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Locating Metric Coloring on the Cherry Blossom, Sun Flower, and Closed Dutch Windmill Graphs

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Abstract

Locating metric coloring is a variation of metric coloring in graphs that integrates vertex coloring with the uniqueness of metric representations. In this coloring, each vertex in a connected graph $G = (V, E)$ is assigned a color such that the distance vectors to each color class are distinct for every pair of different vertices. Let $c : V(G) \to \{1, 2, \ldots, k\}$ be a coloring function (not necessarily proper). The coloring c is called a locating metric coloring if, for any two distinct vertices $u, v \in V$, their distance vectors $r(u) = (d(u, V_1), d(u, V_2), \ldots, d(u, V_k))$ and $r(v) = (d(v, V_1), d(v, V_2), \ldots, d(v, V_k))$ are distinct. So it is obtained $\Pi = \{C_1, C_2, \ldots, C_k\}$ represents the partition of vertices by color classes. Thus, for every vertex, the distance vector $r(v|\Pi) = (d(V, C_1), d(V, C_2), \ldots, d(V, C_k))$ are different. Vertices may share the same color, whether adjacent or not, as long as their metric representations are unique. The smallest number of colors required for such a coloring is called the locating metric chromatic number, denoted $\chi_{lm}(G)$. This study focuses on analyzing locating metric coloring for three specific graphs: the Cherry Blossom graph $CB_{n,n}$, the Sun Flower graph $SF_{n,n}$, and the Closed Dutch Windmill graph CD_n. These graphs were chosen due to the absence of prior research on their locating metric coloring properties. The research method combines pattern recognition and a deductive-axiomatic approach. The proof process begins by determining lower bounds, followed by the construction of upper bounds through coloring function analysis. The resulting locating metric chromatic numbers for each graph are then established.

Keywords: Locating metric coloring; cherry blossom graph; sun flower graph; closed dutch windmill graphs.

1 Introduction

Graph theory is a fundamental branch of mathematics with applications in various fields, including computer science, engineering, and network analysis [1]. One of the important developments in graph coloring is Locating Metric Coloring (LMC), which not only assigns colors to vertices but also ensures that each vertex has a unique metric representation based on its distances to vertices of each color class [2]. Graph theory is divided into several topics, including graph coloring and locating metric coloring [3]. Graph coloring is the assignment of colors to vertices, edges, or regions so that neighbouring elements do not have the same color[4]. Graph coloring itself includes 3 (three) types, namely vertex coloring, side coloring, and region coloring [5]. Meanwhile,

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Locating Metric Coloring is a concept in graph theory that introduces a new approach to vertex coloring [6]. Unlike traditional graph coloring, where adjacent vertices in a connected graph must receive different colors, locating metric coloring emphasizes the uniqueness of the metric representation of each vertex [7] [8]. Let $G = (V, E)$ be a graph consisting of a vertex set $V(G)$ and an edge set $E(G)$. A locating metric coloring is a vertex coloring of the graph that must satisfy two main conditions: first, each vertex $v \in V(G)$ must have a unique metric representation; and second, the vertex coloring, denoted as a function $c : V(G) \to \{1, 2, \ldots, k\}$, may assign the same color to different vertices, as long as their metric representations remain distinct (the coloring does not have to be proper) [9]. The metric representation of a vertex v is defined as a distance vector $r(v) = [d(v, u_1), d(v, u_2), \ldots, d(v, u_k)]$ where $d(v, u)$ denotes the shortest distance between vertex v and vertex u, and $u_1, u_2, \ldots, u_k$ are the reference vertices[10] [11].

One of the main requirements in locating metric coloring is that the metric representation of each vertex v must be unique [12] [13]. This means that if $r(v_1) = r(v_2)$, then the vertices v_1 and v_2 cannot be distinguished solely based on their distances to the reference vertices. Formally, for every pair of vertices $v_1, v_2 \in V$, if $r(v_1) = r(v_2)$, then v_1 must be identical to v_2, which ensures that each vertex has a distinct metric representation. The coloring function c, which maps the vertices in $V(G)$ to a set of colors C, only needs to satisfy the condition that the same color may be assigned to different vertices, as long as their metric representations remain unique. Therefore, even if some vertices share the same color, the uniqueness of their metric representations guarantees that they can be distinguished from each other based on their distances to the reference vertices [14].

The uniqueness of locating metric coloring lies in the distinct distance representation for each vertex, which enables the unique identification of every vertex without requiring a different color for each one [15]. In large networks, it is often difficult to assign a unique color to every vertex; therefore, locating metric coloring offers a more efficient solution while still ensuring clear identification of each vertex [16] [17]. For example, in a communication or sensor network, locating metric coloring allows each vertex to have a unique identity that can be recognized through a combination of distance information and color classes[18]. This supports efficient operations in identification and tracking. The purpose of studying locating metric coloring on graphs is to analyze locating metric coloring, particularly on the Cherry Blossom, Sun Flower, and Closed Dutch Windmill graphs, which have not been previously studied by other researchers.

This study aims to analyze the locating metric coloring on three specific graphs: the Cherry Blossom, Sun Flower, and Closed Dutch Windmill graphs, which have not been previously explored in this context. The method used involves assigning colors to vertices and calculating the distance vectors (metric representations) to ensure that each vertex has a unique identification, even if some colors are reused.

The results show that the locating chromatic number varies depending on the graph structure and number of vertices. For example, in the Cherry Blossom and Sun Flower graphs, locating metric coloring enables efficient vertex distinction with fewer colors than traditional identifying colorings. This uniqueness supports applications such as sensor networks and communication systems, where precise node identification is crucial.

The main contribution of this research is to expand the theory of locating metric coloring by applying it to new graph families and demonstrating its potential for real-world applications in network identification and optimization. Future research may explore algorithmic strategies for automating the coloring process in larger and more complex graphs.

2 Preliminaries

In this section, we provide fundamental definitions, notations, and theoretical results that are essential for the subsequent analysis in this paper. These concepts will form the basis for the theorems and proofs discussed in the **Results and Discussion** section.

2.1 Basic Definitions and Notations

We begin by defining some key concepts that are used throughout this paper.

Definition 1. *Let $G = (V, E)$ be a connected graph with vertex set $V(G)$ and edge set $E(G)$. A locating metric coloring of the graph G is a vertex coloring function $c : V(G) \to \{1, 2, \ldots, k\}$, such that for any two distinct vertices $u, v \in V(G)$, their distance vectors are distinct. Specifically, the distance vectors of vertices u and v are given by:*

$$r(u) = (d(u, V_1), d(u, V_2), \ldots, d(u, V_k))$$

$$r(v) = (d(v, V_1), d(v, V_2), \ldots, d(v, V_k))$$

where $V_1, V_2, \ldots, V_k$ represent the partition of $V(G)$ into k color classes.

Definition 2. *The locating metric chromatic number $\chi_{lm}(G)$ of a graph G is the smallest integer k such that there exists a locating metric coloring of G using k colors.*

In this study, we adopt the standard graph-theoretic notations. The following theorems, will serve as the theoretical framework for determining the lower and upper bounds of $\chi_{lm}(G)$ for the Cherry Blossom graph $CB_{n,n}$, the Sun Flower graph $SF_{n,n}$, and the Closed Dutch Windmill graph CD_n.

2.2 Theoretical Background

The following concepts and results from previous research are essential for the analysis of locating metric coloring. These results will be used to establish the bounds and calculations in the **Results and Discussion** section.

- **Distance Vector**: The distance vector for a vertex v with respect to the color classes $V_1, V_2, \ldots, V_k$ is defined as the vector:

  $$r(v) = (d(v, V_1), d(v, V_2), \ldots, d(v, V_k))$$

  where $d(v, V_i)$ is the distance between vertex v and the set V_i.
- **Metric Representation**: In locating metric coloring, the metric representation of a vertex is a function of the distances to each color class. It is crucial that the metric representations of distinct vertices are unique to ensure the validity of the coloring.
- **Axiomatic Deductive Approach**: This paper employs an axiomatic deductive method, where existing axioms, lemmas, and theorems from graph theory are applied to derive results for the locating metric chromatic number. This approach ensures that the conclusions drawn are logically consistent with the fundamental principles of graph theory.

3 Results and Discussion

This research produces three theorems about locating metric coloring on graphs. The following is the result of the theorem and its proof regarding the locating metric coloring of Cherry Blossom graph $\chi_{lm}(CB_{n,n})$, Sun Flower graph $\chi_{lm}(SF_{n,n})$, dan Closed Dutch Windmill Graph $\chi_{lm}(CD_n)$.

Theorem 1. *Let Cherry Blossom $CB_{n,n}$ be a graph that has n stems and n petals, then $\chi_{lm}(CB_{n,n}) = n + 1$ for $n \geq 3$.*

Proof: The Cherry Blossom graph $CB_{n,n}$ has a vertex set $V(CB_{n,n}) = \{x\} \cup \{x_i, y_i, z_i; 1 \leq i \leq n\}$ and an edge set $E(CB_{n,n}) = \{xx_i, x_iy_i, x_iz_i; 1 \leq i \leq n\} \cup \{y_iz_i; 1 \leq i \leq n\}$. Based on the vertex and edge sets, the cardinalities of the vertex set and edge set of the graph $CB_{n,n}$ are $|V(CB_{n,n})| = 3n + 1$ and $|E(CB_{n,n})| = 4n$, respectively.

It will then be proven that $\chi_{lm}(CB_{n,n}) = n+1$ by showing both the lower and upper bounds, namely $\chi_{lm}(CB_{n,n}) \geq n+1$ and $\chi_{lm}(CB_{n,n}) \leq n+1$. First, the lower bound $\chi_{lm}(CB_{n,n}) \geq n+1$ will be shown using a contradiction. Assume that $\chi_{lm}(CB_{n,n}) < n+1$. Jika $\chi_{lm}(CB_{n,n}) = n$ then there will be two identical representations. For example, consider $CB_{3,3}$, as a result, there will be 2 pairs of identical representations, namely:

- ▷ For the representation $(0, 2, 1)$ it is possessed by the vertices x_1 and y_2 with the same color.
- ▷ For the representation $(2, 0, 1)$ it is possessed by the vertices x_2 and y_3 with the same color.

This condition is inconsistent with the definition of locating metric coloring, therefore a contradiction arises if we assume that $\chi_{lm}(CB_{n,n}) = n$, thus, it is proven that $\chi_{lm}(CB_{n,n}) \geq n+1$. Next, the upper will be proven that $\chi_{lm}(CB_{n,n}) \leq n+1$ by showing a coloring function that defines $c : V(CB_{n,n}) \to \{1, 2, 3, \ldots, n+1\}$, as follows:

$$c(x_i) = i; \quad 1 \leq i \leq n$$

$$c(y_i) = \begin{cases} i; & 1 \leq i \leq n-1 \\ n-i; & i = n \end{cases}$$

$$c(z_i) = \begin{cases} i+1; & 2 \leq i \leq n-1 \\ n; & i = n \end{cases}$$

$$c(x) = n+1$$

Based on the function above, it can be concluded that the representation of each vertex is distinct. The representation of each vertex can be seen in Table 1.

Table 1: Representation of Each in the Graph $CB_{n,n}$

n	**Representation**	**Condition**
x	$(1,1,1,\underbrace{...,0}_{n-1})$	$n=3$
x_1	$(0,2,2,\underbrace{...,1}_{n-2})$	$n=3$
x_2	$(2,0,2,\underbrace{...,1}_{n-3})$	$n=4$
x_1	$(2,0,2,2,\underbrace{...,1}_{n-4})$	$n=5$
$x_{i\geq 4}$	$(2,2,\underbrace{...,0}_{i-2},2,\underbrace{...,1}_{n-(i+1)})$	$n \geq 6$
y_1	$(0,1,2,2,\underbrace{...,1}_{n-3})$	$n=4$
y_2	$(0,2,1,2,\underbrace{...,1}_{n-4})$	$n=5$
y_3	$(0,2,2,1,2,\underbrace{...,1}_{n-5})$	$n=6$
$y_{i\geq 4}$	$(0,2,2,1,2\ \underbrace{...,1}_{n-(i+1)})$	$n \geq 7$
z_1	$(1,0,2,2,\underbrace{...,1}_{n-3})$	$n=4$
z_2	$(1,2,0,2,\underbrace{...,1}_{n-4})$	$n=5$
z_3	$(1,2,2,2,0,2,\underbrace{...,1}_{n-5})$	$n=6$
$z_{i\geq 4}$	$(1,2,2,2,0,2,2,\underbrace{...,1}_{n-(i+2)})$	$n \geq 7$

The coloring function and Table 1 show that $|c(V(CB_{n,n}))| = n+1$, therefore $\chi_{lm}(CB_{n,n}) \leq n+1$. Based on the lower and upper bounds of $\chi_{lm}(CB_{n,n})$, we obtain $n+1 \leq \chi_{lm}(CB_{n,n}) \leq n+1$. Thus, it is proven that $\chi_{lm}(CB_{n,n}) = n+1$ for $n \geq 3$. ■

Example of a coloring of $\chi_{lm}(CB_{n,n})$ for $n = 3$

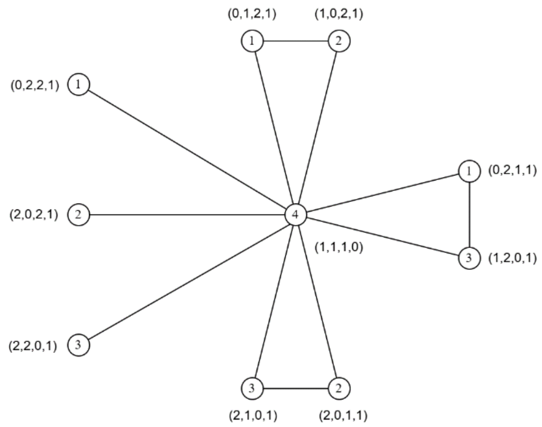

Figure 1: Cherry Blossom Graph for $n = 3$

Theorem 2. *Let Sun Flower graph $SF_{n,n}$ be a graph consisting of n core vertices and n petals, $\chi_{lm}(SF_{n,n}) = n+1$ untuk $n \geq 3$, $\chi_{lm}(SF_{n,n}) = n$ for odd $n, n \geq 5$ and $\chi_{lm}(SF_{n,n}) = n-1$ for even $n, n \geq 4$.*

Proof: The Sun Flower graph $(SF_{n,n})$ has a vertex set $V(SF_{n,n}) = \{x\} \cup \{x_i, y_i; 1 \leq i \leq n\}$ and an edge set $E(SF_{n,n}) = \{xx_i; 1 \leq i \leq n\} \cup \{x_iy_i; 1 \leq i \leq n\}$. Based on the vertex and edge sets, we obtain the cardinality of the vertex and edge sets of the graph $(SF_{n,n})$ respectively $|V(SF_{n,n})| = 2n+1$ and $|E(SF_{n,n})| = 2n$. Next, we will prove that:

$$\chi_{lm}(SF_{n,n}) = \begin{cases} n+1, & \text{for } n = 3 \\ n, & \text{for even } n, n \geq 4 \\ n-1, & \text{for odd } n, n \geq 5 \end{cases}$$

by establishing the lower and upper bounds, namely

$$\chi_{lm}(SF_{n,n}) \geq \begin{cases} \text{n + 1}, & \text{for } n = 3 \\ n, & \text{for even n, n} \geq 4 \\ n-1, & \text{for odd } n, n \geq 5 \end{cases} \quad \text{and} \quad \chi_{lm}(SF_{n,n}) \leq \begin{cases} n+1, & \text{for } n = 3 \\ n, & \text{for even } n, n \geq 4 \\ n-1, & \text{for odd } n, n \geq 5 \end{cases}$$

Case 1. For $n = 3$

First, we will show the lower bound of $\chi_{lm}(SF_{n,n}) = n+1$ for $n = 3$ by using a contradiction. Assume that $\chi_{lm}(SF_{n,n}) < n+1$. If $\chi_{lm}(SF_{n,n}) = n$ then there will be identical representations. For example, take $SF_{3,3}$, as a result, there will be one pair of vertices with the same code, namely:

- For representation (0,1,1) it is shared by the vertices x and x_1 with the same color.

This condition does not satisfy the definition of a locating metric coloring. Therefore, there is a contradiction if we assume that $\chi_{lm}(SF_{n,n}) = n$, which proves that $\chi_{lm}(SF_{n,n}) = n+1$ for $n = 3$. Next, we will prove the upper bound, that $\chi_{lm}(SF_{n,n}) \leq n+1$ by presenting a coloring function that defines $c : V(SF_{n,n}) \to \{1, 2, 3\}$ as follows:

$$c(x_i) = c(y_i) = i, \quad 1 \leq i \leq n$$

$$\begin{aligned} c(x) &= n+1 \\ &= 3+1 \\ &= 4 \end{aligned}$$

Based on the function above, it is obtained that the code for each vertex is distinct, satisfying the locating metric coloring condition. Table 2 illustrates these locating metric representations for each vertex in the graph $SF_3, 3$. Each row displays a vertex, its corresponding code derived from the distances to each color class, and the condition (in this case, $n = 3$) under which the representation is valid. This table confirms that every vertex has a unique representation. The code for each vertex can be seen in Table 2!

Table 2: Representation of Each Vertex in the Graph $SF_{3,3}$

Vertex	Representation	Condition
x	(1,1,1,0)	$n = 3$
x_1	(0,1,1,1)	$n = 3$
x_2	(1,0,1,1)	$n = 3$
x_3	(1,1,0,1)	$n = 3$
y_1	(0,1,2,2)	$n = 3$
y_2	(2,0,1,2)	$n = 3$
y_3	(1,2,0,2)	$n = 3$

The coloring function and Table 2 show that $|c(V(SF_{n,n}))| = n + 1$, so $\chi_{lm}(SF_{n,n}) \leq n + 1$. Based on the lower and upper bounds of $\chi_{lm}(SF_{n,n})$, we obtain:

$$n + 1 \leq \chi_{lm}(SF_{n,n}) \leq n + 1$$

Based on the lower and upper bounds of $\chi_{lm}(SF_{n,n}) = n + 1$ when $n = 3$. ■
Example of coloring for $\chi_{lm}(SF_{n,n})$ when $n = 3$.

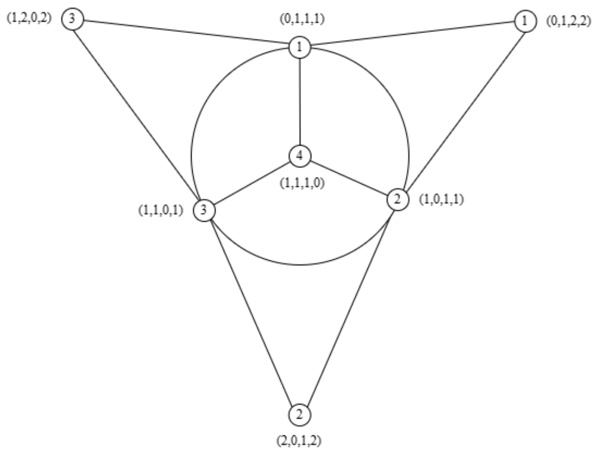

Figure 2: Sun Flower Graph for $n = 3$

Case 2. For even n, $n \geq 4$
First, we will show the lower bound of $\chi_{lm}(SF_{n,n}) = n$ for even n in $n \geq 4$ by using a contradiction. Assume that $\chi_{lm}(SF_{n,n}) < n$. If $\chi_{lm}(SF_{n,n}) = n - 1$ then there will be identical representations. For example, take $SF_{4,4}$, as a result, there will be 3 pairs of identical representations, namely:

- For the representation $(0, 1, 1)$ it is shared by the vertices x_1 and y_3 with the same color.
- For the representation $(1, 0, 1)$ it is shared by the vertices x_2 and y_3 with the same color.
- For the representation $(1, 1, 0)$ it is shared by the vertices x and y_4 with the same color.

This condition does not satisfy the definition of a locating metric coloring. Therefore, there is a contradiction if we assume that $\chi_{lm}(SF_{n,n}) = n - 1$, which proves that $\chi_{lm}(SF_{n,n}) \geq n$. Next, we will prove the upper bound, that $\chi_{lm}(SF_{n,n}) \leq n$ by presenting a coloring function that defines $c : V(SF_{n,n}) \to \{1, 2, 3, \ldots, n\}$ as follows:

$$c(x_i) = i, \quad 1 \le i \le n$$
$$c(y_i) = 1, \quad i \equiv 1 \pmod 2, \quad 1 \le i \le n$$
$$c(y_i) = \frac{n}{2} + 1, \quad i \equiv 0 \pmod 2, \quad 2 \le i \le n$$
$$c(x) = n$$

Based on the function above, it is obtained that the code for each vertex is distinct, satisfying the locating metric coloring condition. Table 2 illustrates these locating metric representations for each vertex in the graph $SF_{4,4}$. Each row displays a vertex, its corresponding code derived from the distances to each color class, and the condition (in this case $n = 3$) under which the representation is valid. This table confirms that every vertex has a unique representation.

Table 3: Representation of Each Vertex in the Graph $SF_{n,n}$ for even n

Vertex	Representation	Condition
x	$(1, 1, 1, \underbrace{..., 0}_{n-2})$	$n \ge 5$, where n is even
x_1	$(0, 1, 1, 1)$	$n = 4$
	$(0, 1, 2, 2, 1)$	$n = 6$
	$(0, 1, 2, 2, 1, 2, 1)$	$n = 8$
x_2	$(1, 0, 1, 1)$	$n = 4$
	$(1, 0, 1, 2, \underbrace{..., 1}_{n-5})$	$n \ge 6$
x_3	$(1, 1, 0, 1)$	$n = 4$
	$(1, 1, 0, 1, 2, 1)$	$n = 6$
	$(1, 1, 0, 1, 2, \underbrace{..., 1}_{n-5})$	$n \ge 8$
y_1	$(0, 1, 2, 2)$	$n = 4$
	$(0, 1, 2, 2, 3, 2)$	$n = 6$
	$(0, 1, 2, 3, 3, 3, 3, 2)$	$n = 8$
y_2	$(2, 0, 1, 2)$	$n = 4$
	$(2, 0, 1, 2, 3, 2)$	$n = 6$
	$(2, 0, 1, 2, 3, 3, 3, 3, 2)$	$n = 8$
y_3	$(0, 2, 1, 1)$	$n = 4$
	$(0, 2, 1, 1, 2, 2)$	$n = 6$
	$(0, 2, 1, 1, 2, 3, 3, 2)$	$n = 8$

The coloring function and Table 3 show that $|c(V(SF_{n,n}))| = n$, so $\chi_{lm}(SF_{n,n}) \le n$. Based on the lower and upper bounds of $\chi_{lm}(SF_{n,n})$, we obtain $n \le \chi_{lm}(SF_{n,n}) \le n$. Thus, it is proven that $\chi_{lm}(SF_{n,n}) = n$, when n is even, $n \ge 4$. ■
Example of coloring for $\chi_{lm}(SF_{n,n})$ when n is even, $n \ge 4$.

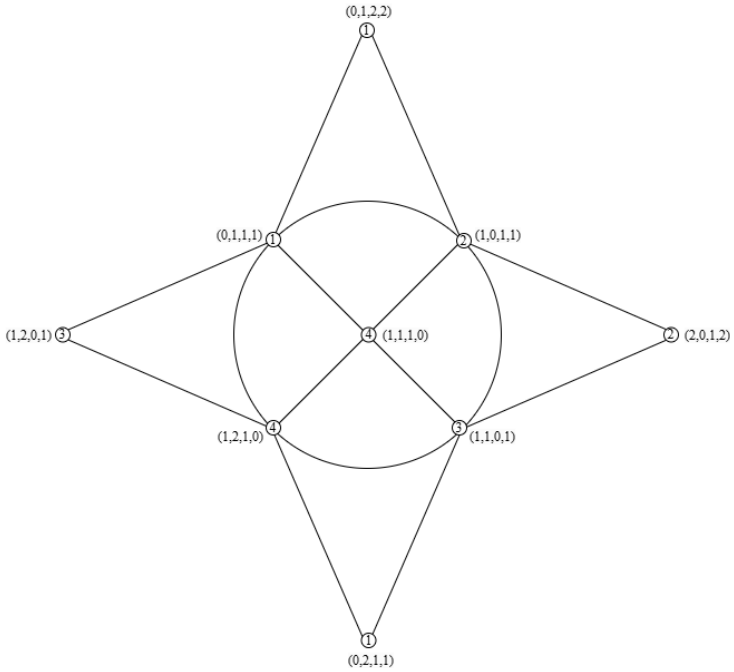

Figure 3: Sun Flower Graph for even n, specifically $n = 4$

Case 3. For odd n, with $n \geq 5$
First, we will show the lower bound of the locating metric chromatic number of the sunflower graph $\chi_{lm}(SF_{n,n}) = n - 1$ for odd $n \geq 5$ by using proof by contradiction. Assume the opposite of what we want to prove. That is, suppose $\chi_{lm}(SF_{n,n}) < n - 1$. If $\chi_{lm}(SF_{n,n}) = n - 2$ then there will exist vertices with the same color representations. For example, let us consider $SF_{n,n}$ = 5. As a result, there will be 4 pairs of vertices with identical representations, namely:

- For the representation $(0, 1, 1)$ it is shared by the vertices x_1 and x_4 with the same color.
- For the representation $(0, 1, 2)$ it is shared by the vertices y_1 and y_4 with the same color.
- For the representation $(0, 1, 0)$ it is shared by the vertices x_2 and x_5 with the same color.
- For the representation $(1, 1, 0)$ dimiliki oleh titik x dan x_3 dengan warna yang sama.

This condition does not satisfy the definition of a locating metric coloring. Therefore, a contradiction arises from assuming $\chi_{lm}(SF_{n,n}) = n - 2$, and it follows that $\chi_{lm}(SF_{n,n}) \geq n - 1$. Next, we will prove the upper bound, that $\chi_{lm}(SF_{n,n}) \leq n-1$ by constructing a coloring function defined as $c : V(SF_{n,n}) \rightarrow \{1, 2, 3, \ldots, n - 1\}$ as follows:

$$c(x_i) = c(y_i) = \begin{cases} n & \text{if } 1 \leq n \leq 5 \\ 5 & \text{if } n = 6 \\ n-1 & \text{if } i = n \end{cases}$$

$$c(x) = n - 1$$

Based on the function above, it is obtained that the code for each vertex is distinct, satisfying the locating metric coloring condition. Table 4 illustrates these locating metric representations for each vertex in the graph $SF_{5,5}$. Each row displays a vertex, its corresponding code derived from the distances to each color class, and the condition (in this case, $n = 5$) under which the representation is valid. This table confirms that every vertex has a unique representation.
The coloring function and Table 4 show that $|c(V(SF_{n,n}))| = n - 1$, so $\chi_{lm}(SF_{n,n}) \leq n - 1$. Based on the lower and upper bounds of $\chi_{lm}(SF_{n,n})$, we obtain $n - 1 \leq \chi_{lm}(SF_{n,n}) \leq n - 1$. Thus, it is proven that $\chi_{lm}(SF_{n,n}) = n - 1$ for odd $n \geq 5$. ■

Table 4: Vertex Representations in the Graph $SF_{n,n}$ for odd n

n	Representation	Condition
x	$(1,1,1,\underbrace{...,0}_{n-5})$	$n \geq 5$, where n is odd
x_1	(0,1,2,1)	$n = 5$
	$(0,1,2,\underbrace{...,1}_{n-5})$	$n \geq 7$
x_2	(1,0,1,1)	$n = 5$
	$(1,0,1,2,\underbrace{...,1}_{n-6})$	$n \geq 7$
x_3	(2,1,0,1)	$n = 5$
	(2,1,0,1,2,1)	$n = 7$
	$(1,1,0,1,2,\underbrace{...,1}_{n-7})$	$n \geq 9$
y_1	(0,1,2,2)	$n = 5$
	$(0,1,2,3,\underbrace{...,2}_{n-6})$	$n \geq 7$
y_2	(2,1,1,2)	$n = 5$
	(2,0,1,2,3,2)	$n = 7$
	$(2,0,1,2,3,\underbrace{...,2}_{n-7})$	$n \geq 9$
y_3	(3,2,0,2)	$n = 5$
	(3,2,0,1,2,2)	$n = 7$
	(3,2,0,1,2,3,3,2)	$n = 9$

Example of a locating metric coloring $\chi_{lm}(SF_{n,n})$ for odd n, $n \geq 5$.

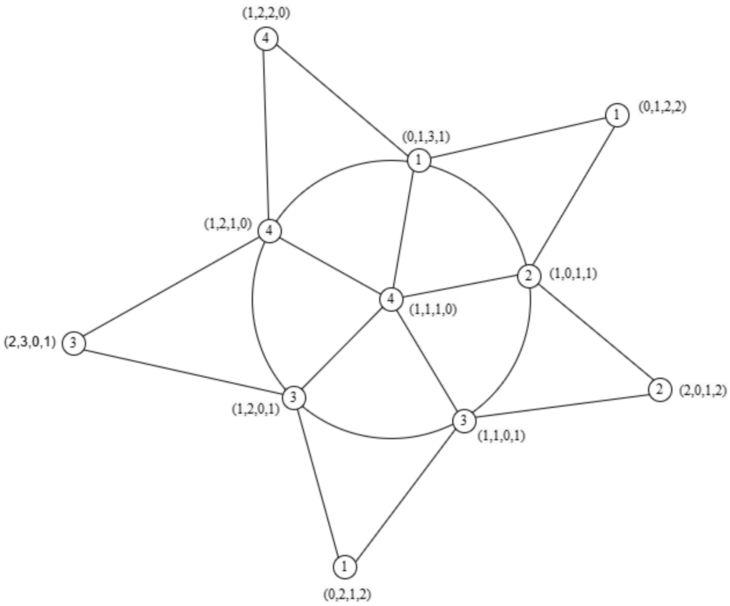

Figure 4: Sun Flower Graph for odd $n = 5$

Theorem 3. *Let the Closed Dutch Windmill Graph* (CD_n) *be a graph with* n *blades, where* $n \geq 3$*. Then,* $\chi_{lm}(CD_n) = n + 1$*.*

Proof: The Closed Dutch Windmill Graph (CD_n) has the vertex set $V(CD_n) = \{x\} \cup \{x_i, y_i, z_i : 1 \leq i \leq n\}$ and the edge set $E(CD_n) = \{xx_i, xz_i : 1 \leq i \leq n\} \cup \{y_i x_i, y_i z_i : 1 \leq i \leq n\}$. Based on the vertex set and the edge set, the cardinalities of the vertex set and the edge set of the graph

(CD_n) are $|V(CD_n)| = 3n + 1$ and $|E(CD_n)| = 4n$.

It will then be proved that $\chi_{lm}(CD_n) = n + 1$ by showing both the lower and upper bounds, namely $\chi_{lm}(CD_n) \geq n + 1$ and $\chi_{lm}(CD_n) \leq n + 1$. First, we will show the lower bound of $\chi_{lm}(CD_n) \geq n + 1$ using proof by contradiction. Assume that $\chi_{lm}(CD_n) < n + 1$. If $\chi_{lm}(CD_n) = n$ then there will exist identical representations. As an example, consider CD_3, which results in 3 pairs of identical representations, namely:

- For the representation $(0, 1, 1)$ it is shared by the vertices y_1 and x_2, with the same color.
- For the representation $(0, 1, 0)$ it is shared by the vertices z_1 and y_2, with the same color.
- For the representation $(1, 1, 1)$ it is shared by the vertices x and y_3, with the same color.

This condition does not satisfy the definition of locating metric coloring, therefore a contradiction arises from assuming $\chi_{lm}(CD_n) = n$, and it follows that $\chi_{lm}(CD_n) \geq n + 1$. Next, we will prove that $\chi_{lm}(CD_n) \leq n + 1$ by defining a coloring function $c : V(CD_n) \rightarrow \{1, 2, 3, \ldots, n + 1\}$ as follows:

$$c(x_i) = \begin{cases} i & ; 1 \leq i \leq n - 1 \\ n - i & ; i = n \end{cases}$$

$$c(y_i) = i;\ 1 \leq i \leq n$$

$$c(z_i) = \begin{cases} n + 1 & ; 1 \leq i \leq n - 1 \\ n & ; i = n \end{cases}$$

The coloring function and Table 5 show that $|c(V(CD_n))| = n + 1$, so $\chi_{lm}(CD_n) \leq n + 1$. Based on the lower and upper bounds of $\chi_{lm}(CD_n)$, we obtain $n + 1 \leq \chi_{lm}(CD_n) \leq n + 1$. Therefore, it is proven that $\chi_{lm}(CD_n) = n + 1$ for $n \geq 3$. ■

Example of a locating metric coloring for $\chi_{lm}(CD_n)$ when $n = 3$.

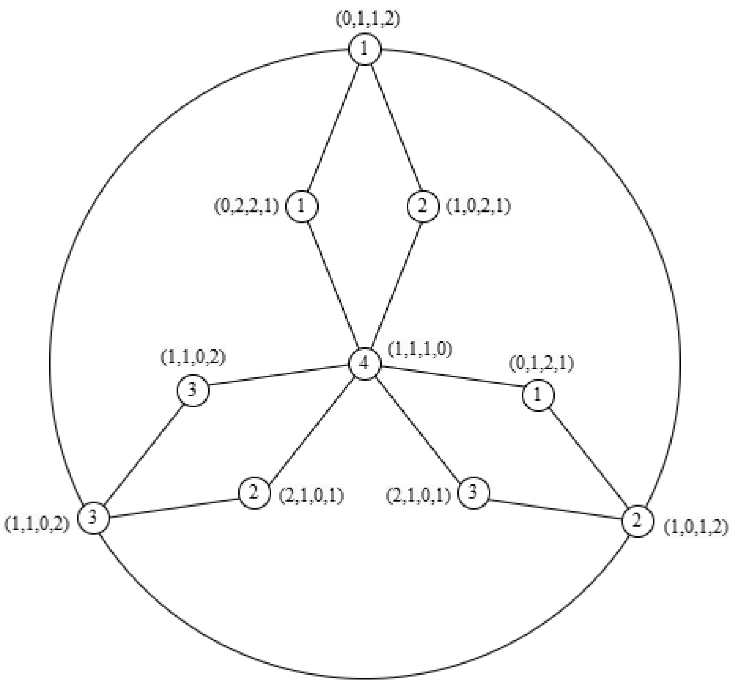

Figure 5. The Closed Dutch Windmill Graph

Based on the function above, it is obtained that the code for each vertex is distinct, satisfying the locating metric coloring condition. Table 2 illustrates these locating metric representations for each vertex in the graph CD_3. Each row displays a vertex, its corresponding code derived from the distances to each color class, and the condition (in this case, n=3 $n = 3$) under which the representation is valid. This table confirms that every vertex has a unique representation.

Table 5: Vertex Representations in the Graph CD_n

n	Representation	Condition
x	$(1, 1, 1, \underbrace{..., 0}_{n-5})$	$n \geq 3$
x_1	$(0, 2, 2, 2, \underbrace{..., 0}_{n-5})$	$n \geq 3$
x_2	$(2, 0, 2, 2, \underbrace{..., 0}_{n-4})$	$n \geq 4$
x_3	$(2, 2, 0, 2, \underbrace{..., 1}_{n-4})$	$n \geq 5$
$x_{i\geq 4}$	$(2, \underbrace{..., 0}_{i-2}, 2, \underbrace{..., 1}_{n-(n-1)})$	$n \geq 6$
y_1	$(0, 1, 2, \underbrace{..., 1}_{n-3})$	$n \geq 4$
y_2	$(0, 2, 1, 2, \underbrace{..., 1}_{n-4})$	$n \geq 5$
y_3	$(0, 2, 2, 1, 2, \underbrace{..., 1}_{n-5})$	$n \geq 6$
z_1	$(1, 0, 2, \underbrace{..., 1}_{n-3})$	$n \geq 4$
z_2	$(2, 1, 0, 2, \underbrace{..., 1}_{n-4})$	$n \geq 5$
z_3	$(2, 1, 0, 2, 2, \underbrace{..., 1}_{n-3})$	$n \geq 5$

4 Conclusion

Based on the results of the research above, new theorems have been successfully formulated regarding the locating metric coloring of three specific graph families: the Cherry Blossom graph ($CB_{n,n}$), the Sunflower graph ($SF_{n,n}$), and the Closed Dutch Windmill graph (CD_n). The main findings demonstrate that each of these graphs satisfies locating metric coloring conditions under specific parameters, and the exact locating chromatic number $\chi_{lm}(G)$ for each graph has been determined or bounded accordingly.

These results contribute significantly to the advancement of graph theory, particularly in the subfield of locating metric coloring. The study not only strengthens theoretical foundations but also offers potential applications in areas such as network verification, chemical graph theory, and information security, where unique identification of nodes based on distances is essential.

Furthermore, this research opens new directions for future studies. One potential area is the exploration of locating metric coloring in dynamic or weighted graphs. Additionally, further investigations can be conducted into algorithmic approaches for finding locating colorings, or generalizing the current results to broader graph classes, such as multipartite or random graphs.

Overall, this work can serve as a valuable reference for other researchers conducting similar studies, and as a guideline for extending the application of locating metric coloring in both theoretical and practical contexts.

Open Problem: Based on previous studies, the topic of locating metric coloring is still relatively new, and many types of graphs remain unexplored. Readers are encouraged to continue research on this topic by using different types of graphs and analyzing the locating metric coloring resulting from various graph operations.

CRediT Authorship Contribution Statement

Khusnul Hotimatus Agustina: Conceptualization, Methodology, Original Draft Writing.
Arika Indah Kristiana: Data Curation, Formal Analysis, Writing–Review & Editing.
Dafik: Visualization, Review Writing, Data Preparation, Validation.

Declaration of Generative AI and AI-assisted technologies

No generative AI or AI-assisted technologies were used in the preparation of this manuscript. All analyses, theorem proofs, and writing were conducted solely by the authors without the assistance of generative or AI-assisted technologies.

Declaration of Competing Interest

The authors declare that there are no potential or actual conflicts of interest that could influence the outcomes or interpretation of this research. There are no financial or non-financial relationships affecting this research.

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Data and Code Availability

The data and code used in this study are available upon reasonable request from the corresponding author.

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