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Pricing Double Barrier Options with Time-Varying Interest using Standard, Antithetic, and Control Variate Monte Carlo

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Abstract

This study develops an integrated framework for pricing double barrier options under time-varying interest rates by combining ARIMA-based forecasting with Monte Carlo simulations. Monthly U.S. Treasury Bill rates from 2019–2025 are modeled using the ARIMA(2,2,0) process to generate dynamic risk-free rates, which are incorporated into three Monte Carlo approaches standard, antithetic variate, and control variate. Tesla Inc. stock prices are used as the underlying asset modeled through Geometric Brownian Motion. The integration of ARIMA-based dynamic rates within the Monte Carlo framework enables more realistic pathwise discounting and improves simulation convergence. The results show that the control variate method provides the most accurate and stable estimates for knock-in call options, whereas the antithetic variate technique yields superior accuracy for knock-in put, knock-out call, and knock-out put options. Overall, the combined use of ARIMA-forecasted interest rates and variance-reduction techniques enhances the precision and stability of double barrier option valuation under dynamic financial conditions.

Keywords: antithetic variate; ARIMA; control variate; double barrier option; Monte Carlo simulation

1 Introduction

Investment plays a crucial role in the modern economy, functioning both as a channel for productive capital allocation and as a means to enhance financial well-being. In recent years, participation in capital markets has grown substantially worldwide. In Indonesia, the number of investors increased by 18.01% in 2023 compared to the previous year, exceeding seven million Single Investor Identifications (SID) by mid-2025 [1]. Globally, more than 100 million individuals now use stock trading applications, with an annual growth rate of around 20% [2]. Despite this rapid expansion, market volatility continues to pose significant risks for investors.

Derivative instruments, particularly options, serve as important tools for hedging and speculation. Among them, double barrier options have gained increasing attention because their payoff depends on whether the underlying asset price breaches predetermined upper or lower barriers. There are two main types: knock-in options, which become active only after a barrier is hit, and knock-out options, which become void once the barrier is reached. Due to their relatively lower premiums compared to vanilla options, barrier options are often preferred in markets with moderate volatility [3], [4].

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However, pricing barrier options is challenging due to their path-dependent nature, where the option value depends on the entire price trajectory rather than only the terminal price at maturity [5]. This characteristic limits the analytical applicability of classical models such as the Black-Scholes framework. Consequently, Monte Carlo simulation has become a widely used approach for pricing barrier options because it captures realistic market behavior. Nevertheless, standard Monte Carlo simulations often suffer from slow convergence and high estimation variance. To address this, variance reduction techniques such as antithetic variates and control variates are commonly employed to enhance computational efficiency [6].

Previous studies have demonstrated that the control variate method improves convergence for knock-in options, while the antithetic variate approach effectively reduces variance in knock-out options [7], [8]. Further extensions include models incorporating stochastic volatility using the Heston framework [9] and non-constant interest rates via the Cox-Ingersoll-Ross (CIR) process [10], both of which improve pricing realism and stability.

Despite these advancements, most prior research analyzes either knock-in or knock-out options separately. This study addresses that gap by evaluating both option types simultaneously under a time-varying interest rate framework. The research introduces an integrated simulation approach that combines ARIMA-based interest rate forecasting with three Monte Carlo methods: standard, antithetic variate, and control variate, to assess their relative performance. Using Tesla Inc. (TSLA) stock prices as the underlying asset, this study aims to improve pricing accuracy and provide a more realistic model of option valuation in dynamic financial markets.

2 Methods

This study adopts an experimental quantitative approach based on numerical simulation to estimate the price of double barrier options under time-varying interest rates. The simulation is conducted using three Monte Carlo techniques: the standard method, antithetic variate, and control variate. This simulation-based approach is employed due to the absence of closed-form analytical solutions for double barrier options with non-constant interest rates, necessitating numerical methods to obtain accurate estimates [3], [6].

2.1 Data and Sources

The study utilizes secondary data comprising two main types. First, daily stock price data of Tesla Inc. (TSLA) from August 23, 2023, to August 23, 2025, obtained from Yahoo Finance [11]. This data is used to model the stochastic price process in the simulations. Second, monthly U.S. Treasury Bill rates from 2019 to 2025, sourced from the official website of the U.S. Department of the Treasury [12], are used to forecast non-constant interest rates through the ARIMA method.

2.2 Return and Normality Testing

Daily stock returns are computed using the logarithmic transformation:

$$R_t = \ln\left(\frac{S_t}{S_{t-1}}\right), \tag{1}$$

where S_t represents the stock price on day-t [3]. Prior to simulation, the return data is tested for normality using the Kolmogorov–Smirnov (K–S) test to ensure it follows a normal distribution. The test statistic is defined as:

$$D = \max|F_n(x) - F_0(x)|, \tag{2}$$

where $F_n(x)$ is the empirical distribution function, and $F_0(x)$ is the theoretical distribution. The data is considered normally distributed if the D-value is less than the critical value at a given significance level [13].

2.3 Modeling Non-Constant Interest Rates Using ARIMA

Capturing the temporal variability of interest rates involves employing the Autoregressive Integrated Moving Average (ARIMA) model, denoted as ARIMA(p, d, q), which is used to generate monthly forecasts of interest rates serving as inputs for the Monte Carlo simulation.

2.3.1 Stationarity Testing

Stationarity of the data is assessed with respect to its mean. The Augmented Dickey–Fuller (ADF) test is employed to evaluate whether the mean of the series is constant over time. The test is based on the following regression model:

$$\Delta X_t = \gamma X_{t-1} + \sum_{j=2}^{p} \beta_j \Delta X_{t-j+1} + w_t, \tag{3}$$

where the null hypothesis (H_0): $\gamma = 0$ indicates the presence of a unit root, implying non-stationarity, while the alternative hypothesis (H_1): $\gamma < 0$ indicates mean stationarity. The series is considered stationary if the ADF test statistic is smaller than the corresponding Dickey–Fuller critical value [14]. If non-stationarity is detected, differencing is applied until the mean becomes stable:

$$\nabla X_t = X_t - X_{t-1}, \tag{4}$$

2.3.2 Model Identification and Parameter Estimation

Model identification is performed using autocorrelation (ACF) and partial autocorrelation (PACF) functions. A cutoff in ACF typically suggests an MA(q) structure, while a PACF cutoff points to an AR(p) structure. If both decay gradually, a combined ARMA or ARIMA model is likely suitable. The parameters are estimated through Maximum Likelihood Estimation (MLE), and significance tests are applied to retain only meaningful coefficients. Model selection is guided by the lowest values of the Akaike Information Criterion (AIC) and the Bayesian Information Criterion (BIC):

$$AIC = -2\ln L + 2k, \tag{5}$$

$$BIC = -2\ln L + k\ln n, \tag{6}$$

where L is the log-likelihood and k is the number of estimated parameters [15].

2.3.3 Diagnostic Testing

Post-estimation diagnostics are carried out to validate model adequacy. The Ljung–Box test is employed to detect any remaining autocorrelation in the residuals [15]:

$$Q(m) = n(n+2)\sum_{k=1}^{m} \frac{\hat{\rho}_k^2}{n-k}. \tag{7}$$

In addition, a t-test on the residual mean is performed to ensure its expectation is zero [13]:

$$t = \frac{\bar{\varepsilon}}{s_\varepsilon/\sqrt{n}}. \tag{8}$$

2.3.4 Forecasting and Evaluation of Accuracy

The ARIMA model produces interest rate forecasts through conditional expectation:

$$\hat{X}_{t+h|t} = E(X_{t+h}|X_t, X_{t-1}, \ldots), \tag{9}$$

with 95% confidence intervals defined as:

$$\hat{X}_{t+h|t} \pm 1.96\hat{\sigma}_h. \tag{10}$$

Forecast accuracy is assessed via the Mean Absolute Percentage Error (MAPE):

$$MAPE = 100\% \times \frac{1}{n}\sum_{t=1}^{n}\left|\frac{X_t - \hat{X}_t}{X_t}\right|. \tag{11}$$

MAPE values below 20% indicate high predictive accuracy, while those exceeding 50% suggest poor forecasting performance [15],[16],[17].

2.4 Double Barrier Option

Double barrier options are a class of exotic derivatives characterized by two threshold prices: a lower barrier (H_L) and an upper barrier (H_U). These options either activate (knock-in) or become void (knock-out) depending on whether the underlying asset price crosses the set barriers[18].

The corresponding payoff functions for various option types are:

$$Knock\ Out\ Call = \begin{cases} \max(S_T - K, 0), & S_t \in (H_L, H_U),\ \forall t \in [0,T], \\ 0, & \text{otherwise}, \end{cases} \tag{12}$$

$$Knock\ In\ Call = \begin{cases} \max(S_T - K, 0), & S_t \notin (H_L, H_U),\ \exists t \in [0,T], \\ 0, & \text{otherwise}, \end{cases} \tag{13}$$

$$Knock\ Out\ Put = \begin{cases} \max(K - S_T, 0), & S_t \in (H_L, H_U),\ \forall t \in [0,T], \\ 0, & \text{otherwise}, \end{cases} \tag{14}$$

$$Knock\ In\ Put = \begin{cases} \max(K - S_T, 0), & S_t \notin (H_L, H_U),\ \exists t \in [0,T], \\ 0, & \text{otherwise}. \end{cases} \tag{15}$$

where S_T is the asset price at maturity and K is the strike price [5]. To address the effect of discrete monitoring, a continuity correction is applied. Monitoring is performed daily ($n = 252 \times T$) and applies the discrete-to-continuous adjustment factor $0.5826\,\sigma\sqrt{T/n}$:

$$H^* = H\exp(\pm 0.5826\,\sigma\sqrt{T/n}), \tag{16}$$

where the positive sign is used for upper barriers and the negative for lower ones, with the initial barrier level defined as $H_0 = \max(S_0, K)$ and varied according to $H = H_0(1 + s\%)$, where s denotes the percentage shift applied to move the barrier upward or downward relative to H_0.

A key identity in the literature relates knock-in and knock-out options to a vanilla option:

$$\text{Knock-In Option} + \text{Knock-Out Option} = \text{Vanilla Option} \tag{17}$$

This equation highlights the complementary nature of knock-in and knock-out structures [3].

2.5 Monte Carlo Simulation

Monte Carlo methods are employed to compute option prices by simulating numerous stochastic price paths for the underlying asset. This study compares three simulation techniques: standard Monte Carlo, antithetic variates, and control variates.

2.5.1 Standard Monte Carlo Method

The Monte Carlo method is employed to estimate option prices by generating a total of n standard normally distributed random variables, denoted as Z_i. The stock price evolution, which is inherently *path-dependent*, is modeled using the following stochastic process:

$$S_i(t_{j+1}) = S_i(t_j)\exp\left(\left(r_j - \tfrac{1}{2}\sigma^2\right)\Delta t + \sigma\sqrt{\Delta t}Z_{i,j}\right). \tag{18}$$

The random variables Z_i are derived from uniformly distributed pseudo-random numbers $U_i \sim U(0,1)$, which are then transformed into standard normal variables. This transformation can be accomplished using either the inverse cumulative distribution function $Z = \Phi^{-1}(U)$ or the Box–Muller method:

$$Z_{i,2j-1} = \sqrt{-2\ln U_{i,j}^{(1)}}\cos(2\pi U_{i,j}^{(2)}), \tag{19}$$

$$Z_{i,2j} = \sqrt{-2\ln U_{i,j}^{(1)}}\sin(2\pi U_{i,j}^{(2)}). \tag{20}$$

Using the generated Z values, the stock price paths are simulated under time-varying interest rates, and the corresponding discounted payoff values of the double-barrier options are computed. For call and put options, the discounted payoffs are respectively defined as:

$$C_i = e^{-\sum_{j=0}^{n-1} r_j\,\Delta t}(\text{payoff call}), \tag{21}$$

$$P_i = e^{-\sum_{j=0}^{n-1} r_j\,\Delta t}(\text{payoff put}). \tag{22}$$

In this study, the time-varying interest rates r_j obtained from the ARIMA forecasts are specified on a monthly basis and mapped onto the simulation grid by assigning each forecasted value to the corresponding daily steps within that month. Both the drift term $(r_j - 0.5\sigma^2)\Delta t$ and the discount factor $\exp(-\sum_j r_j\Delta t)$ use the same mapped sequence r_j, ensuring consistency between the stochastic price evolution and the present-value calculation. The estimated option prices are then obtained by taking the average over all simulated paths:

$$\hat{C}_n = \frac{1}{n}\sum_{i=1}^{n} C_i, \tag{23}$$

$$\hat{P}_n = \frac{1}{n}\sum_{i=1}^{n} P_i. \tag{24}$$

In addition to pricing estimation, the Monte Carlo simulation also provides an estimate of the standard error, which quantifies the uncertainty of the result:

$$\widehat{SE}(\hat{\alpha}_n) = \frac{s_f}{\sqrt{n}}, \tag{25}$$

$$s_f = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\left(f(U_i) - \hat{\alpha}_n\right)^2}, \tag{26}$$

$$\hat{\alpha}_n = \frac{1}{n}\sum_{i=1}^{n} f(U_i). \tag{27}$$

Here, s_f represents the sample standard deviation, and n denotes the number of simulated price paths [6].

2.5.2 Antithetic Variate Method

The antithetic variate method utilizes paired random variables Z_i and $-Z_i$ to reduce the variance of the estimator. The stock prices at time $t + \Delta t$ for each pair are expressed as follows:

$$S_i^{(+)}(t + \Delta t) = S_i(t) \exp\left[(r_j - \tfrac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t}\, Z_i\right], \quad (28)$$

$$S_i^{(-)}(t + \Delta t) = S_i(t) \exp\left[(r_j - \tfrac{1}{2}\sigma^2)\Delta t - \sigma\sqrt{\Delta t}\, Z_i\right]. \quad (29)$$

The option price estimator for this method is computed as the mean of the payoffs obtained from both paths:

$$\widehat{V}_{Anti} = \frac{1}{N}\sum_{i=1}^{N} \frac{C_i^{(+)} + C_i^{(-)}}{2}. \quad (30)$$

By using antithetic pairs, this method significantly decreases the variance of the simulation results without altering the expected value of the estimator [6].

2.5.3 Control Variate Method

In the *control variate* approach, a European vanilla option is employed as a control variable whose analytical expected value is known from the Black–Scholes framework (Hull, 2021). The theoretical price of the European option is expressed as:

$$C_{BS} = S_0 N(d_+) - K e^{-rT} N(d_-), \quad (31)$$

where:

$$d_\pm = \frac{\ln(S_0/K) + (r \pm \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}. \quad (32)$$

In the control variate method, a European vanilla option is used as the control variable because its analytical price is known from the Black–Scholes model. The control variate estimator is defined as:

$$\theta_{CV}^{(i)} = C_i + \theta(C_{BS} - Y_i), \quad (33)$$

where C_i denotes the simulated payoff of the double barrier option, Y_i represents the simulated payoff of the corresponding vanilla option computed along the same simulated price path, and C_{BS} is the theoretical price of the vanilla option under the Black–Scholes model. Here, θ corresponds to θ_c for call options and θ_p for put options. In the numerical implementation, these coefficients are estimated directly from simulated payoffs as

$$\theta_c = \frac{\mathrm{Cov}(C, C_E)}{\mathrm{Var}(C_E)}, \quad (34)$$

$$\theta_p = \frac{\mathrm{Cov}(P, P_E)}{\mathrm{Var}(P_E)}, \quad (35)$$

where C_E and P_E denote the simulated payoffs of the vanilla call and put options, respectively. The final estimator of the option price is computed as:

$$\widehat{V}_{CV} = \frac{1}{N}\sum_{i=1}^{N} \theta_{CV}^{(i)}, \quad (36)$$

This method proves to be effective because the double-barrier and European option payoffs exhibit a strong linear dependence, which enables the control variate adjustment (through the coefficient to substantially reduce the estimation variance [6]

2.6 Notation

The main variables and symbols used throughout this study are summarized in Table 1.

Table 1: Summary of notation used throughout the paper.

Symbol	Description
S_t	Stock price at time t
S_0	Initial stock price
K	Strike price
T	Time to maturity (in years)
r_j	Time-varying risk-free rate at step j
σ	Annualized volatility
Δt	Time increment per simulation step (T/n)
n	Number of time steps in each simulation ($252 \times T$ for daily monitoring)
M	Number of Monte Carlo simulation paths
H_L, H_U	Lower and upper barrier levels
H^*	Continuity-corrected barrier level
Z_i	Standard normal random variable at step i
$N(\cdot)$	Standard normal cumulative distribution function (CDF)
c, p	Call and put option values
C_i, P_i	Discounted payoffs for call and put in simulation path i
D	Discount factor $\exp(-\sum_j r_j \Delta t)$
r	Equivalent constant annual rate for Black–Scholes benchmark

2.7 Research Stages

1. **Data Collection and Preparation:** Collect daily closing prices of Tesla Inc. (TSLA) as the underlying asset and monthly U.S. Treasury Bill rates as the risk-free interest rate inputs. The data are cleaned, aligned by date, and converted into consistent time series formats for further analysis.
2. **Interest Rate Modeling (ARIMA):** Model non-constant interest rates using the ARIMA(p, d, q) framework. The procedure includes testing for stationarity using the Augmented Dickey–Fuller (ADF) test, identifying and estimating parameters via ACF and PACF with Maximum Likelihood Estimation (MLE), conducting diagnostic checks using the Ljung–Box Q-test and residual t-test, and evaluating forecast accuracy through the Mean Absolute Percentage Error (MAPE).
3. **Parameter Estimation:** Calculate daily log-returns and assess their distributional properties using the Kolmogorov–Smirnov normality test. The estimated parameters — volatility (σ), initial price (S_0), strike price (K), maturity (T), lower and upper barriers (L, U), number of time steps (n), and number of simulations (M) — are used as inputs in the option pricing model.
4. **Monte Carlo Simulation:** Estimate the fair value of double-barrier options using three Monte Carlo approaches: standard, antithetic variates, and control variates. Each simulation employs daily monitoring with time-varying interest rates r_t, drift and discounting based on the same rate mapping, and a continuity correction for discrete barriers.
5. **Result Evaluation:** Compare the simulation outcomes in terms of pricing accuracy, standard error, and computational efficiency. The results are analyzed to determine which Monte Carlo method provides the most accurate, stable, and computationally efficient valuation for double-barrier options.

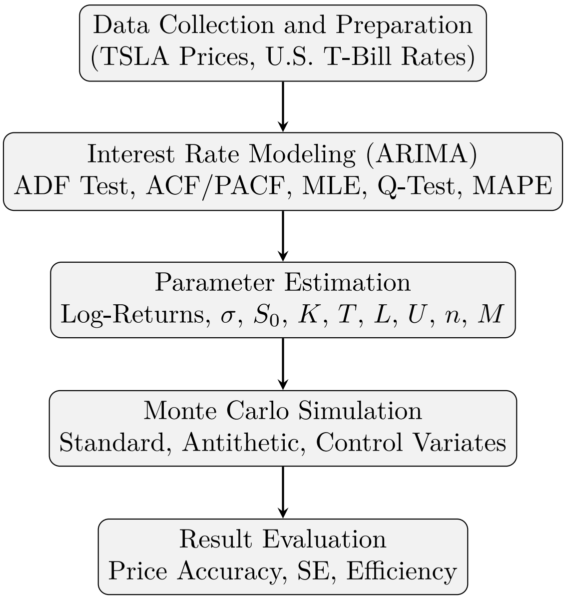

Figure 1: Flowchart of the research methodology.

3 Results and Discussion

This section presents the results of the data analysis, interest rate modeling using the ARIMA method, and the valuation of double barrier options via Monte Carlo simulation. Three simulation techniques are implemented: standard Monte Carlo, antithetic variates, and control variates.

3.1 Results

The dataset employed in this study comprises the daily closing prices of Tesla Inc. (TSLA) shares from August 23, 2023, to August 23, 2025, and monthly USD interest rates from 2019 to 2025. The Tesla stock prices have an average of \$260.51 with a standard deviation of \$70.13, ranging from \$142.05 to \$479.86. The Kolmogorov–Smirnov test yields a statistic of 0.0567 with a p-value of 0.0763 (> 0.05), indicating that the log returns are normally distributed. In addition, a sensitivity analysis varying the drift (μ) and volatility (σ) parameters by $\pm 10\%$ shows terminal price deviations below 0.1%, confirming that the model outputs are stable and supporting the robustness of the GBM assumption. The monthly interest rates, with a mean of 2.58% and a standard deviation of 2.08% (ranging from 0.01% to 5.44%), are used to construct a time-varying risk-free rate series r_j through an ARIMA forecasting model.

The first step in the ARIMA modeling process is to test the stationarity of the interest rate series using the Augmented Dickey–Fuller (ADF) test. The original interest rate data is found to be non-stationary ($p\text{-}value = 0.7125 > 0.05$). After the first differencing, the series remains non-stationary ($p\text{-}value = 0.1432 > 0.05$), but it becomes stationary after the second differencing ($p\text{-}value = 0.0000 < 0.05$). This result indicates that the interest rate data follows an ARIMA$(p, 2, q)$ process.

Table 2: ADF Stationarity Test Results for Interest Rate Data

Condition	**ADF Statistic**	**p-value**	**Decision**
Before differencing	-1.8742	0.7125	Non-stationary
After 1st differencing	-2.9864	0.1432	Non-stationary
After 2nd differencing	-8.3471	0.0000	Stationary

The ACF and PACF plots indicate that the differenced interest rate data follow a low-order process. Several ARIMA configurations were estimated to determine the most appropriate model specification. Based on the information criteria and forecasting performance, the ARIMA(3,2,0) model yields the lowest AIC value of -572.5303 and BIC -565.9162, while the ARIMA(2,2,0) model provides the smallest MAPE value of 14.2294%. Both models demonstrate adequate residual diagnostics, confirming their suitability for modeling the interest rate dynamics.

Table 3: ARIMA Model Estimation and Evaluation Results

Model	Significant Parameters	AIC	BIC	MAPE (%)
ARIMA(1,2,0)	$\phi_1 = -0.5108$ ($p = 0.0000$)	-558.9354	-554.5260	16.2472
ARIMA(1,2,1)	θ_1 significant; ϕ_1 not significant	-571.9345	-565.3205	26.9804
ARIMA(2,2,0)	ϕ_1, ϕ_2 significant ($p < 0.05$)	-567.8182	-561.2041	14.2294
ARIMA(2,2,1)	θ_1 significant; ϕ_1, ϕ_2 not significant	-571.3260	-562.5072	18.7809
ARIMA(3,2,0)	ϕ_1, ϕ_3 significant; ϕ_2 not significant	-572.5303	-565.9162	23.4533
ARIMA(3,2,1)	ϕ_1, θ_1 significant; ϕ_2, ϕ_3 not significant	-571.3389	-562.5202	14.5108

The ARIMA(2,2,0) model is selected as the best specification based on its parameter significance, residual diagnostics, and forecasting performance. Although ARIMA(3,2,0) yielded the lowest information criteria values (AIC = –572.5303; BIC = –565.9162), ARIMA(2,2,0) was preferred for its superior out-of-sample accuracy (lower MAPE) and significant coefficients. Both autoregressive terms ($\phi_1 = -0.7144$, $\phi_2 = -0.3983$) are statistically significant ($p < 0.05$), and the residuals pass the Ljung–Box test (p-*value* > 0.05), confirming no autocorrelation and white-noise behavior. With a MAPE of 14.2294%, the model demonstrates reliable predictive accuracy, validating its suitability for forecasting the risk-free interest rate.

The following table presents the four-month-ahead forecasts for USD interest rates generated using the ARIMA(2,2,0) model.

Table 4: Forecasted Monthly USD Interest Rates (ARIMA(2,2,0))

Period	Forecast	Lower 95%	Upper 95%
2025-09-30	0.0433	0.0380	0.0485
2025-10-31	0.0437	0.0351	0.0522
2025-11-30	0.0436	0.0314	0.0559
2025-12-31	0.0437	0.0266	0.0609

The forecasted USD interest rates remain relatively stable over the next four months, with values ranging between 0.0433 and 0.0437. The 95% confidence intervals indicate a moderate-to-high uncertainty band, widening slightly over time, from [0.0380–0.0484] in September to [0.0266-0.0608] in December 2025. This suggests that while short-term USD interest rates are stable, small fluctuations may occur due to market factors. These forecasted rates are subsequently used as the time-varying risk-free rate (r_j) inputs in the Monte Carlo simulation.

Table 5: Simulation Parameters for the Double Barrier Option

Parameter	Symbol	Value
Initial stock price	S_0	340.0100
Strike price	K	395.2300
Constant risk-free rate (annualized, BS benchmark)	r	0.0435
Time-varying risk-free rates (monthly, annualized)	$\{r_j\}$	0.0432, 0.0433, 0.0437, 0.0436, 0.0437
Volatility	σ	0.6293
Time to maturity	T	0.3333
Upper barrier	U	606.7100
Lower barrier	L	166.1200
Number of simulations	M	50–500,000

The simulation parameters are determined to ensure that each variable reflects actual market conditions as accurately as possible. The initial stock price ($S_0 = 340.0100$) corresponds to Tesla Inc.'s closing price on August 23, 2025. The strike price ($K = 395.2300$) is derived from the average strike price of options expiring on December 23, 2025. The risk-free rate sequence $\{r_j\}$ is obtained from the ARIMA(2,2,0) forecasts of monthly U.S. Treasury Bill rates.

For the Black–Scholes benchmark, a constant equivalent annual rate ($r = 0.0435$) is used for discounting. In the Monte Carlo simulations, however, the time-varying interest rate sequence $\{r_j\}$ is applied consistently to both the drift term $(r_j - 0.5\sigma^2)\Delta t$ and the discount factor $\exp(-\sum_j r_j \Delta t)$ to ensure coherence between price evolution and present value computation. Each monthly r_j value (August–December 2025) is assigned to all daily steps within its corresponding month, maintaining consistency between the stochastic process and discounting in the simulation grid.

The annual volatility ($\sigma = 0.6293$) is calculated from the standard deviation of daily log returns, reflecting the high volatility of Tesla's stock. The option's time to maturity is four months ($T = \frac{4}{12}$ year), from August 23 to December 23, 2025. The barrier levels are determined using a continuous barrier adjustment approach, considering both volatility and the time horizon. The upper barrier is set at $U = 606.71$, and the lower barrier at $L = 166.12$. The number of iterations (M) ranges from 50 to 500,000 to examine convergence behavior across all three simulation methods: standard Monte Carlo, antithetic variate, and control variate.

Once the parameters are defined, the next step is to compute the benchmark option prices using the Black–Scholes model. These values serve both as theoretical reference points and control variables in the control variate method. Based on the selected parameters, the vanilla option prices are calculated using the Black–Scholes formula as follows:

$$C = S_0 N(d_1) - Ke^{-rT} N(d_2) = 340.0100(0.4238) - 395.2326(0.9856)(0.2892) = 31.4469$$

$$P = Ke^{-rT} N(-d_2) - S_0 N(-d_1) = 395.2326(0.9856)(0.7108) - 340.0100(0.5762) = 80.9759$$

Therefore, the benchmark prices are \$31.4469 for the call option and \$80.9759 for the put option. These values are used as the expected control values in the control variate approach. The next stage is to conduct Monte Carlo simulations to estimate the prices of double barrier options using stochastic modeling techniques. This method involves simulating numerous random paths of the underlying asset price under the assumption of Geometric Brownian Motion (GBM).

The simulation results for knock-in options using the standard Monte Carlo method are shown in Table 6. As the number of iterations (M) increases from 50 to 500,000, the estimated option values become more stable, and the standard error decreases consistently for both call and put options.

Table 6: Knock-in Option Simulation Results (Standard Monte Carlo, r_j time-varying)

M	Call – Price	Call – Error	Put – Price	Put – Error
50	17.1852	9.7394	9.2029	6.4422
1,000	15.4085	2.0349	14.9147	1.8167
10,000	18.6020	0.7338	12.7517	0.5298
100,000	17.1193	0.2163	12.9278	0.1681
500,000	17.3310	0.0981	12.8924	0.0751

The simulation results for knock-out options are presented in Table 7.

Table 7: Knock-out Option Simulation Results (Standard Monte Carlo, r_j time-varying)

M	Call – Price	Call – Error	Put – Price	Put – Error
50	5.3873	3.0110	93.6382	9.4297
1,000	15.3160	1.1882	67.7029	2.1324
10,000	13.9665	0.3490	66.5776	0.6710
100,000	14.2007	0.1121	67.8121	0.2146
500,000	14.0720	0.0500	68.1174	0.0959

Tables 6 and 7 present the results of standard Monte Carlo simulations for *double barrier knock-in* and *knock-out* options under time-varying interest rates (r_j). It is evident that increasing the number of iterations (M) consistently reduces the standard error and yields more stable option price estimates.

For the knock-in option, the call price converges around \$17.3310 with a standard error of 0.0981, while the put option stabilizes near \$12.8924 with a standard error of 0.0751. For the knock-out option, the call price approaches \$14.0720 with a standard error of 0.0500, and the put price converges at approximately \$68.1174 with a standard error of 0.0959 when $M = 500{,}000$.

These results confirm that as the number of Monte Carlo iterations increases, both knock-in and knock-out option prices exhibit strong convergence and numerical stability, supporting the theoretical expectation of the Monte Carlo method's consistency.

Beyond the standard Monte Carlo approach, simulations were also conducted using the antithetic variate technique. The simulation results for the knock-in option under this method are shown in Table 8.

Table 8: Knock-in Option Simulation Results (Monte Carlo with Antithetic Variates, r_j time-varying)

M	Call – Price	Call – Error	Put – Price	Put – Error
50	15.6684	6.4728	11.7113	5.0380
1,000	17.5238	1.4895	13.1417	1.1552
10,000	17.1708	0.4679	12.7905	0.3624
100,000	17.2851	0.1499	12.7839	0.1149
500,000	17.2892	0.0670	12.7614	0.0513

The results for the knock-out option using the same method are displayed in Table 9.

Table 9: Knock-out Option Simulation Results (Monte Carlo with Antithetic Variates, r_j time-varying)

M	Call – Price	Call – Error	Put – Price	Put – Error
50	12.3151	2.8811	67.6338	3.5334
1,000	13.2903	0.6932	67.6197	0.8446
10,000	14.4136	0.2329	68.2897	0.2663
100,000	14.0667	0.0725	68.1624	0.0837
500,000	14.0817	0.0324	68.1951	0.0374

The antithetic variate method improves convergence by using symmetrically distributed random pairs that offset deviations. As shown in Tables 8 and 9, increasing the number of simulations (M) reduces standard errors and stabilizes option price estimates.

For the knock-in option, the call price converges to about $17.2892(SE = 0.0670)andtheputto12.7614$ (SE = 0.0513). For the knock-out option, the call approaches $14.0817(SE = 0.0324)andtheput68.1951$ (SE = 0.0374) at 500,000 iterations. These results confirm that the antithetic variate method effectively lowers variance while preserving expected values, yielding faster and more accurate convergence in double barrier option pricing under time-varying interest rates.

Additionally, the control variate technique was applied, utilizing the linear dependence between the double-barrier and vanilla options. The knock-in simulation results using this method are shown in Table 10.

Table 10: Knock-in Option Simulation Results (Monte Carlo with Control Variates, $r(t)$ time-varying)

M	Call – Price	Call – Error	Put – Price	Put – Error
50	19.6045	3.8255	9.7405	5.8367
1,000	18.7256	0.9811	12.1419	1.4170
10,000	17.7293	0.3267	11.9850	0.4540
100,000	17.4429	0.1036	12.6472	0.1455
500,000	17.4145	0.0462	12.6767	0.0652

Meanwhile, the knock-out option results using the control variate method are displayed in Table 11.

Table 11: Knock-out Option Simulation Results (Monte Carlo with Control Variates, $r(t)$ time-varying)

M	Call – Price	Call – Error	Put – Price	Put – Error
50	11.8424	3.8255	71.2354	5.8367
1,000	12.7213	0.9811	68.8341	1.4170
10,000	13.7176	0.3267	68.9909	0.4540
100,000	14.0040	0.1036	68.3288	0.1455
500,000	14.0324	0.0462	68.2993	0.0652

The simulation results for the knock-in option under this method are presented in Table 10. Meanwhile, the knock-out option results using the control variate method are displayed in Table 11.

Table 12: Control Variate Coefficients (θ_c, θ_p) for Knock-in and Knock-out Options under Time-varying Interest Rates

M	Knock-in		Knock-out	
	θ_c	θ_p	θ_c	θ_p
50	0.9215	0.2819	0.0785	0.7181
1,000	0.8448	0.3212	0.1552	0.6788
10,000	0.8286	0.3187	0.1714	0.6813
100,000	0.8235	0.3365	0.1765	0.6635
500,000	0.8221	0.3372	0.1779	0.6628

The *control variate* method effectively reduces the variance of Monte Carlo simulations by linking double-barrier and European vanilla options as control variables. As shown in Tables 10 and 11, increasing the number of simulations (M) steadily decreases the standard error, reaching below 0.05 at 500,000 iterations. The estimated knock-in prices converge to 17.4145 (call) and 12.6767 (put), while the knock-out prices stabilize at 14.0324 (call) and 68.2993 (put), demonstrating accurate and consistent valuation under time-varying interest rates.

Table 12 reports the estimated control variate coefficients (θ_c, θ_p), which quantify the optimal linear adjustment factors obtained from the ratio $\mathrm{Cov}(C, Y)/\mathrm{Var}(Y)$. For knock-in options, θ_c values range from 0.8210 to 0.9215, whereas knock-out options exhibit smaller θ_c (0.0785–0.1779) but larger θ_p (0.6628–0.7181). These magnitudes indicate a stronger linear dependence between the knock-in call and its corresponding vanilla option payoff, while for knock-out structures the put payoff acts as a more effective control variable. The approximate equality of standard errors between knock-in and knock-out options reflects their complementary payoff structures under the shared control-variate scheme, resulting in similar sampling variances.

Finally, Table 13 compares the performance of standard, antithetic, and control variate Monte Carlo methods at $M = 500{,}000$, confirming the superior stability and precision of the control variate approach across all option types.

Table 13: Comparison of Monte Carlo Methods at $M = 500{,}000$ Iterations

Option Type	Method	Price	Error
Knock-In Call	Standard Monte Carlo	17.3310	0.0981
	Antithetic Variate	17.2892	0.0670
	Control Variate	17.4145	0.0462
Knock-In Put	Standard Monte Carlo	12.8924	0.0751
	Antithetic Variate	12.7614	0.0513
	Control Variate	12.6767	0.0652
Knock-Out Call	Standard Monte Carlo	14.0720	0.0500
	Antithetic Variate	14.0817	0.0324
	Control Variate	14.0324	0.0462
Knock-Out Put	Standard Monte Carlo	68.1174	0.0959
	Antithetic Variate	68.1951	0.0374
	Control Variate	68.2993	0.0652

In the Black–Scholes framework, a constant equivalent annual rate of 4.35% ($r = 0.0435$) is used, obtained from the average of the ARIMA(2,2,0) monthly forecasts. By contrast, the Monte Carlo simulations apply time-varying monthly rates $\{r_j\}$, yielding a more realistic pathwise discounting.

At $M = 500{,}000$ (Table 13), knock-in calls price higher than knock-out calls ($17.3310 vs. $14.0720), while knock-in puts are far below knock-out puts ($12.8924 vs. $68.1174), reflecting opposite activation/deactivation at the barrier. Across methods, all converge; antithetic variates deliver the smallest error for KO puts (0.0374), and control variates give consistently low errors (e.g., KI call 0.0462; KO put 0.0652) without extra iterations.

Table 14: Most Accurate Method by Option Type

Option Type	Method with Minimum Error	Price (USD)	Error
Knock-In Call	Control Variate	17.4145	0.0462
Knock-In Put	Antithetic Variate	12.7614	0.0513
Knock-Out Call	Antithetic Variate	14.0817	0.0324
Knock-Out Put	Antithetic Variate	68.1951	0.0374

As presented in Table 14, the antithetic variate method produces the most accurate results for three of the four option types—knock-in put, knock-out call, and knock-out put—achieving the lowest standard errors of 0.0513, 0.0324, and 0.0374, respectively. This indicates that the antithetic variate approach is particularly effective for barrier options whose payoffs respond asymmetrically to price movements, as it efficiently cancels out random fluctuations through paired simulations.

In contrast, the control variate method performs best for the knock-in call option, where the payoff exhibits a strong linear dependence on the corresponding vanilla call payoff. The use of the vanilla option as a control variable effectively captures this dependence and substantially reduces the variance, yielding the smallest standard error of 0.0462. Therefore, while the antithetic variate method demonstrates superior accuracy for most barrier structures, the control variate technique proves more advantageous for options whose payoffs closely track the underlying asset's directional movements. Collectively, these findings confirm that the choice of variance reduction method should align with the payoff structure and dependence characteristics of each option type.

The performance differences among the three variance-reduction methods can be explained by their theoretical properties. The antithetic variate method is most effective for knock-out and knock-in put options because their payoffs are nonlinear and asymmetrically affected by downward movements in the underlying asset. By generating negatively correlated random paths (Z and -Z), this method cancels out stochastic noise around the mean, producing smoother convergence. In contrast, the control variate approach performs better for knock-in call options because the double-barrier payoff exhibits a strong linear dependence on the corresponding European call value under the same simulated path. This dependence allows the control variate adjustment through the estimated coefficient to effectively offset systematic bias and reduce the estimator's variance. The standard Monte Carlo method, which lacks such a dependence-based adjustment, converges more slowly, confirming the theoretical advantage of variance-reduction techniques for path-dependent payoffs.

To ensure the reliability of these results, a parity check was conducted based on Equation 17, which states that the sum of the knock-in and knock-out option prices should equal the corresponding vanilla option price under the Black–Scholes framework. Table 15 presents the numerical validation of this relationship, confirming that the simulated values adhere closely to the theoretical parity condition. In this validation, no rebate is assumed, and the strike price and maturity are identical across the knock-in, knock-out, and vanilla options. Parity is considered numerically satisfied when the difference between the combined knock-in and knock-out values and the corresponding vanilla option value remains within a tolerance of $\varepsilon = 0.1$.

Table 15: Parity Verification: Knock-In/Knock-Out vs. Vanilla Option (Black–Scholes)

Option Type	Knock-In	Knock-Out	Sum (In + Out)	Vanilla (B–S)	Difference
Call	17.4145	14.0817	31.4962	31.4469	≈ 0.0493
Put	12.7614	68.1951	80.9565	80.9759	≈ 0.0194

The parity verification in Table 15 shows that the combined knock-in and knock-out option prices (31.4962 for the call and 80.9565 for the put) are nearly identical to their corresponding Black–Scholes values (31.4469 and 80.9759), with deviations well below 0.1. This strong numerical consistency confirms that the Monte Carlo simulations particularly those employing control variate and antithetic variate methods accurately preserve the theoretical parity between barrier and vanilla options. Consequently, the comparison and method recommendations presented earlier are validated as both statistically reliable and theoretically sound.

3.2 Discussion

The results confirm that all three Monte Carlo approaches standard, antithetic variate, and control variate produce double barrier option prices that converge toward the theoretical Black–Scholes values. Increasing the number of simulations consistently reduces standard errors and enhances numerical stability. Among the methods, the control variate performs best for knock-in call options due to its high control coefficient ($\theta_c \approx 0.8221$), reflecting a strong linear dependence between the barrier and vanilla payoffs, while the antithetic variate yields the most accurate results for knock-in put, knock-out call, and knock-out put options by effectively balancing random fluctuations in asymmetric payoffs.

These findings align with prior studies demonstrating the efficiency of variance reduction techniques in improving simulation accuracy [6]. The parity verification further supports the validity of the barrier option theory [3], with deviations between simulated and theoretical values below 0.05. Moreover, incorporating a time-varying interest rate modeled using ARIMA(2,2,0) enhances realism compared to the traditional constant-rate assumption [10], allowing the framework to capture dynamic macroeconomic conditions.

Nevertheless, this study has limitations: the constant volatility assumption may not fully reflect market behavior, and the focus on Tesla Inc. restricts generalizability. Future research should integrate stochastic volatility models (e.g., Heston) or quasi-Monte Carlo methods to improve convergence, and extend the analysis to multiple assets for broader validation. In conclusion, combining dynamic interest rates with variance reduction techniques substantially improves the precision and stability of double barrier option pricing, offering a robust and practical foundation for more adaptive financial modeling.

4 Conclusion

This study evaluates three Monte Carlo simulation techniques standard, antithetic variate, and control variate for pricing double barrier options under time-varying interest rates generated from an ARIMA(2,2,0) model. The results show that all three approaches produce option prices converging toward the theoretical Black-Scholes benchmark, with decreasing standard errors as the number of simulations increases. The control variate method performs best for knock-in call options, while the antithetic variate technique yields the most accurate estimates for knock-in put, knock-out call, and knock-out put options. Parity validation confirms the consistency between simulated barrier option prices and their corresponding vanilla values, verifying both numerical accuracy and theoretical soundness.

Future research may enhance this framework by incorporating stochastic volatility models such as Heston or adopting quasi–Monte Carlo techniques to improve convergence rates. Expanding the analysis to multiple assets and alternative interest rate processes would also broaden the model's applicability in dynamic financial markets.

CRediT Author Contributions

Bella Cindy Thalita: Conceptualization, Methodology, Software, Formal analysis, Writing – original draft.
Isnani Darti : Supervision, Validation, Writing – review & editing, Funding acquisition.

Statement on the Use of Artificial Intelligence (AI) and AI-Assisted Technologies

During the research and manuscript preparation process, the author used ChatGPT (OpenAI) in a limited capacity, specifically for validating code syntax and verifying computational results in Python, particularly for the Monte Carlo simulations and ARIMA modeling. All analyses, interpretations, and manuscript writing were conducted independently by the author, who has thoroughly reviewed and verified all content and bears full responsibility for the scientific accuracy and integrity of this work.

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Ethical Considerations

This research does not involve human subjects, animals, or sensitive data, and therefore does not require ethical approval from any institutional review board. All data used are publicly available and freely accessible.

Data and Code Availability

The historical stock price data of Tesla Inc. (TSLA) were obtained from Yahoo Finance at `https://finance.yahoo.com/quote/TSLA/history`, and the monthly U.S. Treasury Bill rates were sourced from the official website of the U.S. Department of the Treasury at `https://home.treasury.gov/resource-center/data-chart-center/interest-rates/TextView?type=daily_treasury_bill_rates&field_tdr_date_value=2025`.

The Python scripts used for ARIMA modeling and Monte Carlo simulations that support the findings of this study are available from the corresponding author (Isnani Darti, email: isnanidarti@ub.ac.id) upon reasonable request.

The data and code cannot be shared publicly due to institutional data management policies but may be provided for academic use upon request and subject to confidentiality agreements.

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