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Option Pricing Using Modification of Black Scholes Merton Model with GJR-GARCH

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Abstract

Option pricing is a crucial aspect of risk management and investment strategy, particularly in markets with dynamic and asymmetric volatility. The classical Black–Scholes–Merton (BSM) model has limitations because it assumes constant volatility, making it less able to represent actual market conditions. This study aims to empirically interpret the performance of the Black–Scholes–Merton (BSM) and Fractional Black–Scholes–Merton (FBSM) models by integrating volatility estimated using the GJR-GARCH (1,1) model.The estimated volatility from the GJR-GARCH model is used as input to the BSM and FBSM models to calculate the price of a European call option. The data used are the daily closing prices of Apple Inc. (AAPL) shares from January 7, 2022, to March 7, 2025. The performance of each model is evaluated based on the Mean Absolute Percentage Error (MAPE) value by comparing theoretical option prices with market option prices.The results showed that the FBSM model with GJR-GARCH (1,1) volatility produced the lowest error rate with a MAPE value of 4.08%, while the BSM model with historical volatility produced a MAPE of 17.25%. These findings suggest that the use of asymmetric volatility and long memory characteristics can improve the accuracy of option price estimation, especially for stocks with high volatility.

Keywords: Black Scholes Merton, Fractional Black Scholes Merton, GJR-GARCH, European *call*.

1. Introduction

Option pricing is an important aspect of finance, particularly in managing risk and designing effective investment strategies. Options, as one of the derivative instruments, give the holder the right to buy (*call*) or sell *(put)* an asset at a certain price and time without any obligation to exercise the right [1]. This makes options a popular *hedging* and speculation tool in the capital market.

The Black-Scholes-Merton (BSM) model is a frequently used model in options pricing. This model assumes that asset prices fluctuate according to geometric Brownian motion. However, in practice, these assumptions often do not match the reality of dynamic and volatile markets. One of the phenomena ignored by the BSM model is the leverage effect, which is when market volatility tends to increase sharply after a decline in asset prices [2].

To overcome these limitations, the *Generalized Autoregressive Conditional Heteroscedasticity* (GARCH) model was developed which is able to model volatility dynamically [3]. However, the

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standard GARCH model does not consider asymmetry. Therefore, the Glosten Jagannathan-Runkle GARCH (GJR-GARCH) model was developed to capture asymmetric effects in market volatility and provide more accurate estimates of market risk [2].

In addition, fractional approaches are also beginning to be applied in financial models to account for the *(long-range dependence)* of financial data.The Fractional Black Scholes Merton model (FBSM) using fractional Brownian motion (fBm) has shown advantages in handling complex and non stationary markets [4].

Some previous studies have shown that the integration of the GJR GARCH model into the option pricing model can provide more precise estimates, especially in dynamic and asymmetric market conditions [5]. On the other hand, research [6] also shows that modifying BSM through a fractional approach can improve the accuracy of option pricing, especially in emerging markets. However, most of these studies still examine the two approaches separately and not many have integrated asymmetric volatility with a fractional-based option pricing framework.

Based on this background, this study aims to integrate volatility estimated using the GJR-GARCH model into FBSM framework for option pricing. This study does not focus on developing a new theoretical formulation, but rather on an empirical evaluation of the combination of two existing approaches. Specifically, this study aims to assess whether the integration of asymmetric volatility and long-term memory dynamics can produce more accurate European call option price estimates compared to the classical BSM model and its variants that use historical volatility.

2. Methods

This study compares two option pricing models, namely BSM and FBSM, with GJR-GARCH volatility.

2.1. Data

The dataset used in this study consists of 793 daily observations of Apple Inc. (AAPL) stock prices spanning the period from January 7, 2022 to March 7, 2025, obtained from https://finance.yahoo.com/. The option dataset comprises European call option prices, corresponding strike prices, and time to maturity, also collected from Yahoo Finance. The observed stock price on March 22, 2025 was USD 218.27. The prevailing risk-free interest rate, represented by the U.S. central bank rate, was 4.5%. The option maturity date was April 25, 2025, corresponding to 35 days to expiration, or $T = \frac{35}{365} = 0.09589$ years. The strike price applied in this study was USD 215.

2.2. Process

2.2.1. Stock Return

Stock returns are calculated using logarithmic returns to ensure data stationarity and suitability with volatility modeling, with the formula [7]:

$$R_t = \ln\left(\frac{S_t}{S_{t-1}}\right) \tag{1}$$

Description:
S_t : stock price at time t
S_{t-1} : stock price at time $t-1$

2.2.2. ARIMA (p,d,q) Model

The ARIMA (p,d,q) model is used as a mean equation to eliminate autocorrelation in stock returns before volatility modeling. The ARIMA order is selected based on the smallest Akaike Information Criterion (AIC) value and a diagnostic examination of the residuals, which indicates the absence of autocorrelation. Based on these criteria, the ARIMA(3,0,3) model was selected and used in the development of the GJR-GARCH model.

2.2.3. GJR-GARCH (p,q) Volatility Model

In financial data, not all data show a uniform volatility response to certain shocks. Some data show differences in the magnitude of volatility changes when there are movements in return values, known as asymmetric effects or leverage effects. Data that has asymmetric volatility is modeled with the Asymmetric GARCH model. The existence of this can be tested using the Sign bias test.

The GJR-GARCH (p,q) model is able to measure volatility due to the differential impact of good and bad news. The GJR-GARCH (p,q) model differs from the standard GARCH model in that it has components that make it asymmetric [8]. The GJR-GARCH (p,q) model can be expressed using the following equation [9]:

$$\sigma_t^2 = \omega + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2 + \sum_{i=1}^{q} \gamma_i \, l_{t-i} \varepsilon_{t-i}^2 \tag{2}$$

with:

$$l_{t-i} = \begin{cases} 1, & \text{if } \varepsilon_{t-i} < 0 \\ 0, & \text{if } \varepsilon_{t-i} \geq 0 \end{cases}$$

Description:

ω	: Constant
α_i	: ARCH Parameters
β_j	: GARCH Parameters
γ_i	: Leverage Parameters
ε_{t-i}	: Error term
l_{t-i}	: dummy variables

Model selection is done by considering the *Akaike Information Criterion* (AIC) value, where the best model is indicated by the smallest AIC value. The AIC formula is expressed as follows [10]:

$$\text{AIC}(P) = n \ln \hat{\sigma}_b^2 + 2M \tag{3}$$

2.2.4. Volatility

Volatility is a measure of stock price fluctuations. Volatility is directly proportional to option prices, meaning that as volatility increases, option prices tend to increase as well. The annual volatility of stock prices can be calculated using the following Equation [11]:

$$\sigma = \sqrt{\alpha \ \cdot \frac{\sum_{t=1}^{n} (R_t - \bar{R}_t)^2}{n-1}}$$

The number of trading days in a year is a. If the data is daily, then the trading period is also daily, namely 252 days.

2.2.5. Hurst Exponent Estimation

The Hurst parameter$H$ is used in the FBSM model to represent the long memory nature of stock price movements. The value of H is in the interval $0 < H < 1$ with the following interpretation [4]:

1. $H = 0,5$ exhibits standard Brownian motion behavior
2. $H > 0,5$ shows the presence of long memory (persistent behavior)
3. $H < 0,5$ exhibiting anti-persistent behavior

2.2.6. Black Scholes Merton (BSM)

1. Black Scholes Merton
   The BSM model for pricing European-type options is a derivative of the Black-Scholes model with several basic assumptions: stock prices conform to a lognormal distribution with normal mean and volatility, short selling is permitted, continuous trading at a constant and risk-free interest rate [12], and the absence of transaction costs, taxes, or dividend payments. The BSM model can be stated as [13]:

$$C(S,t) = S \cdot N(d_1) - e^{-r(T-t)} \cdot K \cdot N(d_2) \tag{4}$$

   with

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \tag{5}$$

   and

$$d_2 = \frac{\ln\left(\frac{S}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \tag{6}$$

2. Fractional Black Scholes Merton (FBSM)
   The BSM model modified by incorporating the Hurst parameter and based on fractional Brownian motion is known as the FBSM model. The Hurst parameter plays a role in representing the long-term memory properties of financial data. With Hurst parameters $\lambda_H = 2H(T-t)^{2H-1}$. The mathematical form of this model for call options is given in Eq [4]:

$$C(S,t) = S \cdot N(d_1) - e^{-r(T-t)} \cdot K \cdot N(d_2) \tag{7}$$

   with

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{1}{2}\lambda_H\sigma^2\right)(T-t)}{\sigma\sqrt{\lambda_H(T-t)}} \tag{8}$$

   and

$$d_2 = \frac{\ln\left(\frac{S}{K}\right) + \left(r - \frac{\lambda_H}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{\lambda_H(T-t)}}, \quad \text{with} \quad \lambda_H = 2H(T-t)^{2H-1} \tag{9}$$

The selection of BSM and FBSM models with historical volatility and GJR-GARCH, is determined based on the smallest MAPE criteria so as to produce the most accurate call option price estimates. The MAPE value can be found with the equation [14]:

$$\text{MAPE} = \left(\frac{1}{n}\sum_{t=1}^{n}\left|\frac{Y_t - \hat{Y}_t}{Y_t}\right|\right) \times 100\% \tag{10}$$

Description:
- Y_t : actual value in data t
- $\hat{Y}_t$: forecast value on t data
- n : Lots of data

3. Results and Discussion

This research was conducted through several stages, including stock return calculation, descriptive statistical analysis, stationarity testing, ARIMA model identification, heteroscedasticity testing, volatility modeling using the GJR-GARCH model, and option pricing using the BSM and FBSM models. The results obtained from each stage were then analyzed to determine the model's ability to detect volatility dynamics and improve the accuracy of option price estimation.

3.1. Descriptive Statistics

The data used is the descriptive price of the closing price (*close*) of AAPL shares for the period January 7, 2022 to March 7, 2025 obtained by https://finance.yahoo.com/ and accessed on March 22, 2025. The data used is 793, Calculation of APPL daily stock returns with Eq. (1) is as follows:

$$\begin{aligned} R_1 &= \ln\left(\frac{239.07}{235.33}\right) \\ &= \ln(1.015893) \\ &= 0.015768 \end{aligned}$$

Based on Table 1, the 1st stock data return is 0.015768 until the 792nd return is 0.000116. So that the total return amounted to 792.

Table 1: Calculation of return results

No	Stock Returns
1	0.015768
2	-0.001741
3	-0.000806
⋮	⋮
792	0.000116

Furthermore, descriptive analysis is an important initial stage to provide an overview of the data. A summary of descriptive statistics of AAPL stock returns for the period January 7, 2022 to March 7, 2025 is shown in Table 2.

Table 2: Descriptive Statistics

CLOSE ($)	**Value**
Mean	0.000414
Skewness	0.080678
Kurtosis	5.188935
Observations	792

The descriptive statistics of returns show 792 observations with an average of 0.000414 with an almost symmetrical distribution with a skewness of 0.080678. The Kurtosis value of 5.188935 indicates a leptokurtic distribution with potential outliers in stock returns.

3.2. Stationary

Next, a stationary test is carried out on the return data with the time series graph formed and a more precise examination using the Augmented Dickey-Fuller Test (ADF). Following Fig. 2, it is known that APPL stock returns show a stationary graph.

The ADF test obtained $P - value = 0,0000 < \alpha(0,05)$ then the hypothesis H_0 is rejected, it can be concluded that the return data is stationary.

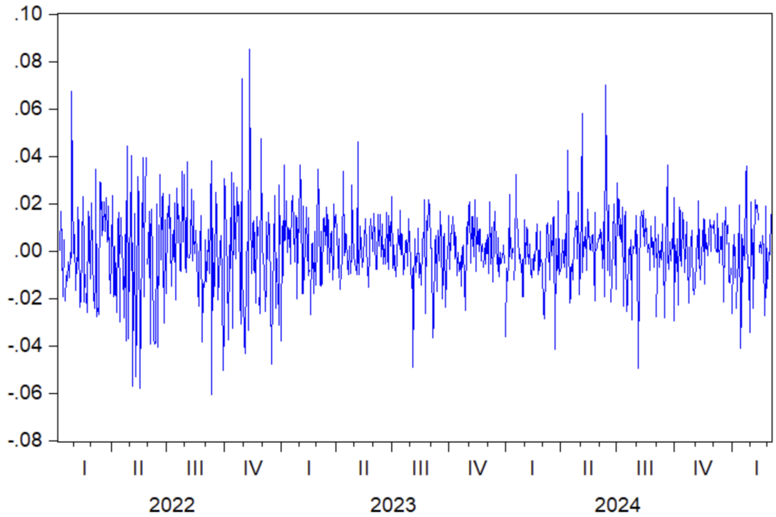

Fig. 1: Return graph

3.3. Identification of ARIMA (p,d,q)

Determination of the best (p,d,q) model can be obtained through the automatic ARIMA forecasting function automatically in the forecast package. The best ARIMA model formed from APPL stock return data using the automatic ARIMA forecasting function is ARIMA (3,0,3). The ARIMA (3,0,3) model is carried out parameter estimation. The results can be seen in Table 3.

Table 3: ARIMA (p,d,q) model estimation results

Model	Parameter	Coefficient	P-value	AIC
ARIMA(3,0,3)	φ_1	0.313817	0.0000	
	φ_2	0.298883	0.0000	
	φ_3	-0.948575	0.0000	
	θ_1	-0.333982	0.0000	-5.316741
	θ_2	-0.332635	0.0000	
	θ_3	0.998452	0.0000	
ARIMA(1,0,1)	φ_1	-0.980265	0.4999	
	θ_1	1.000000	0.0000	-5.305919

With a significance level of 0.05, Table 3 shows that the $p-value$ ARIMA (3,0,3) of ARIMA (3,0,3) is smaller than 0.05, so it can be concluded that it has significant parameters.

The next stage is model verification by testing autocorrelation,normality test, and heteroscedacity test. Testing autocorrelation by looking at the $p-value$ of the ACF and PACF graphs. The results obtained by all values $p-value$ are greater than the significant level of 0.05, so the hypothesis H_0 fails to be rejected, namely the residual data does not have autocorrelation.

While testing the normal distribution of residuals using the Jarque Bera test. Based on Fig. 2, it is known that the value of $p-value$ in the Jarque-Bera test for AAPL return data residuals is $0,0000 < \alpha(0,05)$, thus rejecting H_0 and it can be concluded that the AAPL return data residuals are not normally distributed.

Heteroscedasticity testing on the residuals of the ARIMA (3,0,3) model to determine whether there is an ARCH (q)/GARCH(p,q) effect.

Table 4: ARIMA (3,0,3) heteroscedasticity test results

Model	Statistic	P-value	Decision
ARIMA(3,0,3)	9.126566	0.0025	H_0 rejected

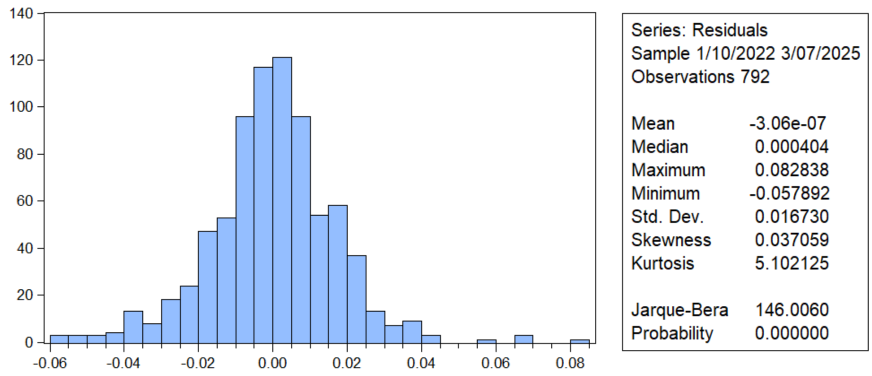

Fig. 2: ARIMA (3,0,3) normality test plot

Based on Table 4 shows that the $p - value 0,00025 < \alpha(0,05)$, the hypothesis H_0 is rejected so it can be concluded that there is an ARCH/GARCH effect.

3.4. GJR-GARCH (p,q) Volatility Model

To identify alternative models, ARCH (q) or GARCH (p,q) models with small orders are commonly considered [15]. The initial models formed with mean ARMA (3,3) are ARCH (1), ARCH (2), GARCH (1,1), GARCH (1,2), and GARCH (2,2). After seeing the smallest AIC value and estimating parameters through a significance test that has a conclusion that all parameters are significant with a $p - value 0,00025 < \alpha(0,05)$ the best model is GARCH (1.1).

Table 5: ARCH (q)/GARCH (q) estimation results

Model	Parameter	Coefficient	P-value	Decision
GARCH(1,1)	α_1	0.037823	0.0034	Significance
	β_1	0.955922	0.0000	Significance

The next step is to see if there is a leverage effect or asymmetric effect in the GARCH model. Testing for asymmetric effects in this study was carried out with the Sign Bias Test. The following results of testing the asymmetric effect with the Sign Bias Test can be seen in Table 6.

Table 6: Sign Bias test results

Parameter	P-value
Sign Bias	0.0000
Negative Sign Bias	0.1121
Positive Sign Bias	0.0000
Joint Effect	0.0000

The sign bias test results in Table 6 show that Negative Size Bias is not significant ($p - value = 0.1121$), while Sign Bias, Positive Size Bias, and Join Effect are significant at $\alpha = 5\%$. This confirms the existence of volatility asymmetry that cannot be captured by GARCH(1,1), so the analysis continues with the GJR-GARCH model.

GJR-GARCH models formed with mean ARMA (3,3) are GJR-GARCH (1,1), GJR GARCH (1,2) and GJR-GARCH (2,2). The best model selection is done by estimating parameters and seeing the smallest AIC value. The parameter estimation results of the GJR-GARCH model are shown in Table 7.

Table 7: GJR-GARCH(p,q) estimation results

Model	Parameter	Coefficient	P-value	AIC
GJR-GARCH(1,1)	α_0	2.41×10^{-6}	0.1366	-5.453030
	α_1	0.008596	0.5143	
	γ_1	0.056173	0.0092	
	β_1	0.954707	0.0000	
GJR-GARCH(1,2)	α_0	1.84×10^{-6}	0.4416	-5.453012
	α_1	0.009532	0.5011	
	γ_1	0.031462	0.4407	
	β_1	1.345613	0.0906	
	β_2	-0.377255	0.6186	
GJR-GARCH(2,2)	α_0	0.000104	0.0006	-5.402901
	α_1	0.025471	0.4865	
	α_2	0.133153	0.1363	
	γ_1	-0.060413	0.0111	
	γ_2	0.280023	0.0375	
	β_1	0.457657	0.0927	
	β_2	0.026609	0.8824	

Based on Table 7, it is obtained that GJR-GARCH (1,1) is a GJR-GARCH model that has the smallest AIC value with all parameters having a p-value <0.05 so that all parameters in GJR-GARCH (1,1) are significant. So the best GJR-GARCH model to measure and forecast the volatility of APPL stock returns is GJR-GARCH (1,1). The following is the equation of the GJR-GARCH (1,1) model obtained:

$$\sigma_t^2 = 2.41 \times 10^{-6} + (0.008596)\,\varepsilon_{t-1}^2 + 0.954707\,\sigma_{t-1}^2 + 0.056173\,\varepsilon_{t-1}^2 I_{t-1} \tag{11}$$

Based on the estimation results of equation 15, the GJR-GARCH (1,1) model exhibits dynamic and asymmetric volatility characteristics. The β_j parameter value of 0.954707, which is close to one, indicates high volatility persistence, meaning that volatility shocks in previous periods have a long-term influence on current volatility. This condition is commonly found in financial market data and reflects that volatility does not immediately return to its average level after a shock.

The γ_i parameter is positive and statistically significant, confirming the presence of a leverage effect, where a stock price decline can lead to a larger increase in volatility than a positive shock of the same magnitude. This finding suggests that the GJR-GARCH model has a better ability to capture volatility asymmetry than the symmetric GARCH model, making it more relevant for use as a volatility input in option pricing. Next, forecasting for the next period is carried out where all the data is 793 and the purpose of forecasting is to find out the next period, namely 794. The forecasting results are as follows:

$$\begin{aligned}
\sigma_{794}^2 &= 2{,}41 \times 10^{-6} + (0{,}008596)\varepsilon_{t-1}^2 + 0{,}954707\sigma_{t-1}^2 \\
&\quad + 0{,}056173\varepsilon_{t-1}^2 I_{t-1} \\
&= 2{,}41 \times 10^{-6} + (0{,}008596)\varepsilon_{793}^2 + 0{,}954707\sigma_{793}^2 \\
&\quad + 0{,}056173\varepsilon_{793}^2 I_{793} \\
&= 2{,}41 \times 10^{-6} + (0{,}008596)(0{,}01498)^2 \\
&\quad + 0{,}954707(0{,}000259) + 0{,}056173(0{,}01498)^2(0) \\
&= 2{,}41 \times 10^{-6} + (0{,}008596)(0{,}000224) + \\
&\quad 0{,}954707(0{,}000259) + 0{,}056173(0{,}000224)(0) \\
&= 0{,}00000241 + 0{,}00000193 + 0{,}000247 + 0 \\
&= 0{,}00025134 \\
\sigma_{794}^2 &\approx 0{,}000251
\end{aligned} \tag{12}$$

Forecasting for the next period, namely 795-829 using the formula

$$\sigma^2_{t+n} = \omega + \left(\alpha + \beta + \frac{1}{2}\gamma\right)\sigma^2_{t+n-1} \tag{13}$$

with:
ω : 0.00000241
α : 0.008596
β : 0.954707
γ : 0.056173
Here are the forecast results

Table 8: forecast calculation results

period to	Forecast
795	0.0002518
796	0.0002521
797	0.0002523
⋮	⋮
829	0.0002588

So the average variance of GJR-GARCH Period 794-829 is

$$\begin{aligned}
\sigma^2_{\text{imp}} &= \frac{1}{N}\sum_{k=1}^{N} E_t\left[\sigma^2_{t+k}\right] \\
&= \frac{1}{35}\Big[0.0002516 + 0.0002518 + 0.0002521 + 0.0002523 + 0.0002526 + 0.0002528 + 0.0002530 \\
&\quad + 0.0002533 + 0.0002535 + 0.0002537 + 0.0002539 + 0.0002542 + 0.0002544 + 0.0002546 \\
&\quad + 0.0002548 + 0.0002550 + 0.0002553 + 0.0002555 + 0.0002557 + 0.0002559 + 0.0002561 \\
&\quad + 0.0002563 + 0.0002565 + 0.0002567 + 0.0002569 + 0.0002571 + 0.0002573 + 0.0002575 \\
&\quad + 0.0002577 + 0.0002579 + 0.0002581 + 0.0002583 + 0.0002584 + 0.0002586 + 0.0002588\Big] \\
&= \frac{1}{35}\Big[0.0089381\Big] \\
&= 0.0002554
\end{aligned} \tag{14}$$

The average value of the GJR-GARCH(1,1) variance for the period 794-829 is 0.0002554. So the Volatility value uses the Eq. (14).

$$\begin{aligned}
\text{Volatilitas} &= \sqrt{\text{Mean Varians GJR-GARCH} \times 252} \\
&= \sqrt{0.0002554 \times 252} \\
&= \sqrt{0.064361} \\
&= 0{,}25369469 \\
&\approx 0.253695
\end{aligned} \tag{15}$$

Then the volatility results with the GJR-GARCH (1,1) variance are $0,253695$ And the Historical volatility value is 0.270240.

3.5. Black Scholes Merton (BSM)

Call option pricing was calculated using the BSM and FBSM models, using two volatility approaches: historical volatility and GJR-GARCH (1,1) estimated volatility.

In the FBSM model, option pricing was calculated by incorporating the Hurst exponent H, The Hurst exponent was estimated using the Rescaled Range (R/S) analysis method. This method evaluates the long-memory property of the log-return series by examining the scaling behavior of the rescaled range statistic. The estimation was implemented using the hurstexp() function from the pracma package in R. The estimated Hurst exponent is H=0.5390113.The estimated value was $H = 0.5390113$, indicating a long-term dependence (long memory) on stock return movements. This value indicates that stock price dynamics are not entirely random, making the fractional approach relevant.

Table 9: GJR-GARCH (1,1) volatility results

No	S_0	Strike Price	Time (Year)	Volatility	Call Option Price	Model Type
1	218.27	200	0.095890	0.270240 0,253695	20.34 20.11	BSM
				0.270240 0,253695	20.14 19.95	FBSM
2	218.27	215	0.095890	0.270240 0,253695	9.51 9.08	BSM
				0.270240 0,253695	9.14 8.73	FBSM
3	218.27	225	0.095890	0.270240 0,253695	4.85 4.43	BSM
				0.270240 0,253695	4.49 4.08	FBSM
4	218.27	230	0.095890	0.270240 0,253695	3.29 2.91	BSM
				0.270240 0,253695	2.96 2.61	FBSM
5	218.27	235	0.095890	0.270240 0,253695	2.15 1.83	BSM
				0.270240 0,253695	1.88 1.59	FBSM

Based on the results in Table 9, it can be seen that the FBSM model consistently produces lower option prices than the BSM model at the same volatility level, using both historical volatility and GJR-GARCH (1,1) volatility. This difference is influenced by the presence of the Hurst exponent ($H = 0.5390113$) in the FBSM model, which represents the persistent behavior of AAPL stock returns.

An H value greater than 0.5 indicates that stock returns tend to maintain their direction over the long term. Taking this property into account, the FBSM model produces a different effective variance adjustment than the classic BSM model, which assumes memoryless price movements. Consequently, the option price estimates generated by the FBSM model are more conservative and more reflective of actual market dynamics.

Furthermore, the use of GJR-GARCH(1,1) volatility estimates provides option price estimates that are relatively closer to market prices than historical volatility. This demonstrates that the integration of asymmetric volatility and long memory can improve the performance of option pricing models.

Empirically, a comparison between theoretical option prices and market option prices indicates overpricing and underpricing in several option contracts. Overpricing occurs when the market price exceeds the theoretical price, while underpricing occurs when the market price is below the theoretical price. This analysis aims to evaluate the relative performance of the BSM and FBSM models in representing market option prices.

The selection of the smallest error value using the MAPE value is carried out to evaluate the relative performance of the BSM and FBSM models for pricing call options. Table 10 shows the MAPE values of call options in the BSM and FBSM models using GJR-GARCH (1,1) volatility and Historical Volatility.

Table 10: MAPE VALUE

K	**Options in the Market**	**Historical BSM**	**FBSM Historical**	**BSM GJR-GARCH (1,1)**	**FBSM GJR-GARCH (1,1)**
200	20.10	20.34	20.11	20.14	19.95
215	9.50	9.51	9,08	9,14	8.73
225	4.33	4.85	4.43	4.49	4.08
230	2.54	3.29	2.91	2.96	2.61
235	1.50	2.15	1.83	1.88	1.59
MAPE	-	17.25%	8.65%	9.86%	4.56%

Based on Table 10, the comparison of the call option price with the BSM and FBSM models obtained the smallest MAPE value in the FBSM model using GJR-GARCH (1,1) volatility with MAPE value of 4.56%. This confirms that the existence of the Hurst exponent and dynamic volatility plays an important role in improving the accuracy of option price estimation.

4. Conclusion

This study demonstrates that integrating the asymmetric volatility of the GJR-GARCH (1,1) model into the BSM and FBSM models can improve the pricing accuracy of European call options on Apple Inc. (AAPL) shares. Empirical evaluation using the Mean Absolute Percentage Error (MAPE) indicates that the FBSM model with GJR-GARCH volatility yields the lowest error rate compared to other models. Theoretically, this increased accuracy can be explained by two main mechanisms. First, the GJR-GARCH model is able to capture the leverage effect, which is a greater volatility response to negative shocks than to positive shocks, thus better aligning volatility estimates with the characteristics of the AAPL stock market. Second, the application of the fractional approach to the FBSM model with a Hurst exponent value of $H > 0.5$ indicates the existence of long-term persistence (long memory) in stock return movements, which cannot be accommodated by the standard Brownian motion assumptions of the classical BSM model. The combination of these two characteristics produces more realistic option price estimates that are closer to market prices. However, despite the insights provided by the GJR-GARCH model in option pricing, this study acknowledges several limitations. First, it assumes a constant risk-free rate (r), which may not fully capture real-world market dynamics. Second, the scope of this study is limited to a single asset, namely Apple Inc. (AAPL), which suggests the need for further empirical testing across different sectors and asset classes. Lastly, transaction costs were excluded from the analysis. Future research may consider incorporating stochastic interest rate models and accounting for market frictions, such as transaction costs and slippage, to enhance the practical applicability of the model.

CRediT Authorship Contribution Statement

Author One: Conceptualization, Methodology, Software, Formal Analysis, Investigation, Resources, Data Curation, Writing–Original Draft Preparation, Visualization, Funding Acquisition. **Author Two:**Conceptualization, Validation, Writing–Review & Editing, Supervision, Project Administration, Funding Acquisition.

Declaration of Generative AI and AI-assisted technologies

The authors used the AI application Deepl to help translate several sections of the manuscript from Indonesian to English.

Declaration of Competing Interest

The authors declare no competing interests.

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Data and Code Availability

Data is available online at the website finance.yahoo.com[1]

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