a trinomial tree, and the subsequent steps are binomial trees. The trinomial step at the beginning is used to capture asset price dynamics more flexibly (up, down, or constant) and is especially important if there are features such as barriers. The subsequent steps utilize the binomial method to simplify the calculations. If the sequence is inverted, the initial flexibility is lost and the complexity increases. The initial segment of the binomial tree, emanating from node A,B,C when Δt_1. The cell width is equivalent to the time-step length of the tree Δt_1 and the cell height of the grid is $\sigma\sqrt{\Delta t_1}$. Notation σ indicates the volatility of asset prices, and the height of the grid in the binomial tree reflects the magnitude of price deviations that may occur in a single time step. This is used to ascertain the structure of the tree and the accuracy of option price calculations. The second part of the binomial tree is initiated from node Y,G,H,I,J,K when $T_1+\Delta t_2$. The cell width is equivalent to the time-step length of the tree Δt_2 and the cell height of the grid is $\sigma\sqrt{\Delta t_2}$). The third binomial tree section begins at node Z,L,M,N,O,P,Q when T_1+T_2. The cell width is equivalent to the time-step length of the tree Δt_3 and the cell height of the grid is $\sigma\sqrt{\Delta t_3}$).

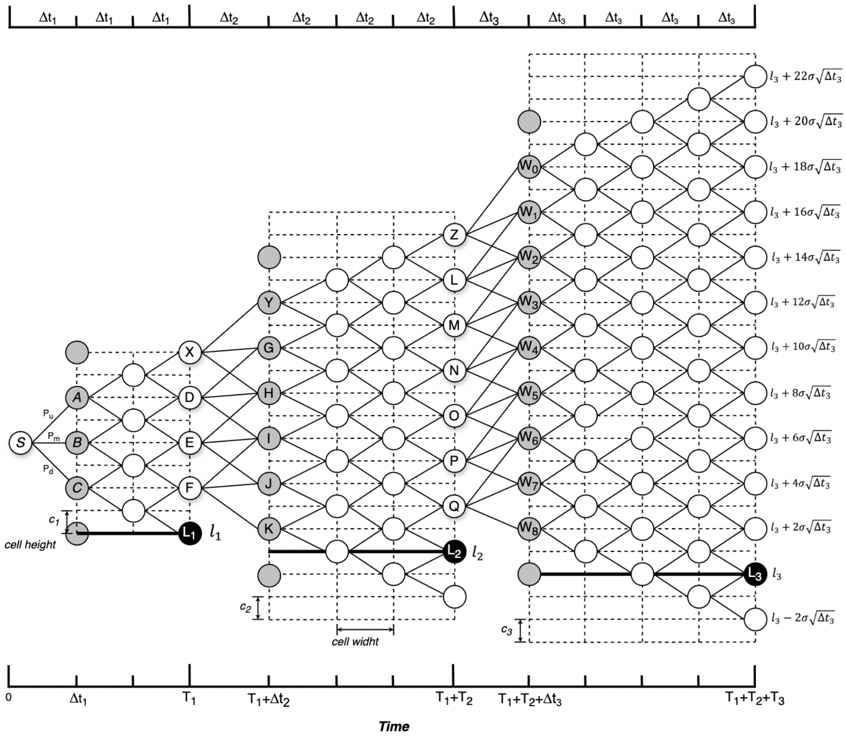


Figure 1: Structure of the Bino-Trinomial Tree for Pricing European Down-and-Out Call Options with Multi-Step Moving Barriers

To ensure the efficacy and accuracy of option prices obtained, it is imperative that the BTT satisfy the following two properties: firstly, it must have nodes that coincide with each discrete barrier, and secondly, it must possess valid trinomial tree branching probabilities. This approach is adopted to ensure the model's capacity to manage non-linearity errors. In the context of single-barrier systems, the alignment of nodes to coincide with each barrier is typically achieved by defining the time step for monitoring as $\Delta t_x \equiv \frac{T_x}{N_x}$ for x depending on the monitoring time and N an integer. To fulfill the two properties mentioned earlier and ascertain the option value, the following steps are presented.

In the first step, the focus must be placed on the part of the bBTT at $[0, T_1]$, which means that the bBTT initiates from node S. To construct the bBTT for the single barrier option, it is necessary to select candidate nodes A, B, and C from other nodes that guarantee a valid probability of p_u, p_m, and p_d $(0 \leq p_u, p_m, p_d \leq 1)$. The selection of candidate nodes is determined by defining the stock price for node X as S_X and the V-log-price of stock price V' as $\ln\left(\frac{V'}{V}\right)$. Therefore, it can be concluded that the V-log-price of z is equivalent to the stock price of Ve^z. At the initial monitoring instance, the S_S-log-price of the barrier L_1 is designated as $l_1 = \ln\left(\frac{L_1}{S_S}\right)$. It is imperative to consider that the value of S_S-log-price at each gray node in the trinomial tree step Δt_x is

$$\begin{cases} l_x + 2j\sigma\sqrt{\Delta t_x} & \text{; if the number of steps of CRR is even} \\ l_x + (2j+1)\sigma\sqrt{\Delta t_x} & \text{; else} \end{cases} \tag{1}$$

for an integer j.

The increase and decrease factors in the CRR binomial are represented by $\sigma\sqrt{\Delta t_x}$ and $-\sigma\sqrt{\Delta t_x}$, respectively. Subsequently, the mean and variance functions are defined as follows.

$$\begin{aligned} \mu(x) &\equiv \left(r - \frac{\sigma^2}{2}\right)x \\ \mathrm{var}(x) &\equiv \sigma^2 x \end{aligned} \tag{2}$$

The mean and variance functions of the S_S-log-price at nodes A, B, and C are represented by $\mu(\Delta t_1)$ and $\mathrm{var}(\Delta t_1)$, respectively. The parameter r denotes the risk-free interest rate. The difference between adjacent S_S-log-price values (e.g., nodes A and B) is $2\sigma\sqrt{\Delta t_x}$. Therefore, it can be concluded with a degree of certainty that there is a unique vertex located within the interval $\left[\mu(\Delta t_x) - \sigma\sqrt{\Delta t_x}, \mu(\Delta t_x) + \sigma\sqrt{\Delta t_x}\right]$, select node B as the vertex. The S_S-log-price at B is denoted by $\hat{\mu}$, which is the S_S-log-price that is closest to $\mu(\Delta t_x)$. Subsequently, the S_S-log-price at A and C are $\hat{\mu} + 2\sigma\sqrt{\Delta t_1}$ and $\hat{\mu} - 2\sigma\sqrt{\Delta t_1}$, respectively. The subsequent step is to define.

$$\begin{aligned} \beta &\equiv \mu - \mu(\Delta t_x) \\ \alpha &\equiv \mu + 2\sigma\sqrt{\Delta t_x} - \mu(\Delta t_x) = \beta + 2\sigma\sqrt{\Delta t_x} \\ \gamma &\equiv \mu - 2\sigma\sqrt{\Delta t_x} - \mu(\Delta t_x) = \beta - 2\sigma\sqrt{\Delta t_x} \end{aligned} \tag{3}$$

The equation resulting from the first equation is $\beta \in \left[-\sigma\sqrt{\Delta t_x}, \sigma\sqrt{\Delta t_x}\right]$, with $\alpha > \beta > \gamma$. The calculation of the branching probability is obtained through the solution of the following system of equations.

$$\begin{bmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} p_u \\ p_m \\ p_d \end{bmatrix} = \begin{bmatrix} \mathrm{var}(\Delta t_x) \\ 0 \\ 1 \end{bmatrix} \tag{4}$$

The Eq. 4 can be solved through the application of Cramer's rule. The resulting equation is as follows.

$$\begin{aligned} det &= (\beta - \alpha)(\gamma - \beta)(\gamma - \alpha) < 0 \\ det_m &= (\alpha\gamma + \mathrm{var}(\Delta t_x))(\alpha - \gamma) \\ det_d &= (\alpha\beta + \mathrm{var}(\Delta t_x))(\beta - \alpha) \\ det_u &= (\beta\gamma + \mathrm{var}(\Delta t_x))(\gamma - \beta) \end{aligned}$$

Therefore, the branching probability values are given by the following equations: $p_u = \frac{det_u}{det}, p_m = \frac{det_m}{det}$, and $p_d = \frac{det_d}{det}$. Subsequently, the validity of the branching probability results will be demonstrated. It can be proven that $p_u, p_m, p_d \geq 0$. Since $det < 0$, it is sufficient to demonstrate that $det_u, det_m, det_d \leq 0$. Given that $\alpha > \beta > \gamma$, it is sufficient to demonstrate that $\beta\gamma + \mathrm{var}(\Delta t_x) \geq 0$, $\alpha\gamma + \mathrm{var}(\Delta t_x) \leq 0$, and $\alpha\beta + \mathrm{var}(\Delta t_x) \geq 0$. See.

$$\begin{aligned} \beta\gamma + \mathrm{var}(\Delta t_1) &= \beta^2 - 2\beta\sigma\sqrt{\Delta t_1} + \sigma^2\Delta t_1 = \left(\beta - \sigma\sqrt{\Delta t_1}\right)^2 \geq 0 \\ \gamma + \mathrm{var}(\Delta t_1) &= \beta^2 - 4\sigma^2\Delta t_1 + 2\sigma^2\Delta t_1 = \beta^2 - 2\sigma^2\Delta t_1 \leq 0 \\ \alpha\beta + \mathrm{var}(\Delta t_1) &= \beta^2 + 2\beta\sigma\sqrt{\Delta t_1} + \sigma^2\Delta t_1 = \left(\beta + \sigma\sqrt{\Delta t_1}\right)^2 \geq 0 \end{aligned}$$

It has been determined that the branching opportunities are valid.

2.1.2 Window Barrier

In constructing a bBTT model with an up-and-out window barrier , similar steps can be used as when constructing a bBTT model with a down-and-out barrier. This refers to research conducted by Dai and Lyuu [18]. However, the difference lies in the fact that for the down-and-out barrier option, the knock-out condition occurs if the asset price falls below the lower boundary throughout the entire period until maturity, so each node must be checked against the lower log-barrier value $l = ln\left(\frac{L}{S_S}\right)$. Conversely, for the up-and-out window barrier option, the knock-out condition only applies within a specific time interval and occurs if the asset price rises above the upper limit H, with the log-barrier $l = ln\left(\frac{H}{S_S}\right)$.

2.2 Barrier Option Framework

Barrier options are a class of path-dependent options whose payoff depends on the price movement of the underlying asset when it attains a predetermined barrier price level during the life of the option. In this study, a European call option is selected based on exercise time. This option entails the right to purchase shares at the strike price within an agreed period, and it can only be exercised at maturity. Concurrently, the employed option is the barrier knock-out option. There are two payoff conditions at maturity time T , strike price K, final asset price S_T (at maturity), initial asset price S_0 (at time zero), lower barrier L, and upper barrier H, given by [23]:

a. Down-and-Out

$$payoff = \begin{cases} \max(S_T - K, 0) & ; \text{if } S_0 > L \\ 0 & ; \text{else.} \end{cases} \tag{5}$$

b. Up-and-Out

$$payoff = \begin{cases} \max(S_T - K, 0) & ; \text{if } S_0 < H \\ 0 & ; \text{else.} \end{cases} \tag{6}$$

2.3 Forward Construction of Stock Prices

2.3.1 Multi-Step Moving Barrier

In determining the stock price $S_{j,i}$ employing a forward phase. At the trinomial step t_1, the stock prices at nodes A, B, and C are $a = bu^2$, $b = S_S e^{\hat{\mu}}$, and $c = bd^2$, respectively. In the following stage of the procedure, the binomial step stock price is to be calculated at the initial monitoring time. This is to be done in connection with the trinomial step stock price that was obtained earlier.

After converting the log-price value into the actual stock price at the first monitoring time, the BTT is built at the second monitoring time using the stock price of the valid node at the first monitoring time as the starting point. At the second monitoring time, the BTT grows at $[T_1, T_1 + T_2]$. As previously established, to ensure that BTT is aligned with L_2, the cell width is set to $\Delta t_2 \equiv \frac{T_2}{N_2}$. Additionally, the S_D-log-price of the barrier L_2 is selected, with l_2 defined as $ln(\frac{L_2}{S_D})$. Using analogous steps, determine the nodes that are above L_2, and select the successors of the nodes X,D,E, and F when $T_1 + \Delta t_2$. To ensure that the branching opportunity is valid, consider the node D as a representative point. The S_D-log-price of the light grey vertex can be expressed in Eq. 1, where l_x and Δt_x are replaced by l_2 dan Δt_2, respectively. Furthermore, a single light gray vertex of S_D-log-price will exists within the interval $\left[\mu(\Delta t_2) - \sigma\sqrt{\Delta t_2}, \mu(\Delta t_2) + \sigma\sqrt{\Delta t_2})\right]$. For instance, the vertex indicated here as "node H" in Figure 1. The S_D-log-price for node H is defined as $\hat{\mu}$ The S_D-log-price of the other D successors is set to $\hat{\mu}(\Delta t_2) + \sigma\sqrt{\Delta t_2}$ and $\hat{\mu}(\Delta t_2) - \sigma\sqrt{\Delta t_2}$, respectively. The parameters α, β, and γ should be defined as in Eq. 3, with $\mu(\Delta t_x)$and Δt_x replaced by $\mu(\Delta t_2)$ and Δt_2, respectively. In addition, the branching probability of node D can be solved by Eq. 4, where var(Δt_x) is replaced by var(Δt_2). However, in this case, there is not only one bBTT that has a triple branching probability. However, given the uniformity in time-step values across each section, the branching probability values will remain constant for the other bBTTs. It is sufficient to demonstrate a single valid triple of branching probabilities, with the remaining triples being adjusted accordingly. Subsequent to the substantiation of the validity of the branching opportunities, the stock price should be constructed as was done in the initial monitoring time step.

The construction of the BTT for the third monitoring period is essentially analogous to the construction of the BTT for the second monitoring period. The objective is to ensure that a single layer of nodes comes into contact with the L_3 barrier. Subsequently, it is imperative to ascertain that the S_X-log-price of each bBTT possesses a valid branching probability value. This can be accomplished by designating one of the bBTTs that possesses a triple valid branching probability and making the requisite adjustments to the others. The subsequent step involves the construction of a CRR tree, with nodes W_0, W_1, W_2, and so forth serving as the foundation for this structure. It is evident that this model can be calibrated to align with the strike price by manipulating the cell width, designated as Δt_3, which is the time step.

2.3.2 Window Barrier

In general, the construction of the stock price for an up-and-out window barrier is analogous to the construction of the stock price for a bBTT model with a down-and-out barrier. The primary distinction lies in the direction of the log-barrier comparison, as delineated in Subsubsection 2.1.2., as well as in the monitoring time frame that imposes limitations on the application of the knockout condition, which will be addressed in greater detail in Subsubsection 2.4.2.

2.4 Backward Induction for Option Valuation

2.4.1 Multi-Step Moving Barrier

After constructing the stock price tree using the BTT model, the subsequent step involves determining the option price through the backward induction method. This method involves the calculation of the option value from the maturity time to the initial time, with consideration given to the movement of stock prices and the application of barrier conditions that undergo change at each monitoring time interval. In the initial stage, the option price is determined at maturity based on the standard payoff for a down-and-out European call option, as illustrated in Eq. 5. Subsequently, a backward calculation is initiated, originating from the most recent monitoring time and progressing towards the initial monitoring time. In the interval $[T_1+T_2+\Delta t_3, T_1+T_2+T_3]$, the stock price is constrained to moving in either an upward or a downward direction. Consequently, the option value is calculated based on the binomial tree formula [24]. Eq. 7 presents the binomial tree formula when the barrier type is European call down and out.

$$c_{j,i} = \begin{cases} e^{-r\Delta t}\left(p_u c_{j+1,i+1} + p_d c_{j,i+1}\right), & \text{if } S_{j,i} > L \\ 0, & \text{else.} \end{cases} \tag{7}$$

L is adjusted to the monitoring time, for instance, when $[T_1+T_2+\Delta t_3, T_1+T_2+T_3]$, the employed barrier is designated as L_3, along with Δt adjusted to Δt_3. The stock price factor exhibits an increase in value as indicated by the expression $u = e^{\sigma\sqrt{\Delta t}}$, while the stock price factor demonstrates a decrease in value as indicated by the expression $d = -e^{\sigma\sqrt{\Delta t}}$. The values of the up and down probabilities in the binomial model are given by $p_u = \frac{e^{r\Delta t}-d}{u-d}$, and $p_d = 1-p_u$, respectively.

In the subsequent stage, during the interval $(T_1+T_2, T_1+T_2, \Delta t_3]$, the stock price can increase, remain constant, or decrease. Consequently, the option value is calculated using the K-R trinomial tree formula [25]. Eq. 8 presents the trinomial tree formula with the type of a European down-and-out call option.

$$c_{j,i} = \begin{cases} e^{-r\Delta t}\left(p_u c_{j+1,i+1} + p_m c_{j,i+1} + p_d c_{j-1,i+1}\right) & ;\text{if } S_{j,i} > L \\ 0 & ;\text{else.} \end{cases} \tag{8}$$

L is adjusted to the monitoring time. For instance, when $(T_1+T_2, T_1+T_2+\Delta t_3]$, the barrier used is L_3, and the Δt used is Δt_3. The stock price factor exhibits an increase of An increase of $u = e^{\lambda\sigma\sqrt{\Delta t}}$ and a decrease of $d = e^{-\lambda\sigma\sqrt{\Delta t}}$, with the condition that $\lambda = \frac{\eta}{\eta_0}$, where $\eta = \frac{\ln S_0 - \ln L}{\sigma\sqrt{\Delta t}}$, $\eta_0 = \lfloor \eta \rfloor$.

According to the trinomial model, the probability values of increase, remain unchanged, and decrease are determined by $p_u = \frac{1}{2\lambda^2} + \frac{\mu\sqrt{\Delta t}}{2\lambda\sigma}$, $p_m = 1-\frac{1}{\lambda^2}$, and $p_d = \frac{1}{2\lambda^2} - \frac{\mu\sqrt{\Delta t}}{2\lambda\sigma}$, respectively, where $\mu = r - \frac{1}{2}\sigma^2$.

This backward phase is performed until a value of $C_{0,0}$ is obtained. This process is carried out in a manner analogous to the previous steps, with the monitoring period being taken into consideration and the model being adjusted accordingly.

2.4.2 Window Barrier

As illustrated in Figure 2, it is assumed that there are $t_1 > 0$ and $t_2 < T$ within the interval $[0,T]$. Subsequent to the construction of the stock price utilizing the bBTT model, it is imperative to ascertain the validity of the triple branching probability and to ensure that the stock price node layer comes into contact with the barrier.

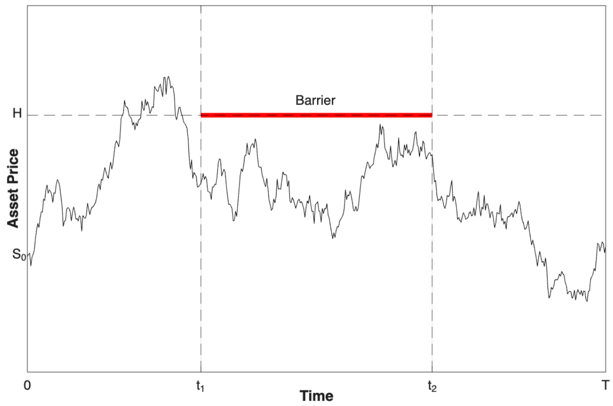


Figure 2: Schematic Representation of Stock Price Dynamics in a European Up-and-Out Call Option with Window Barrier Feature

In this instance, the single window barrier option calculation should be performed using the backward phase. In the absence of any barriers during the interval $[T,t_2]$, the payoff is to be calculated using the payoff formula inherent to the binomial model. Subsequently, in the event that an upper barrier is present within the interval $[t_1,t_2)$, the option must be calculated using the formula established within the binomial model for a European up-and-out call option. Subsequently, within the interval $[0,t_1)$, where the barrier is no longer in effect, the option price is to be calculated employing the binomial method. This is then followed by the trinomial method, which will yield the option value at the initial time. Eq. 9 presents the binomial tree formula for a European up-and-out call option.

$$c_{j,i} = \begin{cases} e^{-r\Delta t}\left(p_u c_{j+1,i+1} + p_d c_{j,i+1}\right) & ;\text{if } S_{j,i} < H \\ 0 & ;\text{else.} \end{cases} \tag{9}$$

In contrast, Eq. 10 presents the trinomial tree formula with the type of European up-and-out call option [26].

$$c_{j,i} = \begin{cases} e^{-r\Delta t}\left(p_u c_{j+1,i+1} + p_m c_{j,i+1} + p_d c_{j-1,i+1}\right) & ;\text{if } S_{j,i} < H \\ 0 & ;\text{else.} \end{cases} \tag{10}$$

3 Results and Discussion

This section presents the results of numerical tests conducted to evaluate the reliability of the Bino-Trinomial Tree (BTT) model in determining the price of barrier options under complex conditions. Two testing scenarios are discussed. The initial experiment focused on assessing various barrier options, specifically European down-and-out call options with a multi-step moving barrier. The second test was conducted on European up-and-out call options with a window barrier. Subsequent analysis focused on three key metrics: price accuracy, convergence behavior, and computational efficiency.

3.1 A Test of the BTT Model for Multi-Step Single Moving Barrier Options

This section discusses the pricing of European down-and-out call options with multi-step moving barriers using the Bino-Trinomial Tree (BTT) model and verifies it with benchmarks obtained from the Black-

Scholes-Merton (BSM) analytical approach. Two experiments were conducted: constant barriers and decreasing barriers.

3.1.1 Experiment involved the use of a constant barrier

Initial experiments were conducted with the following basic parameters: $S_0 = 95, K = 100, L = 90, T = 1, \sigma = 0.25$, and $r = 0.1$. According to the BSM formula, the theoretical value of the down-and-out European call option is 5.9968. In addition, the BTT model is implemented with a segmentation of the monitoring time into three phases, designated as $T_0 = 0.25, T_1 = 0.5$, and $T_2 = 1$. The selection of these time points is based on a periodic monitoring scheme that represents the quarterly practice in the options market. This scheme reflects the actual condition in which the monitoring barrier is discrete. The number of steps is structured as $N_0 = N_1$, and $N_2 = 2N_0$, ensuring the time distribution remains consistent with the interval length of each period. This approach maintains the accuracy of the stochastic process discretization, prevents local numerical errors, and supports the stability and convergence of the BTT scheme to the analytical solution. In this case, the barrier values are kept constant. That is, $L_0 = L_1 = L_2 = 90$.

As illustrated in Figure 3, the convergence pattern of option values calculated using the BTT model is in accordance with the benchmark for down-and-out European call options with a constant barrier. In this scenario, $L_0 = L_1 = L_2 = 90$. The graph illustrates that as the number of steps in the simulation increases, the option values generated by the BTT model increasingly approach the BSM benchmark value of 5.9968. Option value fluctuations are more significant at smaller step counts but become increasingly stable and approach the benchmark value after approximately $1500 - 2000$ steps.

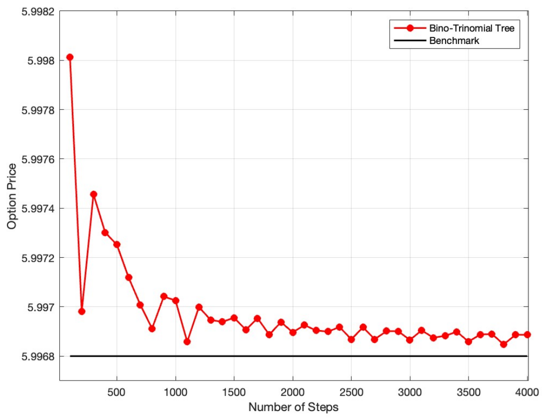


Figure 3: Convergence of the Bino-Trinomial Tree Model to the Black–Scholes-Merton Solution in Pricing European Down-and-Out Call Options with Constant Barrier ($L_0 = L_1 = L_2 = 90$)

Table 1 provides quantitative data that corroborates the observations presented in the graph. A discernible trend emerges wherein the absolute error between the option values derived from the BTT model and the BSM benchmark undergoes a consistent decrease as the number of steps increases. For instance, at 100 steps, the error is 0.00121, while at 4000 steps, the error shrinks to only 0.00009. This finding suggests that the BTT model demonstrates convergent properties toward the analytical BSM solution as the discretization resolution is elevated.

The Root Mean Square Error (RMSE) value of the BTT model simulation relative to the BSM reference value is 0.000274, as indicated by the calculation results. The low RMSE value indicates a negligible average squared deviation between the numerical results and the analytical solution. This finding suggests that the BTT model is not only numerically convergent but also possesses a high degree of accuracy and reliability in its representation of complex barrier option values.

The integration of graphical and numerical analysis demonstrates the viability of the BTT model as a reliable and effective numerical approach for the valuation of barrier options, particularly in the context of down-and-out call options with a fixed barrier.

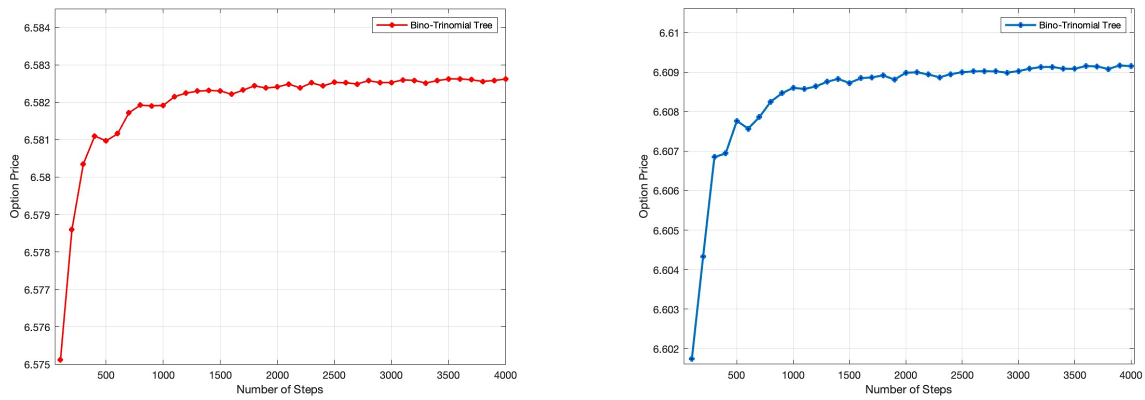
Table 1: Summary of Bino-Trinomial Tree Model Performance for Multi-Step European Down-and-Out Call Options with Constant Barrier ($L_0 = L_1 = L_2 = 90$)

Number of Steps	Option Price	Benchmark	Absolute Error
100	5.998012	5.9968	0.00121
500	5.997252	5.9968	0.00045
1000	5.997026	5.9968	0.00023
2000	5.996897	5.9968	0.00010
3000	5.996866	5.9968	0.00007
4000	5.996887	5.9968	0.00009

3.1.2 Experiment with decreasing barrier

To assess the model's sensitivity to temporal variations in barrier levels, two simulation scenarios were implemented. In the first scenario, the barriers were set as $L_0 = 90$, $L_1 = 80$, and $L_2 = 70$, while in the second scenario, the values were $L_0 = 90$, $L_1 = 70$, and $L_2 = 60$. In both scenarios, all other parameters are kept constant to ensure that the changes originate solely from the barrier variable. The simulation results are presented in Figure 4, which illustrates the convergence behavior of the BTT model with respect to the number of simulation steps for each scenario.

Figure 4 shows that the option price increased rapidly at the initial stage, then steadily reached stability. To verify this stability numerically, a successive difference convergence test was used, which involved calculating the absolute difference between the last four values of the simulation results. If all differences are less than the tolerance threshold of 10^{-4}, the solution is considered to have converged [27].



(a) Scenario 1: $L_0 = 90,\ L_1 = 80$ and $L_2 = 70$ (b) Scenario 2: $L_0 = 90,\ L_1 = 70$ and $L_2 = 60$

Figure 4: Convergence Behavior of the Bino-Trinomial Tree Model for Pricing Multi-Step European Down-and-Out Call Options with Decreasing Moving Barriers

As exhibited in Table 2, for Scenario 1, when the step exceeds 2000, the option price approaches a stable value of approximately 6.5826. Conversely, for Scenario 2, the option value has a propensity to converge to a higher number, which is approximately 6.6091. This tendency is consistent with the principle that the lower the barrier level at the subsequent monitoring point, the smaller the likelihood of a knock-out occurrence. Consequently, the option is more likely to remain active until maturity.

From a probabilistic standpoint, reducing the barrier expands the option's survival area, thereby directly contributing to an increase in the expected value of the payoff at the end of the period. This development serves to enhance the perceived value of the option to its holder. Consequently, the BTT model exhibits not only adequate convergence capabilities but also sensitivity to the temporal characteristics of the barrier, thereby rendering it effective for evaluating dynamic barrier options.

The results of the BTT model test demonstrate that the convergence pattern of the option prices is influenced by the barrier structure used. In the initial experiment with a constant barrier $(90, 90, 90)$,

Table 2: Summary of Bino-Trinomial Tree Model Performance for Multi-Step European Down-and-Out Call Options with Decreasing Moving Barriers

Steps	**Option Price**	
	Scenario 1	**Scenario 2**
100	6.57511	6.60174
500	6.58097	6.60777
1000	6.58192	6.60860
1500	6.58230	6.60872
2000	6.58241	6.60898
2500	6.58253	6.60900
3000	6.58253	6.60902
3500	6.58262	6.60908
4000	6.58262	6.60915

as the number of steps increases, the option price demonstrates a decreasing trend. This is due to the relatively low knock-out risk, which results in an initial approach that overestimates the option value. However, as the discretization becomes more precise, the calculations become more accurate, and the option value approaches the actual price. In contrast, in the second test (barriers $90, 80, 70$) and third test (barriers $90, 70, 60$), where barriers decrease at each monitoring time, option prices instead exhibit an increasing trend. This phenomenon occurs because, when the number of steps is still small, the granularity of price movements is insufficient to represent the complex dynamics. Consequently, many price paths are incorrectly deemed to have touched the barrier and disqualified. This results in option value estimates being lower than they should be. However, as the number of steps increases, estimates of price paths become more precise, and the overly high knock-out probability is rectified, causing the option value to rise toward the convergent value. This finding confirms that barrier structure and the number of steps significantly influence the accuracy of option pricing in a numerical tree-based approach.

3.2 A Test of the BTT Model for Single Window Barrier Options

In this section, the performance of the BTT model for the European up-and-out window call option will be examined. This option type was selected because it reflects more realistic market conditions. The objective of this test is to evaluate the model's capacity to type options with knock-out risk when the asset price rises through the upper barrier (H) at a specified monitoring time. This phenomenon differs from the standard barrier, which remains valid until maturity. In this test, the European up-and-out window call option is utilized with the following parameters:

$$S_0 = 100, K = 90, r = 0.1, \sigma = 0.2, T = 1, H = 110, t_1 = 0.25, t_2 = 0.75$$

The benchmark value is derived from the modified BSM formula method, as outlined in Stoklosa's research [28]. This method yielded a value of 2.20433. Given the continued viability of the K-R trinomial model for testing such barriers, it is selected for comparison. This model is recognized for its stability and accuracy in evaluating barrier options. Concurrently, the BTT model was developed to enhance numerical efficiency and stability, particularly in the context of complex barriers. The comparison is conducted with the same number of steps for both models. The following results were obtained from the convergence:

As illustrated in Figure 5, the convergence pattern of option values from the BTT and K-R trinomial models is evident for a range of discretization steps. It is evident that the option values from both methods converge towards the benchmark value as the number of steps increases, with substantial fluctuations observed only at low numbers of steps. This finding suggests that both methods can be employed to model option values with a degree of accuracy that can be enhanced by selecting an appropriate number of steps.

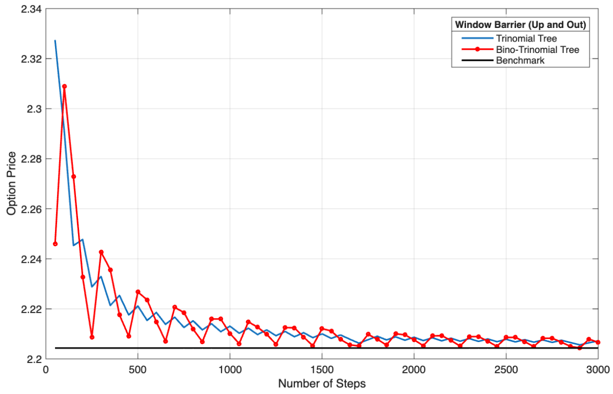


Figure 5: Convergence Analysis of Bino-Trinomial Tree and Trinomial Kamrad-Ritchken Models for a European Up-and-Out Call Option with Window Barrier toward the Analytical Black-Scholes Merton

Currently, Table 3 provides a comprehensive overview of the absolute error values and computation times for each method. The BTT model has been shown to consistently provide estimation values that are more proximate to the benchmark values and that require less computing time than the K-R trinomial method. For example, when $N = 1000$, the BTT error value is 0.0059 with a time of 0.3566 seconds. In comparison, the K-R trinomial method yields an error of 0.0088 with a time of 0.6361 seconds. This pattern persists up to $N = 5000$, and BTT produces an error value of 0.0013 in 17.1792 seconds, compared to the K-R trinomial method, which generates an error value of 0.0018 in 53.8274 seconds.

Table 3: Comparison of Option Price, Error, and Computation Time for Trinomial Tree and Bino-Trinomial Tree Models Applied to a European Up-and-Out Window Call Option

Number of Steps	**Trinomial Tree**			**Bino-Trinomial Tree**		
	Option Price	Error	Computation Time	Option Price	Error	Computation Time
100	2.2911	0.0868	0.0498	2.3088	0.1045	0.0362
500	2.2211	0.0168	0.1058	2.2267	0.0224	0.0814
1000	2.2131	0.0088	0.6361	2.2102	0.0059	0.3566
2000	2.2086	0.0043	4.2086	2.2076	0.0033	1.6034
3000	2.2072	0.0029	12.8533	2.2065	0.0022	4.5544
4000	2.2065	0.0022	28.8375	2.2059	0.0016	9.5192
5000	2.2061	0.0018	53.8274	2.2057	0.0013	17.1792

These results are of particular significance for practitioners and analysts in the capital market, particularly in the context of valuing exotic derivative products such as barrier options with specific time windows. A more efficient and accurate BTT model can be used for faster and more reliable fair value determination, enabling market participants to make hedging and pricing decisions with lower risk and computational costs.

4 Conclusion

This study demonstrates the effectiveness of the bino-trinomial tree model in pricing barrier options with modified structures, including moving and window barriers. The model offers a significant computational advantage while maintaining high pricing accuracy. These characteristics make it particularly suitable for modern derivative markets where efficiency and flexibility are critical. Future research should explore its application to American-style barrier options and stochastic volatility models to further assess its robustness in diverse financial settings.

CRediT Authorship Contribution Statement

Rima Aulia Rahayu: Conceptualization, Methodology, Software, Data Curation, Formal Analysis, Original Writing Draft, Editing and Funding Acquisition. **Fitriani Agustina:** Writing–Review. **Kuntjoro Adji Sidarto:** Supervision.

Declaration of Generative AI and AI-assisted technologies

The author states that generative AI technology was used to assist in preparing this manuscript. ChatGPT (OpenAI) supported the formulation of technical sentences and the organization of the text, and DeepL Translator assisted with translating parts of the manuscript. The author thoroughly reviewed and edited the entire content to ensure accuracy and academic integrity. The final manuscript is the result of the author's intellectual contribution and professional judgment.

Declaration of Competing Interest

The authors declare no competing interests.

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Data Availability

The data supporting the findings of this study are available from the corresponding author upon reasonable request and subject to confidentiality agreements.

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