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Odd Harmonious Labeling of Lotus Flower Graphs

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Abstract

The purpose of this research is to obtain a new graph class definition and to ensure that the new graph class satisfies the properties of odd harmonious labeling. This research method consists of several stages, namely identifying open problems, analyzing data, formulating definitions and theorems, and proving theorems. We define three new classes of flower graphs, namely lotus flower graph, lotus flower graph with pendant, and variation of lotus flower graph. Furthermore, the three new graph classes satisfy the odd harmonious labeling, making them odd harmonious graphs.

Keywords: Lotus flower graph; Odd harmonious graph; Odd harmonious labeling; Variation of lotus flower graph

1 Introduction

Graph labeling has been extensively studied by researchers worldwide. Beyond its theoretical interest, various applications of graph labeling have been discovered, including its use in cryptography, coding theory, and data security systems [1]. Odd harmonious graphs were first introduced by Liang and Bai in 2009 [2]. A graph $G(V, E)$ of order m and size n with $m, n \in \mathbb{Z}^+$ is said to be a odd harmonious graph if there exists an injective vertex labeling $g : V \to \{0, 1, \ldots, 2n-1\}$ such that the induced edge labeling $g^*(ab) = g(a) + g(b)$ is a bijection from E to the set of odd integers $\{1, 3, \ldots, 2n-1\}$ [2].

Several classes of graphs have been proven by researchers to belong to the family of odd harmonious graphs. Abdel-Aal (2013) demonstrated that cyclic snake graphs are odd harmonious [3]. Jeyanthi, Philo, and Sugeng (2015) identified multiple graph classes that fall into the odd harmonious category [4]. Renuka and Balaganesan (2018) showed that several cycles-related graph classes are also odd harmonious [5]. Jesintha and Stanley (2021) proved that the SSG(2) graph is both odd graceful and odd harmonious [6].

Sarasvati, Halikin, and Wijaya (2021) showed that Here we show that graphs constructed by edge comb product of path Pn and cycle on four vertices C_4 or shadow of a cycle of order four $D_2(C_4)$ are odd harmonious the graphs [7]. Philo and Jeyanthi (2021) demonstrated that both line graphs and disjoint unions of graphs belong to the family of odd harmonious graphs [8]. Additional contributions by Jeyanthi and collaborators can be found in various publications, including [9], [10], [11], [12], [13], [14], [15], [16]. Zala et al. (2021) presented several graph classes having both odd and even harmonious properties [17].

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Asumpta et al. (2022) proved that certain families of snake graphs are odd harmonious [18]. Hafez et al. (2023) demonstrated that the converse skew product of graphs is also an odd harmonious graph [19]. Firmansah (2022) showed that the class of string graphs belongs to the category of odd harmonious graphs [20]. Other studies by Firmansah and collaborators are documented in [21], [22], [23], [24], [25], [26].

Lasim, Halikin, and Wijaya (2022) investigated the relationships among harmonious, odd harmonious, and even harmonious graphs [27]. Venkataraman et al. (2022) contributed further by identifying new properties of odd harmonious graphs [28]. Additional relevant studies on this topic include the works of [29], [30], [31], [32], [33], [34], [35], [36], [37], [38].

Between 2020 and 2023, Firmansah developed a series of flower graph classes that are odd harmonious, including the double quadrilateral flower graph [39], and the zinnia flower graph [40]. Furthermore, it was proven that the union of these flower graph classes also results in an odd harmonious graph [41]. Based on these studies, a new class of graphs known as the lotus flower graph is proposed. In accordance with the research objectives, several novel contributions are made, including the formal definitions of the lotus flower graph, the lotus flower graph with pendant, and their variations. It will also be proven that each class of graphs has the properties of odd harmonious graphs.

2 Methods

This section formalizes the mathematical method used to (i) define the proposed flower–type graph families and (ii) prove that each family admits an odd harmonious labeling. We first fix notation and recall the target labeling property, then present a general labeling framework and a proof template that will be instantiated in Section 3 for the three graph classes introduced in this paper.

Preliminaries and Notation

Let $G = (V, E)$ be a finite, simple, connected graph of order $m = |V|$ and size $n = |E|$. A function $g : V \to \{0, 1, \dots, 2n-1\}$ is *injective* if $g(u) \neq g(v)$ whenever $u \neq v$. The induced edge labeling $g^* : E \to \{1, 3, \dots, 2n-1\}$ is defined by

$$g^*(uv) = g(u) + g(v), \qquad \forall\, uv \in E.$$

Graph G is *odd harmonious* if there exists an injective g such that g^* is a bijection from E onto the set of odd integers $\{1, 3, \dots, 2n-1\}$.

Throughout, $\mathbb{Z}_{\geq 0}$ denotes the nonnegative integers, and for a finite set S we write $\bigsqcup$ for a disjoint union. For an arithmetic progression $A = \{a_0 + a_1 t : t \in I\}$ with integer step a_1, we write $\mathrm{step}(A) = a_1$. We use that the sum of two integers is odd iff they have opposite parity.

Problem Setting

For each parameter tuple θ (e.g., $\theta = p$, or (p, k), or (p, r)), we construct a flower–type graph $G_\theta = (V_\theta, E_\theta)$ by an explicit vertex partition and edge pattern (see the Definitions in Section 3). Let $n_\theta = |E_\theta|$. Our task is to design an injective vertex labeling

$$g_\theta : V_\theta \longrightarrow \{0, 1, \dots, 2n_\theta - 1\}$$

such that the induced $g_\theta^* : E_\theta \to \{1, 3, \dots, 2n_\theta - 1\}$ is bijective.

Labeling Framework

The proposed constructions use the following framework.

(F1) Vertex partition. Write $V_\theta = \bigsqcup_{j=0}^{J} \mathcal{V}_j$, where each block $\mathcal{V}_j$ is indexed by one or two discrete parameters (e.g., h or (h,t)) with prescribed index ranges coming from the graph definition. For instance, for $L(p)$ we use

$$\mathcal{V}_0 = \{a_0\}, \quad \mathcal{V}_1 = \{a_h : 1 \le h \le p+2\}, \quad \mathcal{V}_2 = \{b_h : 1 \le h \le p+1\}, \quad \mathcal{V}_3 = \{c_h : 1 \le h \le p\}.$$

(F2) Affine label families with controlled parity. Assign to each block $\mathcal{V}_j$ an *affine arithmetic progression* of the form

$$g_\theta(x) = \alpha_j + \beta_j \cdot \mathrm{index}(x),$$

with integer coefficients (α_j, β_j) chosen so that:

1. The families $\{g_\theta(\mathcal{V}_j)\}_{j=0}^{J}$ are pairwise disjoint (injectivity).
2. Each edge $uv \in E_\theta$ joins vertices whose labels have *opposite parity* (so $g_\theta(u) + g_\theta(v)$ is odd).
3. The image $g_\theta(V_\theta) \subseteq \{0, 1, \ldots, 2n_\theta - 1\}$.

In practice we take $\beta_j \in \{2, 4, \ldots\}$ and alternate the parity of α_j across blocks to force (2), while ranges and offsets are spaced to achieve (1) and (3).

(F3) Edge–sum stratification. Partition the edge set according to its defining pattern, $E_\theta = \bigsqcup_{\ell=1}^{L} \mathcal{E}_\ell$ (e.g., star edges, rim edges, and petal/bridge edges). For each $\mathcal{E}_\ell$ the induced sums $g_\theta^*(\mathcal{E}_\ell)$ form an arithmetic progression of *odd* integers with step $2 \cdot \gcd(\beta_{j(u)}, \beta_{j(v)})$ determined by the meeting blocks. Carefully chosen (α_j, β_j) make these progressions pairwise disjoint and collectively exhaustive.

Proof Template

Fix θ and a labeling g_θ constructed by (F1)–(F3). The odd harmonious property follows from the next lemmas, applied verbatim to each graph class.

Lemma 1 (Injectivity)**.** *If the label blocks $g_\theta(\mathcal{V}_j)$ are pairwise disjoint arithmetic progressions with distinct offsets modulo $\min_j \beta_j$, then g_θ is injective on V_θ.*

Lemma 2 (Oddness of edge sums)**.** *Suppose that for every edge $uv \in E_\theta$ the incident blocks $\mathcal{V}_{j(u)}$ and $\mathcal{V}_{j(v)}$ are assigned opposite parities (i.e., $\alpha_{j(u)} \equiv \alpha_{j(v)} + 1 \pmod 2$ and $\beta_{j(u)}, \beta_{j(v)}$ are even). Then $g_\theta(u) + g_\theta(v)$ is odd for all $uv \in E_\theta$.*

Lemma 3 (Disjointness of edge–sum bands)**.** *Let $E_\theta = \bigsqcup_{\ell=1}^{L} \mathcal{E}_\ell$. If each $\mathcal{E}_\ell$ induces an odd arithmetic progression $S_\ell = \{s_{\ell,0} + 2d_\ell t\}$ with distinct residues modulo $\mathrm{lcm}(2d_1, \ldots, 2d_L)$, then the sets $S_1, \ldots, S_L$ are pairwise disjoint.*

Lemma 4 (Cardinality and coverage)**.** *If $\bigsqcup_{\ell=1}^{L} S_\ell \subseteq \{1, 3, \ldots, 2n_\theta - 1\}$ and $\sum_{\ell=1}^{L} |S_\ell| = n_\theta$, then $\bigsqcup_{\ell=1}^{L} S_\ell = \{1, 3, \ldots, 2n_\theta - 1\}$. Consequently, g_θ^* is a bijection onto the odd set.*

Verification protocol. For each class G_θ we (1) compute $m_\theta = |V_\theta|$ and $n_\theta = |E_\theta|$ from the construction; (2) specify (α_j, β_j) per block as in Section 3; (3) list the induced edge–sum bands S_ℓ for the edge groups; (4) check Lemmas 1–4. This yields a complete, self-contained proof that G_θ is odd harmonious.

Instantiation Plan for the Three Classes

- $L(p)$ **(Definition 1).** Use four blocks $\mathcal{V}_0, \mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3$ with affine families of steps 2 and alternating parities, as already outlined in the proof of Theorem 1. Edge groups correspond to $a_0 - a_h$, $a_h - b_h$, $a_{h+1} - b_h$, $b_h - c_h$, $b_{h+1} - c_h$.

- $L(p, k)$**.** Append a pendant block $\mathcal{V}_4 = \{d_s\}$ with labels shifted beyond the previous maxima so that $g_\theta(\mathcal{V}_4)$ stays disjoint and preserves opposite parity relative to a_0. The new edge group $a_0 - d_s$ produces a terminal odd progression that fills the topmost odd integers.
- $rL(p)$**.** Replace the single c-block by r replicated c^t-blocks with controlled offsets (α_{c^t}) increasing in t. Each replicated petal induces two edge–sum bands; the step and residues are arranged so that all bands across $t = 1, \ldots, r$ are disjoint and the total cardinality equals n_θ.

Remark on template consistency. The concrete formulas given in Section 3 (Equations (1)–(9), (10)–(20), and (21)–(29)) are specific instances of (F2)–(F3). Their offsets and steps are chosen to satisfy Lemmas 1–4 *by construction*: (i) steps are even, (ii) parities alternate across adjacent blocks, (iii) ranges are nonoverlapping and cover the required index spans, (iv) the resulting edge–sum bands partition $\{1, 3, \ldots, 2n_\theta - 1\}$ exactly.

3 Results and Discussion

In this section, three discussions are given, namely lotus flower graph, lotus flower graph with pendant, and variations of lotus flower graph.

3.1 Lotus Flower Graph

Definition 1 (Lotus Flower Graph)**.** *The lotus flower graph $L(p)$ with $p \geq 1, p \in Z^+$ is a graph defined by $V(L(p))$ as the set of vertices*

$$V(L(p)) = \{a_0\} \cup \{a_h \mid 1 \leq h \leq p+2\} \cup \{b_h \mid 1 \leq h \leq p+1\} \cup \{c_h \mid 1 \leq h \leq p\}$$

and $E(L(p))$ as the set of edges

$$\begin{aligned} E(L(p)) = &\{a_0 a_h \mid 1 \leq h \leq p+2\} \cup \{a_h b_h \mid 1 \leq h \leq p+1\} \\ &\cup \{a_{h+1} b_h \mid 1 \leq h \leq p+1\} \cup \{b_h c_h \mid 1 \leq h \leq p\} \cup \{b_{h+1} c_h \mid 1 \leq h \leq p\}. \end{aligned}$$

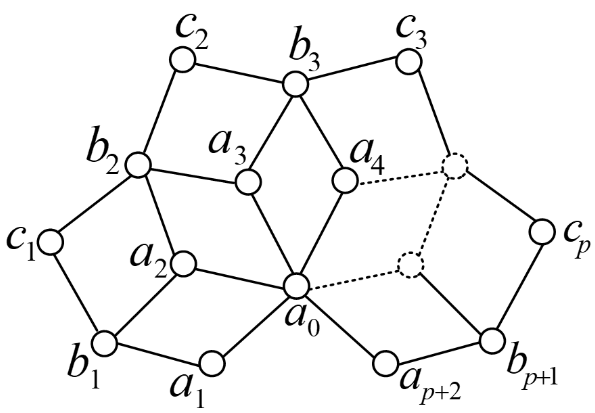

Figure 1: Lotus flower graph $L(p)$

From the construction Fig. 1, we obtain:

$$m = |V(L(p))| = 3p + 4 \quad \text{and} \quad n = |E(L(p))| = 5p + 4.$$

Theorem 1. *Lotus flower graph $L(p)$ with $p \geq 1, p \in Z^+$ is an odd harmonious graph.*

Proof. Defined functions $g : V(L(p)) \to \{0, 1, 2, \ldots, 10p + 7\}$ of the lotus flower graph $L(p)$.

$$g(a_0) = 0 \quad (1)$$
$$g(a_h) = 2h - 1, \quad 1 \le h \le p + 2 \quad (2)$$
$$g(b_h) = 2p + 2h + 2, \quad 1 \le h \le p + 1 \quad (3)$$
$$g(c_h) = 4p + 2h + 3, \quad 1 \le h \le p \quad (4)$$

Based on equation (1), (2), (3) and (4)

$$g(V(L(p)) = \{0\} \cup \{1, 3, 5, \ldots, 2p + 3\} \cup \{2p + 4, 2p + 6, 2p + 8, \ldots, 4p + 4\} \cup \{4p + 5, 4p + 7, 4p + 9, \ldots, 6p + 3\}$$

it is obtained that the function g is injective because it gives different labels to each vertex and $g(V(L(p)) \subseteq \{0, 1, 2, 3, \ldots, 10p + 7\}$.

Defined functions $g^* : E(L(p)) \to \{1, 3, 5, \ldots, 10p + 7\}$ of the lotus flower graph $L(p)$.

$$g^*(a_0 a_h) = 2h - 1, \quad 1 \le h \le p + 2 \quad (5)$$
$$g^*(a_h b_h) = 2p + 4h + 1, \quad 1 \le h \le p + 1 \quad (6)$$
$$g^*(a_{h+1} b_h) = 2p + 4h + 3, \quad 1 \le h \le p + 1 \quad (7)$$
$$g^*(b_h c_h) = 6p + 4h + 5, \quad 1 \le h \le p \quad (8)$$
$$g^*(b_{h+1} c_h) = 6p + 4h + 7, \quad 1 \le h \le p \quad (9)$$

Based on equation (5), (6), (7), (8), and (9)

$$g^*(E(L(p)) = \{1, 3, 5, \ldots, 2p + 3\} \cup \{2p + 5, 2p + 9, 2p + 13, \ldots, 6p + 5\} \cup \{2p + 7, 2p + 11, 2p + 15, \ldots, 6p + 7\} \cup \{6p + 9, 6p + 13, 6p + 17, \ldots, 10p + 5\} \cup \{6p + 11, 6p + 15, 6p + 19, \ldots, 10p + 7\}$$

it is obtained that the function g^* is bijective because it gives different labels to each edge and $g^*(E(L(p)) = \{1, 3, 5, \ldots, 10p + 7\}$. Such that lotus flower graph $L(p)$ is an odd harmonious. □

The following is an example of a lotus flower graph $L(6)$ in Fig. 2 and a lotus flower graph $L(7)$ in Fig. 3

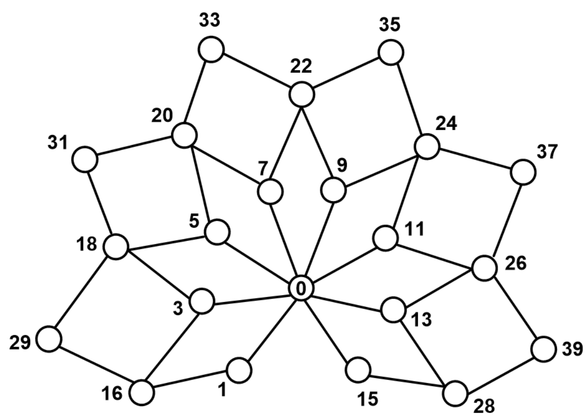

Figure 2: Lotus flower graph $L(6)$

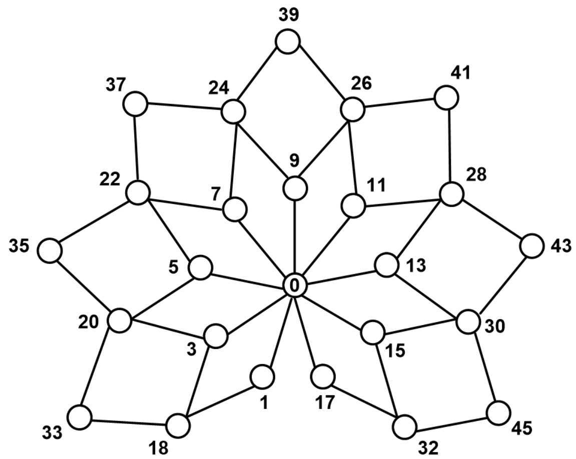

Figure 3: Lotus flower graph $L(7)$

3.2 Lotus Flower Graph with Pendant

Definition 2. *Lotus flower graph with pendant $L(p,k)$ with $p \geq 1, p \in Z^+$ and $k \geq 1, k \in Z^+$ is a graph defined by $V(L(p,k))$ as the set of vertices*

$$V(L(p,k)) = \{a_0\} \cup \{a_h \mid 1 \leq h \leq p+2\} \cup \{b_h \mid 1 \leq h \leq p+1\} \cup \{c_h \mid 1 \leq h \leq p\} \cup \{d_s \mid 1 \leq s \leq k\}$$

and by $E(L(p,k))$ as the set of edges

$$\begin{aligned} E(L(p,k)) = &\{a_0 a_h \mid 1 \leq h \leq p+2\} \cup \{a_h b_h \mid 1 \leq h \leq p+1\} \cup \{a_{h+1} b_h \mid 1 \leq h \leq p+1\} \\ &\cup \{b_h c_h \mid 1 \leq h \leq p\} \cup \{b_{h+1} c_h \mid 1 \leq h \leq p\} \cup \{a_0 d_s \mid 1 \leq s \leq k\}. \end{aligned}$$

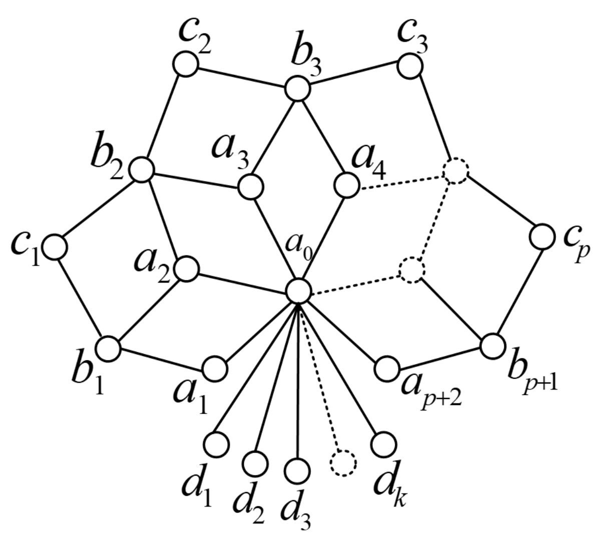

Figure 4: Lotus flower graph with pendant $L(p,k)$

From the construction Fig. 4, we obtain: $m = |V(L(p,k))| = 3p+k+4$ and $n = |E(L(p,k))| = 5p+k+4$.

Theorem 2. *Lotus flower graph with pendant $L(p,k)$ with $p \geq 1, p \in Z^+$ and $k \geq 1, k \in Z^+$ is an odd harmonious graph.*

Proof. Defined functions $g : V(L(p,k)) \to \{0, 1, 2, \ldots, 10p + 2k + 7\}$ of the lotus flower graph with pendant $L(p,k)$.

$$g(a_0) = 0 \quad (10)$$
$$g(a_h) = 2h-1, \quad 1 \le h \le p+2 \quad (11)$$
$$g(b_h) = 2p+2h+2, \quad 1 \le h \le p+1 \quad (12)$$
$$g(c_h) = 4p+2h+3, \quad 1 \le h \le p \quad (13)$$
$$g(d_s) = 10p+2s+7, \quad 1 \le s \le k \quad (14)$$

Based on equation (10), (11), (12), (13) and (14)

$$\begin{aligned} g(V(L(p,k)) =& \{0\} \cup \{1,3,5,\dots,2p+3\} \cup \{2p+4, 2p+6, 2p+8, \dots, 4p+4\} \\ & \cup \{4p+5, 4p+7, 4p+9, \dots, 6p+3\} \cup \{10p+9, 10p+11, \dots, 10p+2k+7\} \end{aligned}$$

it is obtained that the function g is injective because it gives different labels to each vertex and $g(V(L(p,k)) \subseteq \{0,1,2,\dots,10p+2k+7\}$.

Defined functions $g^* : E(L(p,k)) \to \{1,3,5,\dots,10p+2k+7\}$ of the lotus flower graph with pendant $L(p,k)$.

$$g^*(a_0a_h) = 2h-1, \quad 1 \le h \le p+2 \quad (15)$$
$$g^*(a_hb_h) = 2p+4h+1, \quad 1 \le h \le p+1 \quad (16)$$
$$g^*(a_{h+1}b_h) = 2p+4h+3, \quad 1 \le h \le p+1 \quad (17)$$
$$g^*(b_hc_h) = 6p+4h+5, \quad 1 \le h \le p \quad (18)$$
$$g^*(b_{h+1}c_h) = 6p+4h+7, \quad 1 \le h \le p \quad (19)$$
$$g^*(a_0d_s) = 10p+2s+7, \quad 1 \le s \le k \quad (20)$$

Based on equation (15), (16), (17), (18), (19) and (20)

$$\begin{aligned} g^*(E(L(p,k)) =& \{1,3,5,\dots,2p+3\} \cup \{2p+5, 2p+9, 2p+13, \dots, 6p+5\} \\ & \cup \{2p+7, 2p+11, 2p+15, \dots, 6p+7\} \cup \{6p+9, 6p+13, 6p+17, \dots, 10p+5\} \\ & \cup \{6p+11, 6p+15, 6p+19, \dots, 10p+7\} \cup \{10p+9, 10p+11, \dots, 10p+2k+7\} \end{aligned}$$

it is obtained that the function g^* is bijective because it gives different labels to each edge and $g^*(E(L(p,k)) = \{1,3,5,\dots,10p+2k+7\}$. Such that lotus flower graph with pendant $L(p,k)$ is an odd harmonious. □

The following is an example of a lotus flower graph with pendant $L(6,5)$ in Fig. 5 and lotus flower graph with pendant $L(7,3)$ Fig. 6

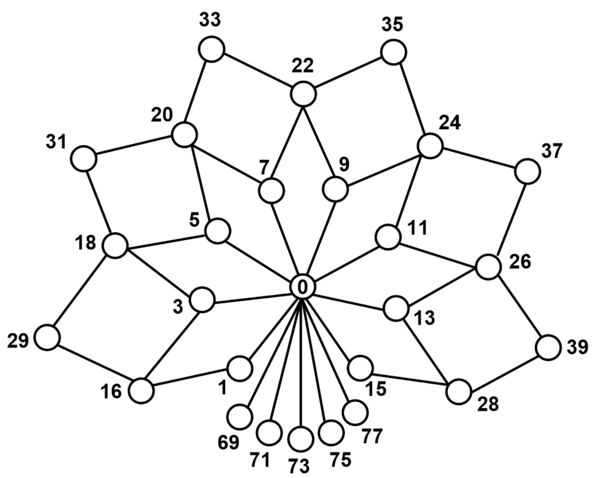

Figure 5: Lotus flower graph with pendant $L(6,5)$

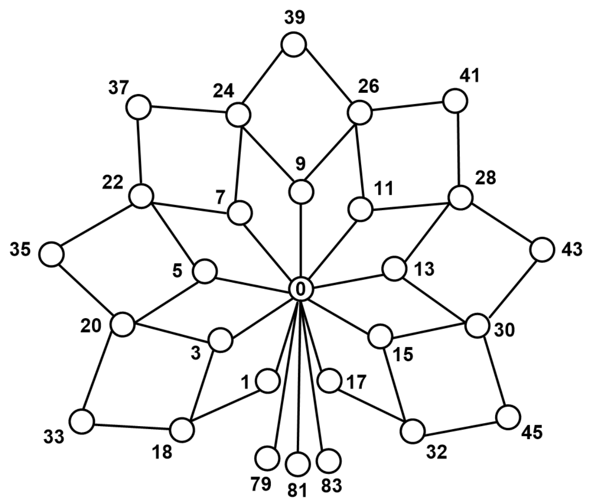

Figure 6: Lotus flower graph with pendant $L(7,3)$

3.3 Variation of Lotus Flower Graph

Definition 3. *Variation of lotus flower graph $rL(p)$ with $p \geq 1, p \in Z^+$ and $r \geq 1, r \in Z^+$ is a graph defined by $V(rL(p))$ as the set of vertices*

$$V(rL(p)) = \{a_0\} \cup \{a_h \mid 1 \leq h \leq p+2\} \cup \{b_h \mid 1 \leq h \leq p+1\} \cup \{c_h^t \mid 1 \leq h \leq p,\ 1 \leq t \leq r\}$$

and $E(rL(p))$ as the set of edges

$$\begin{aligned} E(rL(p)) = &\{a_0 a_h \mid 1 \leq h \leq p+2\} \cup \{a_h b_h \mid 1 \leq h \leq p+1\} \cup \{a_{h+1} b_h \mid 1 \leq h \leq p+1\} \\ &\cup \{b_h c_h^t \mid 1 \leq h \leq p,\ 1 \leq t \leq r\} \cup \{b_{h+1} c_h^t \mid 1 \leq h \leq p,\ 1 \leq t \leq r\}. \end{aligned}$$

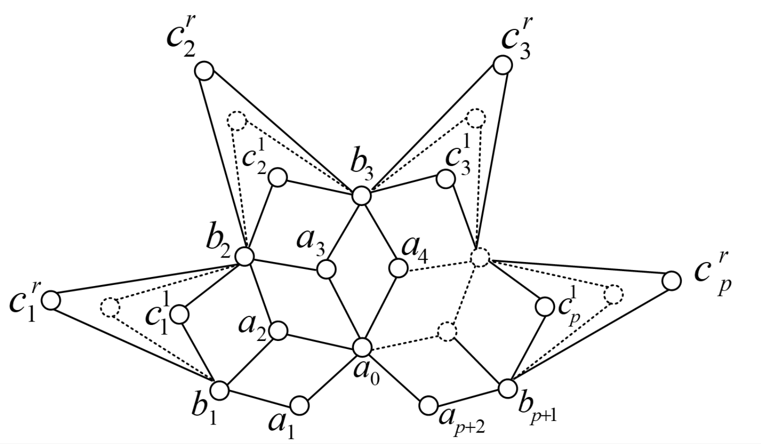

Figure 7: Variation of lotus flower graph $rL(p)$

From the construction Fig. 7, we obtain: $m = |V(rL(p))| = rp + 2p + 4$ and $n = |E(rL(p))| = 2rp + 3p + 4$.

Theorem 3. *Variation of lotus flower graph $rL(p)$ with $p \geq 1, p \in Z^+$ and $r \geq 1, r \in Z^+$ is an odd harmonious graph.*

Proof. Defined functions $g : V(rL(p)) \to \{0, 1, 2, \ldots, 4rp + 6p + 7\}$ of the variation of lotus flower graph $rL(p)$.

$$g(a_0) = 0 \tag{21}$$
$$g(a_h) = 2h - 1, \quad 1 \leq h \leq p + 2 \tag{22}$$
$$g(b_h) = 2p + 2h + 2, \quad 1 \leq h \leq p + 1 \tag{23}$$
$$g(c_h^t) = 4tp + 2h + 3, \quad 1 \leq h \leq p, \ 1 \leq t \leq r \tag{24}$$

Based on equation (21), (22), (23), and (24)

$$\begin{aligned} g(V(rL(p)) =& \{0\} \cup \{1, 3, 5, \ldots, 2p + 3\} \cup \{2p + 4, 2p + 6, 2p + 8, \ldots, 4p + 4\} \\ & \cup \{4p + 5, 4p + 7, 4p + 9, \ldots, 6p + 3, 8p + 5, 8p + 7, 8p + 9, \ldots, 10p + 3, \\ & 12p + 5, 12p + 7, 12p + 9, \ldots, 14p + 3, \ldots, 4rp + 5, 4rp + 7, 4rp + 9, \ldots, 4rp + 2p + 3\} \end{aligned}$$

it is obtained that the function g is injective because it gives different labels to each vertex and $g(V(rL(p)) \subseteq \{0, 1, 2, \ldots, 4rp + 6p + 7\}$.

Defined functions $g^* : E(rL(p)) \rightarrow \{1, 3, 5, \ldots, 4rp + 6p + 7\}$ of the variation of lotus flower graph $rL(p)$.

$$g^*(a_0a_h) = 2h - 1, \quad 1 \leq h \leq p + 2 \tag{25}$$
$$g^*(a_hb_h) = 2p + 4h + 1, \quad 1 \leq h \leq p + 1 \tag{26}$$
$$g^*(a_{h+1}b_h) = 2p + 4h + 3, \quad 1 \leq h \leq p + 1 \tag{27}$$
$$g^*(b_hc_h^t) = 4tp + 2p + 4h + 5, \quad 1 \leq h \leq p, \ 1 \leq t \leq r \tag{28}$$
$$g^*(b_{h+1}c_h^t) = 4tp + 2p + 4h + 7, \quad 1 \leq h \leq p, \ 1 \leq t \leq r \tag{29}$$

Based on equation (25), (26), (27), (28), and (29)

$$\begin{aligned} g^*(E(rL(p)) =& \{1, 3, 5, \ldots, 2p + 3\} \cup \{2p + 5, 2p + 9, 2p + 13, \ldots, 6p + 5\} \\ & \cup \{2p + 7, 2p + 11, 2p + 15, \ldots, 6p + 7\} \\ & \cup \{6p + 9, 6p + 13, 6p + 17, \ldots, 10p + 5, 10p + 9, 10p + 13, 10p + 17, \ldots, 14p + 5, \\ & 14p + 9, 14p + 13, 14p + 17, \ldots, 18p + 5, \ldots, \\ & 4rp + 2p + 9, 4rp + 2p + 13, 4rp + 2p + 17, \ldots, 4rp + 6p + 5\} \\ & \cup \{6p + 11, 6p + 15, 6p + 19, \ldots, 10p + 7, 10p + 11, 10p + 15, 10p + 19, \ldots, 14p + 7, \\ & 14p + 11, 14p + 13, 14p + 19, \ldots, 18p + 7, \ldots, \\ & 4rp + 2p + 11, 4rp + 2p + 15, 4rp + 2p + 19, \ldots, 4rp + 6p + 7\} \end{aligned}$$

it is obtained that the function g^* is bijective because it gives different labels to each edge and $g^*(E(rL(p)) = \{1, 3, 5, \ldots, 4rp + 6p + 7\}$. Such that variation of lotus flower graph $rL(p)$ is an odd harmonious. □

The following is an example of variation of lotus flower graph $5L(6)$ in Fig. 8 and variation of lotus flower graph $4L(7)$ Fig. 9.

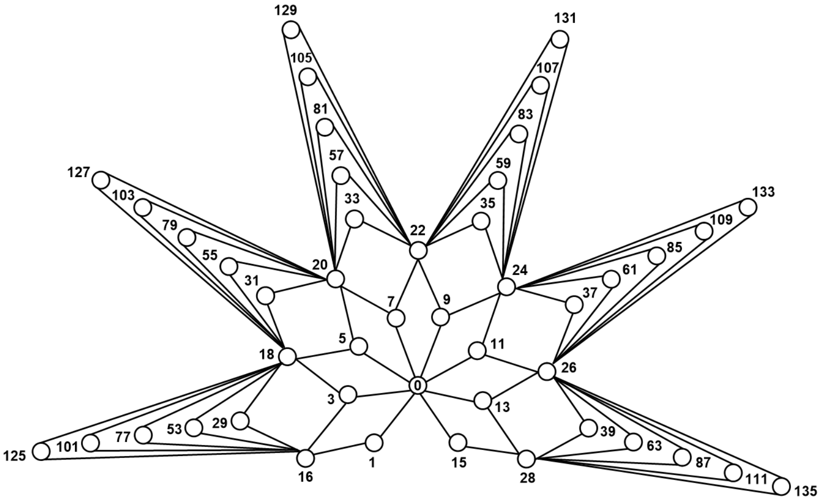

Figure 8: Variation of lotus flower graph $5L(6)$

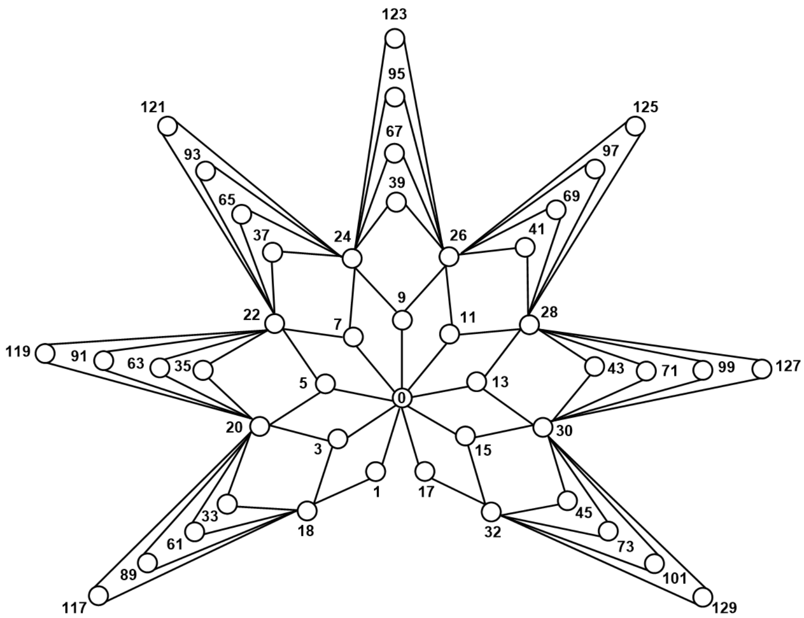

Figure 9: Variation of lotus flower graph $4L(7)$

Based on the results stated in the definition, a new class of graphs is obtained from the flower graph, namely lotus flower graph, lotus flower graph with pendant and variations of lotus flower graph. Based on the results of Theorem 1, Theorem 2, and Theorem 3, it has been proven that lotus flower graph, lotus flower graph with pendant, and variations of lotus flower graph are odd harmonious graphs. This result is in line with the research of [39], [40], and [41].

4 Conclusion and Suggestion

Based on the results obtained in the form of flower graph classes, namely lotus flower graph, lotus flower graph with pendant, and variations of lotus flower graph. Furthermore, it has also been proven that lotus flower graphs, lotus flower graphs with pendant, and variations of lotus flower graphs are odd harmonious graphs. A suggestion for future researchers is to find new graph classes along with their labeling types.

CRediT Authorship Contribution Statement

Fery Firmansah: Conceptualization, Methodology, Writing–Original Draft, Supervision, Project Administration, Funding Acquisition. **Tasari:** Data Curation, Formal Analysis, Writing–Review & Editing. **Muhammad Ridlo Yuwono:** Software, Validation, Visualization.

Declaration of Generative AI and AI-assisted technologies

No generative AI or AI-supported technology was used during the preparation of this manuscript.

Declaration of Competing Interest

The authors declare no conflict of interest.

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Data and Code Availability

The data and code supporting the findings of this study are available from the corresponding author upon reasonable request and subject to confidentiality agreements.

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