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Stability Analysis of Conventional and E-Cigarette Smokers Behavior Model with Saturation Effects

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Abstract

Smoking behavior is a harmful habit that poses serious health risks and has been regarded as a lifestyle by society, regardless of age, gender, or social status. This study develops and analyzes a mathematical model of smoking behavior that classifies between conventional smokers and e-cigarette smokers, incorporates interaction with lung cancer patients, and considers the saturation effect on potential smokers as the number of smokers in the population increases. The method is determining assumptions to create a compartment diagram and construct the model. This model has four equilibrium points. The results show that when $R_{01} < 1, R_{02} < 1$, the smoker-free equilibrium point is asymptotically stable. When $R_{01} < 1, R_{02} > 1$, the endemic equilibrium point of e-cigarette smokers becomes asymptotically stable. When $R_{01} > 1$ and $R_{02} < 1$, the endemic equilibrium point of conventional smokers becomes asymptotically stable. Meanwhile, when $R_{01} > 1$ and $R_{02} > 1$, the endemic equilibrium point of conventional and e-cigarette smokers becomes asymptotically stable. Numerical simulations show that the intensity of smoking transmission and saturation effects influence system dynamics. When transmission rates from both conventional and e-cigarette smokers are lower, the population transitions faster toward a smoker-free population. The saturation effect towards conventional smokers and e-cigarette smokers plays an important role in reducing the conventional and e-cigarette smokers in the human population.

Keywords: E-Cigarette; Lung Cancer; Saturation Effects; Smoking; Stability Analysis

1. Introduction

Smoking behavior is a harmful habit that poses serious health risks and has been regarded as a lifestyle by certain segments of society, regardless of age, gender, or social status. In Indonesia, the prevalence of smoking among people aged 15 years and older remains very high. Statistics Indonesia (Badan Pusat Statistik), reported that in 2025 at least 28.68% of the population aged 15 years and above were smokers [1]. Furthermore, according to the latest report from the Global Youth Tobacco Survey (GATS) in 2021, 3.2% of smokers aged 20-34 years old started smoking for the first time before 10 [2]. Along with the high prevalence of smoking, the behavior has also changed over time, especially with alternative smoking products such as electronic cigarettes.

In the last decade, the emergence of electronic cigarettes has slightly changed smoking behavior patterns in society, especially among adolescents [3], [4]. Jerzyński and Stimson, in their research, assumed that in 2021 there were approximately 82 million electronic cigarette users

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worldwide, showing a rapid increase compared to its initial introduction in 2006 [5]. Electronic Nicotine Delivery Systems (ENDS), often referred to as e-cigarettes or vapes, are battery-powered devices that convert liquid into aerosol vapor. Although these products are often perceived as more appealing due to claims of being safer, offering greater variety, and posing lower health risks, several studies have shown that their use is associated with the initiation of conventional smoking behavior, particularly among adolescents [6], [7]. Moreover, smoking behavior is also often perceived as a means of reducing social and academic stress [8].

Despite the perception that electronic cigarettes are safer than conventional cigarettes, both products remain major risk factors for various non-communicable diseases, particularly lung cancer. Data from the Global Cancer Observatory (GLOBOCAN) report compiled by the International Agency for Research on Cancer (IARC), show that in 2022 Indonesia recorded 34,339 deaths due to lung cancer, making it the leading cause of cancer-related mortality at 14.1%, followed by liver cancer (9.6%) and breast cancer (9.3%) [9]. More than 80% of lung cancer cases are caused by smoking, making it the primary and largely preventable cause of lung cancer deaths [10]. Although further research is still required, exposure to electronic cigarettes has also been strongly associated with indicators of cancer risk [11].

These risks highlight the critical importance of preventing smoking behavior. One approach that can be considered is the saturation effect, which refers to a reduction in individual's motivation to start smoking as they become saturated with smokers in their surroundings. Such an effect may serve as a preventive strategy to reduce smoking behavior and lung cancer deaths associated with cigarette consumption.

Mathematical modeling has been widely applied to various problems. The SIR model was first introduced by Kermack and McKendrick in 1927, and it was initially used more often to model infectious diseases [12]. Currently, the SIR model has been developed to model other problems in life. For instance, Zakiyyah developed the SVPR model to analyze the dynamics of sexual violence by considering recidivist perpetrators [13]. Juhari modified the mathematical model of social media addiction by dividing it into mild and severe levels [14]. Furthermore, Zulaikha dan Putri examined the mathematical stability of the impact of bullying on students' mental health using the SEIR model [15].

Many researchers have also modeled smoking behavior. Meghatria and Belhamiti conducted research to develop a model of smoking behavior in social networks and the risk of developing lung cancer, by dividing the population into potential smokers, smokers, lung cancer patients due to smoking, and individuals who quit smoking [16]. However, their study did not consider the smokers classification and the recovery from lung cancer. Permatasari modeled the spread of lung cancer due to cigarette smoke with the influence of chemotherapy treatment using the SEITR model and classified smokers into light and heavy smokers [17]. Noersena divided smokers into tobacco smokers and electronic cigarette smokers. The mathematical model consisted of four subpopulations: susceptible individuals $S(t)$, tobacco smokers $I_t(t)$, electronic cigarette smokers $I_e(t)$, and individuals who have quit smoking $R(t)$. Noersena developed a smoking behavior model by distinguishing between tobacco smokers and electronic cigarette smokers and analyzing optimal control strategies [18]. However, their study only focuses on the dynamics of smoking behavior without including the impact of diseases such as lung cancer.

Furthermore, Naji and Thirthar developed a SIS-type disease model considering saturated incidence and saturated treatment. Researchers showed that the saturation effect changes the dynamics of the system [19]. However, the study consisted of only two subpopulations: the susceptible population $S(t)$ and the infected population $I(t)$.

Based on the previous literature, the studies focus either on smoking behavior dynamics or smoking-related diseases separately. Therefore, the objective of this study is to develop and analyze a mathematical model of smoking behavior that classifies between conventional smokers and e-cigarette smokers, incorporates interaction with lung cancer patients, and considers the saturation effect on potential smokers as the number of smokers in the population increases.

Empirical evidence indicates that dual use between conventional cigarettes and e-cigarettes does occur in real populations. However, several population studies report that the proportion of persistent dual users remains relatively limited in adult smokers [20], [21]. Considering that most adult smokers tend to maintain stable consumption patterns over the long term and do not engage in sustained dual use, this study classifies conventional cigarette smokers and e-cigarette users into two mutually exclusive groups representing their respective smoking behaviors. This approach enables a clear analytical study of the existence and stability of equilibrium point. Transitions between smoking products (dual use) are not included in the present model and are left for future research.

The model is expected to provide a more realistic representation of smoking behavior and provide insight into the role of saturation effects as a potential approach for reducing smoking initiation and the risk of lung cancer associated with smoking.

2. Methods

This study is literature-based research aimed at developing a mathematical model that describes the behavior of conventional smokers and e-cigarette smokers. The model is formulated as a system of nonlinear ordinary differential equations, based on the assumptions and constraints.

The first step is to conduct a literature study to determine the assumptions, parameters, and constraints used in the research. The second step is to formulate assumptions and constraints to simplify the problem. The third step, is to create a compartment diagram and construct a smoking behavior model into a nonlinear ordinary differential equation system with the assumptions and constraints determined. The next step is to analyze the equilibrium points. Subsequently, the basic reproduction number (R_0) is determined using the Next Generation Matrix. The next step is to analyze the stability of each equilibrium point by linearizing the system using the Jacobian matrix, then analyze the stability using its eigenvalues and basic reproduction number (R_0). Furthermore, numerical simulations are generated using MATLAB, to visualize the population dynamics. Finally, conclusions are drawn based on the analytical and numerical results of the study.

3. Result and Discussion

This section presents the main analytical and numerical results of the proposed smoking behavior model. We begin by formulating the model and describing its underlying assumptions, then proceed to derive the equilibrium points, analyze their stability, and discuss the implications of the numerical simulations.

3.1. Model Formulation

The population is divided into six subpopulations: susceptible or potential smokers $S(t)$, conventional smokers $I_K(t)$, e-cigarette smokers $I_E(t)$, individuals who have quit smoking $R_1(t)$, smokers with lung cancer $P(t)$, and smokers who have recovered from lung cancer $R_2(t)$.

We assume that,

1. recruitment into the population occurs through births at a constant rate α,
2. all compartments experience natural mortality at a rate μ,
3. smokers with lung cancer also experience mortality rates caused by the disease at a rate μ_p,
4. susceptible or potential smokers may initiate conventional or e-cigarette smoking through contact with smokers at transmission rates η_1 and η_2,
5. susceptible or potential smokers experience saturation effects towards conventional and e-cigarette smokers, respectively characterized by parameters β_1 and β_2,
6. transitions between conventional and e-cigarette smoking are not permitted,

7. conventional smokers and e-cigarette smokers quit smoking and move to the quit smoking class at proportion δ and proportion ϵ,
8. conventional and e-cigarette smokers moving to the quit smoking class at rates ξ_1 and ξ_2 or through interactions with lung cancer patients at rates ζ_1 and ζ_2,
9. conventional and e-cigarette smokers may develop lung cancer at rates m and n, with proportions $1-\delta$ and $1-\epsilon$,
10. individuals with lung cancer may recover at rate λ,
11. individuals who have quit smoking may relapse to the susceptible or potential smokers class at rate θ, and
12. all the parameter values are positive.

With these assumptions, the following compartment diagram can be constructed as given in Fig. 1. Based on the compartment diagram in Fig. 1, conventional and e-cigarette smoking

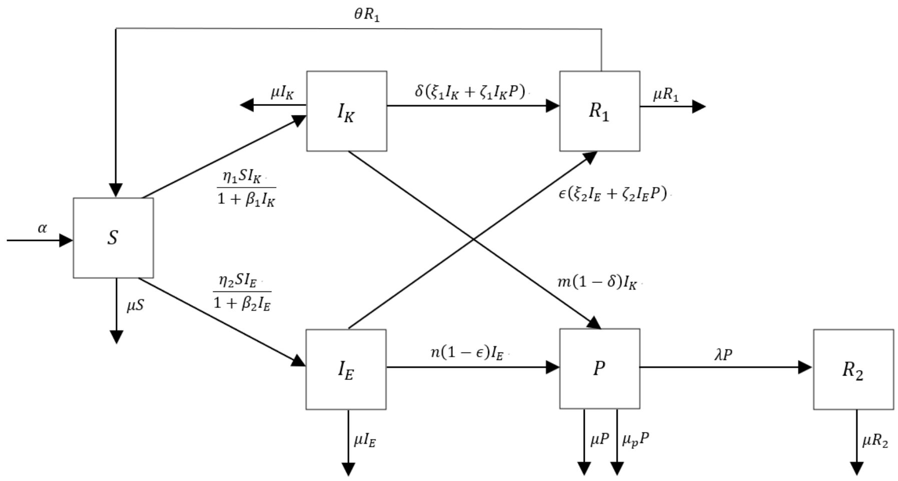


Fig. 1: Compartment Diagram of Smoking Behavior Model

behavior can be modeled by the nonlinear ordinary differential equation system given in Eqs. (1):

$$
\begin{aligned}
\frac{dS}{dt} &= \alpha - \frac{\eta_1 S I_K}{1+\beta_1 I_K} - \frac{\eta_2 S I_E}{1+\beta_2 I_E} + \theta R_1 - \mu S, \\
\frac{dI_K}{dt} &= \frac{\eta_1 S I_K}{1+\beta_1 I_K} - \delta(\xi_1 I_K + \zeta_1 I_K P) - m(1-\delta)I_K - \mu I_K, \\
\frac{dI_E}{dt} &= \frac{\eta_2 S I_E}{1+\beta_2 I_E} - \epsilon(\xi_2 I_E + \zeta_2 I_E P) - n(1-\epsilon)I_E - \mu I_E, \\
\frac{dR_1}{dt} &= \delta(\xi_1 I_K + \zeta_1 I_K P) + \epsilon(\xi_2 I_E + \zeta_2 I_E P) - \theta R_1 - \mu R_1, \\
\frac{dP}{dt} &= m(1-\delta)I_K + n(1-\epsilon)I_E - \lambda P - \mu P - \mu_p P, \\
\frac{dR_2}{dt} &= \lambda P - \mu R_2.
\end{aligned}
\tag{1}
$$

The model is subject to non-negative initial conditions, $S(0) \geq 0, I_K(0) \geq 0, I_E(0) \geq 0, R_1(0) \geq 0, P(0) \geq 0, R_2(0) \geq 0$. $N(t) = S(t) + I_K(t) + I_E(t) + R_1(t) + P(t) + R_2(t)$ denote the total human population at time t.

Theorem 1. *All solutions $S(t), I_K(t), I_E(t), R_1(t), P(t), R_2(t)$ of system Eqs. (1) with non-negative initial conditions remains non-negative for all $t \geq 0$.*

Proof. From system Eqs. (1) we have,

$$\left.\frac{dS}{dt}\right|_{S=0} = \alpha + \theta R_1 \geq 0,$$
$$\left.\frac{dI_K}{dt}\right|_{I_K=0} = 0,$$
$$\left.\frac{dI_E}{dt}\right|_{I_E=0} = 0,$$
$$\left.\frac{dR_1}{dt}\right|_{R_1=0} = \delta(\xi_1 I_K + \zeta_1 I_K P) + \epsilon(\xi_2 I_E + \zeta_2 I_E P) \geq 0,$$
$$\left.\frac{dP}{dt}\right|_{P=0} = m(1-\delta)I_K + n(1-\epsilon)I_E \geq 0,$$
$$\left.\frac{dR_2}{dt}\right|_{R_2=0} = \lambda P \geq 0.$$

It indicates that all rates are non-negative on the boundary of $\mathbb{R}^6_+$. Hence, the solutions are confined to the positive region whenever the initial condition is in the non-negative region $\mathbb{R}^6_+$. □

Theorem 2. *All solutions of system Eqs. (1) are bounded for all $t \geq 0$.*

Proof. Let $N(t) = S(t) + I_K(t) + I_E(t) + R_1(t) + P(t) + R_2(t)$ be the total population. Summing all model equations yields

$$\frac{dN}{dt} = \alpha - \mu N - \mu_p P$$

Since $P(t) \geq 0$ and $\mu_p > 0$, it follows that $-\mu_p P \leq 0$, we obtain

$$\frac{dN}{dt} = \alpha - \mu N - \mu_p P \leq \alpha - \mu N$$

Consider the comparison equation

$$\frac{dN_1}{dt} = \alpha - \mu N_1, \qquad N_1(0) = N(0)$$

whose solution is

$$N_1(t) = \frac{\alpha}{\mu} + ke^{-\mu t}, \text{for } k \text{ constant}$$

Since $N_1(0) = N(0)$ it follows $N(0) = \frac{\alpha}{\mu} + k,$ for k constant. Hence, we have

$$N_1(t) = \frac{\alpha}{\mu} + \left(N(0) - \frac{\alpha}{\mu}\right) e^{-\mu t}$$

By the standard comparison theorem for differential equations, it implies that $N(t) \leq N_1(t)$ for all $t \geq 0$. Consequently,

$$0 \leq N(t) \leq \max\left\{N(0), \frac{\alpha}{\mu}\right\}$$

In particular,

$$\limsup_{t\to\infty} N(t) \leq \frac{\alpha}{\mu}$$

Therefore, all solutions remain in the positively invariant region

$$\Omega = \left\{ (S, I_K, I_E, R_1, P, R_2) \in \mathbb{R}_+^6 : 0 \leq S + I_K + I_E + R_1 + P + R_2 \leq \frac{\alpha}{\mu} \right\}.$$

and the system is bounded, ensuring that population dynamics remain realistic over time. □

The descriptions of the parameters in system Eqs. (1) are provided in Table 1.

Table 1: Parameter Descriptions

Parameter	Description	Unit
α	Recruitment rate	individual/day
μ	Natural death rate	day^{-1}
η_1	Transmission rate of potential smokers to conventional smokers	$(\text{individual} \cdot \text{day})^{-1}$
η_2	Transmission rate of potential smokers to e-cigarette smokers	$(\text{individual} \cdot \text{day})^{-1}$
δ	Proportion of conventional smokers who quit smoking	-
$1 - \delta$	Proportion of conventional smokers who develop lung cancer	-
β_1	Saturation rate of potential smokers toward conventional smoking	individual^{-1}
ξ_1	Quitting rate from conventional smokers moving to the quit smoking class	day^{-1}
ζ_1	Quitting rate from conventional smokers moving to the quit smoking class due to interactions with lung cancer patients	$(\text{individual} \cdot \text{day})^{-1}$
ϵ	Proportion of e-cigarette smokers who quit smoking	-
$1 - \epsilon$	Proportion of e-cigarette smokers who develop lung cancer	-
β_2	Saturation rate of potential smokers toward e-cigarette smoking	individual^{-1}
ξ_2	Quitting rate from e-cigarette smokers moving to the quit smoking class	day^{-1}
ζ_2	Quitting rate from e-cigarette smokers moving to the quit smoking class due to interactions with lung cancer patients	$(\text{individual} \cdot \text{day})^{-1}$
m	Incidence rate of lung cancer in conventional smokers	day^{-1}
n	Incidence rate of lung cancer in e-cigarette smokers	day^{-1}
λ	Recovery rate of lung cancer patients	day^{-1}
μ_p	Lung cancer-induced death rate	day^{-1}
θ	Relapse rate from quit smoking class to the potential smokers class	day^{-1}

3.2. Equilibrium Point and Basic Reproduction Number

Equilibrium point of system Eqs. (1) is obtained when the population remains constant over time, such that

$$\frac{dS}{dt} = \frac{dI_K}{dt} = \frac{dI_E}{dt} = \frac{dR_1}{dt} = \frac{dP}{dt} = \frac{dR_2}{dt} = 0.$$

Based on the constructed mathematical model, four equilibrium points are obtained: the smoker-free equilibrium point, the endemic equilibrium point of e-cigarette smokers, the endemic equilibrium point of conventional smokers, and the endemic equilibrium point of both conventional and e-cigarette smokers.

3.2.1. The Smoker-free Equilibrium Point

The smoker-free equilibrium point represents a critical state where the population is entirely free from smokers, both conventional smokers and e-cigarette smokers. By assuming $I_K = I_E = 0$ and substituting these values into system Eqs. (1), the smoker-free equilibrium point is obtained as follows.

$$E^{(0)} = \left(S^{(0)}, I_K^{(0)}, I_E^{(0)}, R_1^{(0)}, P^{(0)}, R_2^{(0)}\right) = \left(\frac{\alpha}{\mu}, 0, 0, 0, 0, 0\right)$$

Before analyzing another equilibrium point, we derive the basic reproduction number (R_0). The basic reproduction number is obtained using the Next Generation Matrix [22]. In the (1), there are two infected compartments: the conventional smokers subpopulation (I_K) and the e-cigarette smokers subpopulation (I_E).

$$\frac{dI_K}{dt} = \frac{\eta_1 S I_K}{1+\beta_1 I_K} - \delta(\xi_1 I_K + \zeta_1 I_K P) - m(1-\delta)I_K - \mu I_K,$$

$$\frac{dI_E}{dt} = \frac{\eta_2 S I_E}{1+\beta_2 I_E} - \epsilon(\xi_2 I_E + \zeta_2 I_E P) - n(1-\epsilon)I_E - \mu I_E.$$

Let $\mathcal{F}$ represent the rate at which secondary infections increase the i-th infected compartment and $\mathcal{V}$ represent the rate of transfer out of the i-th compartment. Then, we have

$$\mathcal{F} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} \frac{\eta_1 S I_K}{1+\beta_1 I_K} \\ \frac{\eta_2 S I_E}{1+\beta_2 I_E} \end{pmatrix} \tag{2}$$

$$\mathcal{V} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \delta(\xi_1 I_K + \zeta_1 I_K P) + m\,(1-\delta)\,I_K + \mu I_K \\ \epsilon(\xi_2 I_E + \zeta_2 I_E P) + n\,(1-\epsilon)\,I_E + \mu I_E \end{pmatrix} \tag{3}$$

from Eq. (2) and (3), the matrices $\boldsymbol{F}$ and $\boldsymbol{V}$ are obtained

$$\boldsymbol{F} = \begin{pmatrix} \frac{\partial f_1}{\partial I_K} & \frac{\partial f_1}{\partial I_E} \\ \frac{\partial f_2}{\partial I_K} & \frac{\partial f_2}{\partial I_E} \end{pmatrix} = \begin{pmatrix} \frac{\eta_1 S}{(1+\beta_1 I_K)^2} & 0 \\ 0 & \frac{\eta_2 S}{(1+\beta_2 I_E)^2} \end{pmatrix}$$

$$\boldsymbol{V} = \begin{pmatrix} \frac{\partial v_1}{\partial I_K} & \frac{\partial v_1}{\partial I_E} \\ \frac{\partial v_2}{\partial I_K} & \frac{\partial v_2}{\partial I_E} \end{pmatrix} = \begin{pmatrix} \delta(\xi_1 + \zeta_1 P) + m\,(1-\delta) + \mu & 0 \\ 0 & \epsilon(\xi_2 + \zeta_2 P) + n\,(1-\epsilon) + \mu \end{pmatrix}$$

and the $\boldsymbol{V}^{-1}$ is obtained

$$\boldsymbol{V^{-1}} = \begin{pmatrix} \frac{1}{\delta(\xi_1+\zeta_1 P)+m(1-\delta)+\mu} & 0 \\ 0 & \frac{1}{\epsilon(\xi_2+\zeta_2 P)+n(1-\epsilon)+\mu} \end{pmatrix}$$

thus, the NGM matrix is obtained, as follows

$$\boldsymbol{K} = \boldsymbol{FV^{-1}} = \begin{pmatrix} \frac{\eta_1 S}{(1+\beta_1 I_K)^2(\delta(\xi_1+\zeta_1 P)+m(1-\delta)+\mu)} & 0 \\ 0 & \frac{\eta_2 S}{(1+\beta_2 I_E)^2(\epsilon(\xi_2+\zeta_2 P)+n(1-\epsilon)+\mu)} \end{pmatrix} \tag{4}$$

by substituting $E^{(0)}$ to matrix Eq. (4) and solving the characteristic equation by evaluating $\left|\boldsymbol{FV^{-1}} - r\boldsymbol{I}\right| = 0$, the eigenvalues are obtained:

$$R_{01} = \frac{\eta_1\left(\frac{\alpha}{\mu}\right)}{\delta\xi_1 + m\,(1-\delta) + \mu}, \quad R_{02} = \frac{\eta_2\left(\frac{\alpha}{\mu}\right)}{\epsilon\xi_2 + n\,(1-\epsilon) + \mu}.$$

The basic reproduction number is defined as the spectral radius of the matrix, corresponding to its dominant eigenvalue. Since the Next Generation Matrix is block-diagonal, its eigenvalues are given by R_{01} and R_{02}, which represent the reproduction numbers of conventional smokers and e-cigarette smokers respectively. Hence,

$$R_0 = \max\{R_{01}, R_{02}\}$$

3.2.2. The Endemic Equilibrium point of E-cigarette Smokers

The endemic equilibrium point of e-cigarette smokers describes the state in which only e-cigarette smokers are present in the population, while conventional smokers are eradicated. This equilibrium point is characterized by $I_E \neq 0$ and $I_K = 0$. This condition is expressed as $S^{(1)} > 0, I_K^{(1)} = 0, I_E^{(1)} > 0, R_1^{(1)} > 0, P^{(1)} > 0, R_2^{(1)} > 0$. The endemic equilibrium point of e-cigarette smokers is obtained as follows.

$$E^{(1)} = \left(S^{(1)}, I_K^{(1)}, I_E^{(1)}, R_1^{(1)}, P^{(1)}, R_2^{(1)}\right)$$

where

$$\begin{aligned}
S^{(1)} &= \frac{\left(1+\beta_2 I_E^{(1)}\right)\left(A_0 B + A_1 I_E^{(1)}\right)}{B\eta_2},\\
I_K^{(1)} &= 0,\\
I_E^{(1)} &= I_E^{(1)},\\
R_1^{(1)} &= \frac{\epsilon I_E^{(1)}\left(n(1-\epsilon)\zeta_2 I_E^{(1)} + B\xi_2\right)}{B\left(\theta+\mu\right)},\\
P^{(1)} &= \frac{n(1-\epsilon) I_E^{(1)}}{B},\\
R_2^{(1)} &= \frac{\lambda n(1-\epsilon) I_E^{(1)}}{B\mu}.
\end{aligned}$$

with

$$A_0 = \epsilon\xi_2 + n\left(1-\epsilon\right) + \mu, \qquad A_1 = \epsilon n\left(1-\epsilon\right)\zeta_2, \qquad B = \lambda + \mu + \mu_p.$$

Substituting all equilibrium expressions into $\frac{dS}{dt} = 0$ yields a quadratic equation in $I_E^{(1)}$,

$$a_2\left(I_E^{(1)}\right)^2 + a_1 I_E^{(1)} + a_0 = 0 \tag{5}$$

where

$$\begin{aligned}
a_2 &= -\frac{\epsilon n(1-\epsilon)\zeta_2}{\lambda+\mu+\mu_p} + \frac{\theta\epsilon n(1-\epsilon)\zeta_2}{(\lambda+\mu+\mu_p)(\theta+\mu)} - \frac{\mu\beta_2\epsilon n(1-\epsilon)\zeta_2}{\eta_2\left(\lambda+\mu+\mu_p\right)},\\
a_1 &= -(\epsilon\xi_2 + n(1-\epsilon)+\mu) + \frac{\theta\,\epsilon\xi_2}{\theta+\mu} - \frac{\mu}{\eta_2}\left(\frac{\epsilon n(1-\epsilon)\zeta_2}{\lambda+\mu+\mu_p} + \beta_2(\epsilon\xi_2 + n(1-\epsilon)+\mu)\right),\\
a_0 &= \alpha - \frac{\mu}{\eta_2}(\epsilon\xi_2 + n(1-\epsilon)+\mu).
\end{aligned}$$

For the solution to have biologically feasible solution, $I_E^{(1)} > 0$ is required. Therefore, $I_E^{(1)}$ is the positive solution of the Eq. (5),

$$I_E^{(1)} = \frac{-a_1 + \sqrt{a_1^2 - 4a_2 a_0}}{2a_2}$$

The coefficient a_2 can be rewritten as,

$$a_2 = \frac{\epsilon n(1-\epsilon)\zeta_2}{\lambda+\mu+\mu_p}\left(-1 + \frac{\theta}{\theta+\mu} - \frac{\mu\beta_2}{\eta_2}\right)$$

Since, all the parameter values are positive, we have $0 < \frac{\theta}{\theta+\mu} < 1$ it follows that $-1 + \frac{\theta}{\theta+\mu} < 0$. Furthermore $-\frac{\mu\beta_2}{\eta_2} < 0$. Hence, the term in parentheses is negative. It can be concluded that $a_2 < 0$. $I_E^{(1)}$ has real roots if $\Delta = a_1^2 - 4a_2 a_0 > 0$ and has at least one positive solution if $I_{E_1}^{(1)} \cdot I_{E_2}^{(1)} = \frac{a_0}{a_2} < 0$. These conditions are satisfied if $a_0 > 0$. Note that, the constant term of the Eq. (5) satisfies

$$a_0 = \frac{\mu}{\eta_2}\left(\epsilon\xi_2 + n(1-\epsilon)+\mu\right)\left(R_{02}-1\right) > 0$$

Thus, the endemic equilibrium point of e-cigarette smokers exists when $R_{02} > 1$.

3.2.3. The Endemic Equilibrium Point of Conventional Smokers

The endemic equilibrium point of conventional smokers represents the situation in which only conventional smokers persist in the population, while e-cigarette smokers are eradicated. This equilibrium point occurs when $I_K \neq 0$ and $I_E = 0$. Mathematically, this condition is expressed as $S^{(2)} > 0, I_K^{(2)} > 0, I_E^{(2)} = 0, R_1^{(2)} > 0, P^{(2)} > 0, R_2^{(2)} > 0$. The endemic equilibrium point of conventional smokers is obtained as follows.

$$E^{(2)} = \left(S^{(2)}, I_K^{(2)}, I_E^{(2)}, R_1^{(2)}, P^{(2)}, R_2^{(2)}\right)$$

where

$$\begin{aligned}
S^{(2)} &= \frac{\left(1+\beta_1 I_K^{(2)}\right)\left(C_0 B + C_1 I_K^{(2)}\right)}{B\eta_1},\\
I_K^{(2)} &= I_K^{(2)},\\
I_E^{(2)} &= 0,\\
R_1^{(2)} &= \frac{\delta I_K^{(2)}\left(m(1-\delta)\zeta_1 I_K^{(2)} + B\xi_1\right)}{B\left(\theta+\mu\right)},\\
P^{(2)} &= \frac{m(1-\delta)I_K^{(2)}}{B},\\
R_2^{(2)} &= \frac{\lambda m(1-\delta)I_K^{(2)}}{B\mu}.
\end{aligned}$$

with

$$B = \lambda + \mu + \mu_p, \qquad C_0 = \delta\xi_1 + m\left(1-\delta\right) + \mu, \qquad C_1 = \delta m\left(1-\delta\right)\zeta_1.$$

Substituting all equilibrium expressions into $\frac{dS}{dt} = 0$ yields a quadratic equation in $I_K^{(2)}$,

$$b_2\left(I_K^{(2)}\right)^2 + b_1 I_K^{(2)} + b_0 = 0 \tag{6}$$

where

$$\begin{aligned}
b_2 &= -\frac{\delta m(1-\delta)\zeta_1}{\lambda+\mu+\mu_p} + \frac{\theta\,\delta m(1-\delta)\zeta_1}{(\lambda+\mu+\mu_p)(\theta+\mu)} - \frac{\mu\beta_1\delta m(1-\delta)\zeta_1}{\eta_1\left(\lambda+\mu+\mu_p\right)},\\
b_1 &= -\left(\delta\xi_1 + m(1-\delta) + \mu\right) + \frac{\theta\,\delta\xi_1}{\theta+\mu} - \frac{\mu}{\eta_1}\left(\frac{\delta m(1-\delta)\zeta_1}{\lambda+\mu+\mu_p} + \beta_1\left(\delta\xi_1 + m(1-\delta) + \mu\right)\right),\\
b_0 &= \alpha - \frac{\mu}{\eta_1}\left(\delta\xi_1 + m(1-\delta) + \mu\right).
\end{aligned}$$

For the solution to have biological significance, $I_K^{(2)} > 0$ is required. Therefore, $I_K^{(2)}$ is the positive solution of the Eq. (6),

$$I_K^{(2)} = \frac{-b_1 + \sqrt{b_1^2 - 4b_2 b_0}}{2b_2}$$

The coefficient b_2 can be rewritten as,

$$b_2 = \frac{\delta m(1-\delta)\zeta_1}{\lambda+\mu+\mu_p}\left(-1 + \frac{\theta}{\theta+\mu} - \frac{\mu\beta_1}{\eta_1}\right)$$

Since, all the parameter values are positive, we have $0 < \frac{\theta}{\theta+\mu} < 1$ it follows that $-1 + \frac{\theta}{\theta+\mu} < 0$. Furthermore $-\frac{\mu\beta_1}{\eta_1} < 0$. Hence, the term in parentheses is negative. It can be concluded that

$b_2 < 0$. $I_K^{(2)}$ has real roots if $\Delta = b_1^2 - 4b_2b_0 > 0$ and has at least one positive solution if $I_{K_1}^{(2)} \cdot I_{K_2}^{(2)} = \frac{b_0}{b_2} < 0$. These conditions are satisfied if $b_0 > 0$. Note that, the constant term of the Eq. (6) satisfies

$$b_0 = \frac{\mu}{\eta_1}\left(\delta\xi_1 + m(1-\delta) + \mu\right)\left(R_{01} - 1\right) > 0$$

Thus, the endemic equilibrium point of conventional smokers exists when $R_{01} > 1$.

3.2.4. The Endemic Equilibrium Point of Conventional and E-cigarette Smokers

The endemic equilibrium point of conventional and e-cigarette smokers corresponds to the situation in which both types of smokers exist in the population. This equilibrium point arises when $I_K \neq 0$ and $I_E \neq 0$. Mathematically, this condition is expressed as $S^{(3)} > 0, I_K^{(3)} > 0, I_E^{(3)} > 0, R_1^{(3)} > 0, P^{(3)} > 0, R_2^{(3)} > 0$. The endemic equilibrium point $E^{(3)}$ is obtained as follows.

$$E^{(3)} = \left(S^{(3)}, I_K^{(3)}, I_E^{(3)}, R_1^{(3)}, P^{(3)}, R_2^{(3)}\right)$$

where

$$\begin{aligned}
S^{(3)} &= \frac{(F_1 + \delta\zeta_1 G)(1 + \beta_1 I_K^{(3)})}{\eta_1} = \frac{(F_2 + \epsilon\zeta_2 G)(1 + \beta_2 I_E^{(3)})}{\eta_2}, \\
I_K^{(3)} &= I_K^{(3)}, \\
I_E^{(3)} &= I_E^{(3)}, \\
R_1^{(3)} &= \frac{\delta(\xi_1 + \zeta_1 G)I_K^{(3)} + \epsilon(\xi_2 + \zeta_2 G)I_E^{(3)}}{\theta + \mu}, \\
P^{(3)} &= \frac{m(1-\delta)I_K^{(3)} + n(1-\epsilon)I_E^{(3)}}{B}, \\
R_2^{(3)} &= \frac{\lambda\left(m(1-\delta)I_K^{(3)} + n(1-\epsilon)I_E^{(3)}\right)}{B\mu}.
\end{aligned}$$

with

$$B = \lambda + \mu + \mu_p, \qquad F_1 = \delta\xi_1 + m(1-\delta) + \mu,$$

$$F_2 = \epsilon\xi_2 + n(1-\epsilon) + \mu, \qquad G = \frac{m(1-\delta)I_K^{(3)} + n(1-\epsilon)I_E^{(3)}}{\lambda + \mu + \mu_p}.$$

Assume that $R_{01} > 1$ and $R_{02} > 1$. Under these conditions, $E^{(0)}$ is unstable to perturbations in I_K or I_E. Although $E^{(1)}$ exists when $R_{02} > 1$, it becomes unstable if $R_{01} > 1$, because $I_K^{(1)}$ causes $\frac{dI_K}{dt} > 0$. Similarly, $E^{(0)}$ is unstable when $R_{01} > 1$. Although $E^{(2)}$ exists when $R_{01} > 1$, it becomes unstable if $R_{02} > 1$ because I_E can invade, although at low levels. Therefore, all equilibrium points are unstable. Since the feasible region is positive invariant and bounded, solutions cannot diverge to infinity and must approach an equilibrium point within the biologically feasible region. Therefore, the equilibrium $E^{(3)}$ exists when $R_{01} > 1$ and $R_{02} > 1$.

The equilibrium values $I_K^{(3)}$ and $I_E^{(3)}$ are obtained by solving the system. Since the closed-form expressions are difficult to obtain, positive solutions corresponding to $E^{(3)}$ are calculated numerically.

3.3. Stability Analysis

The linear approximation of a nonlinear system near an equilibrium point is given by the Jacobian matrix. The Jacobian matrix is the basis for evaluating local stability through its eigenvalues [23].

3.3.1. Smoker-free Equilibrium Point ($E^{(0)}$)

From system Eqs. (1) and smoker-free equilibrium point, the Jacobian matrix is obtained as follows.

$$J\left(E^{(0)}\right)=\begin{pmatrix}-\mu & a_{12} & a_{13} & \theta & 0 & 0\\ 0 & a_{22} & 0 & 0 & 0 & 0\\ 0 & 0 & a_{33} & 0 & 0 & 0\\ 0 & \delta\xi_1 & \epsilon\xi_2 & a_{44} & 0 & 0\\ 0 & a_{52} & a_{53} & 0 & a_{55} & 0\\ 0 & 0 & 0 & 0 & \lambda & -\mu\end{pmatrix} \tag{7}$$

where
$a_{12}=-\eta_1\left(\frac{\alpha}{\mu}\right), a_{13}=-\eta_2\left(\frac{\alpha}{\mu}\right), a_{22}=\eta_1\left(\frac{\alpha}{\mu}\right)-\delta\xi_1-m\left(1-\delta\right)-\mu, a_{33}=\eta_2\left(\frac{\alpha}{\mu}\right)-\epsilon\xi_2-n\left(1-\epsilon\right)-\mu, a_{44}=-\theta-\mu, a_{52}=m\left(1-\delta\right), a_{53}=n\left(1-\epsilon\right), a_{55}=-\lambda-\mu-\mu_p.$

Based on the Jacobian matrix Eq. (7), six eigenvalues are obtained as follows.
$r_1=-\mu,$
$r_2=\eta_1\left(\frac{\alpha}{\mu}\right)-\delta\xi_1-m\left(1-\delta\right)-\mu,$
$r_3=\eta_2\left(\frac{\alpha}{\mu}\right)-\epsilon\xi_2-n\left(1-\epsilon\right)-\mu,$
$r_4=-\theta-\mu=-\left(\theta+\mu\right),$
$r_5=-\lambda-\mu-\mu_p=-(\lambda+\mu+\mu_p),$
$r_6=-\mu$

The equilibrium point of the linear system is asymptotically stable if the real parts of eigenvalues are negative [24]. Since every parameter is positive, the eigenvalues r_1, r_4, r_5, and r_6 are negative. As a result, the stability of the smoker-free equilibrium point depends on the eigenvalues r_2 and r_3. The eigenvalues r_2 and r_3 must be negative. The system at the smoker-free equilibrium point is locally asymptotically stable if

$$\eta_1<\frac{\mu}{\alpha}\left(\delta\xi_1+m\left(1-\delta\right)+\mu\right) \iff R_{01}=\frac{\eta_1\left(\frac{\alpha}{\mu}\right)}{\delta\xi_1+m\left(1-\delta\right)+\mu}<1,$$

and

$$\eta_2<\frac{\mu}{\alpha}\left(\epsilon\xi_2+n\left(1-\epsilon\right)+\mu\right) \iff R_{02}=\frac{\eta_2\left(\frac{\alpha}{\mu}\right)}{\epsilon\xi_2+n\left(1-\epsilon\right)+\mu}<1.$$

When the stated conditions are satisfied, all eigenvalues evaluated at smoker-free equilibrium point are negative. Thus, this equilibrium point is asymptotically stable. Conversely, if at least one of these threshold conditions is not satisfied, that is, if $R_{01}>1$ or $R_{02}>1$, the smoker-free equilibrium loses its stability, leading to one of the endemic equilibrium points.

3.3.2. Endemic Equilibrium Point of E-Cigarette Smokers ($E^{(1)}$)

The Jacobian matrix of the system evaluated at ($E^{(1)}$) is obtained as follows.

$$J\left(E^{(1)}\right)=\begin{pmatrix}b_{11} & -\eta_1 S^{(1)} & b_{13} & \theta & 0 & 0\\ 0 & b_{22} & 0 & 0 & 0 & 0\\ b_{31} & 0 & b_{33} & 0 & -\epsilon\zeta_2 I_E^{(1)} & 0\\ 0 & b_{42} & b_{43} & b_{44} & \epsilon\zeta_2 I_E^{(1)} & 0\\ 0 & b_{52} & b_{53} & 0 & b_{55} & 0\\ 0 & 0 & 0 & 0 & \lambda & -\mu\end{pmatrix}$$

where $b_{11} = -\frac{\eta_2 I_E^{(1)}}{1+\beta_2 I_E^{(1)}} - \mu, b_{13} = -\frac{\eta_2 S^{(1)}}{(1+\beta_2 I_E^{(1)})^2}, b_{22} = \eta_1 S^{(1)} - \delta(\xi_1 + \zeta_1 P^{(1)}) - m(1-\delta) - \mu,$

$b_{31} = \frac{\eta_2 I_E^{(1)}}{1+\beta_2 I_E^{(1)}}, b_{33} = \frac{\eta_2 S^{(1)}}{(1+\beta_2 I_E^{(1)})^2} - \epsilon(\xi_2 + \zeta_2 P^{(1)}) - n(1-\epsilon) - \mu, b_{42} = \delta(\xi_1 + \zeta_1 P^{(1)}),$

$b_{43} = \epsilon(\xi_2 + \zeta_2 P^{(1)}), b_{44} = -\theta - \mu, b_{52} = m(1-\delta), b_{53} = n(1-\epsilon), b_{55} = -\lambda - \mu - \mu_p.$

By solving $|J(E^{(1)}) - rI| = 0$ the characteristic equation can be obtained, as follows.

$$(-\mu - r)\left(\eta_1 S^{(1)} - \delta(\xi_1 + \zeta_1 P^{(1)}) - m(1-\delta) - \mu - r\right)(r^4 + Y_1 r^3 + Y_2 r^2 + Y_3 r + Y_4) = 0$$

where

$$Y_1 = -\left(b_{55} + b_{44} + b_{33} + b_{11}\right),$$

$$Y_2 = b_{53}\epsilon I_E^{(1)}\zeta_2 + b_{11}b_{33} + b_{11}b_{44} + b_{11}b_{55} - b_{13}b_{31} + b_{33}b_{44} + b_{33}b_{55} + b_{44}b_{55},$$

$$Y_3 = b_{53}b_{11}\epsilon I_E^{(1)}\zeta_2 - b_{44}b_{53}\epsilon I_E^{(1)}\zeta_2 - b_{11}b_{33}b_{44} - b_{11}b_{33}b_{55} - b_{11}b_{44}b_{55} + b_{13}b_{31}b_{44} + b_{13}b_{31}b_{55} - b_{31}b_{43}\theta - b_{33}b_{44}b_{55},$$

$$Y_4 = b_{11}b_{44}b_{53}\epsilon I_E^{(1)}\zeta_2 - b_{31}b_{53}\epsilon I_E^{(1)}\zeta_2 + b_{11}b_{33}b_{44}b_{55} - b_{13}b_{31}b_{44}b_{55} + b_{31}b_{43}b_{55}\theta.$$

From Eq. (3.3.2), we obtain two eigenvalues, $r_7 = -\mu$ which is negative since every parameter are positive and

$$r_8 = \eta_1 S^{(1)} - \delta(\xi_1 + \zeta_1 P^{(1)}) - m(1-\delta) - \mu < 0 \iff \eta_1 < \frac{\delta(\xi_1 + \zeta_1 P^{(1)}) + m(1-\delta) + \mu}{S^{(1)}}$$

Using the definition of the basic reproduction number $R_{01} = \frac{\eta_1 S^{(0)}}{\delta\xi_1 + m(1-\delta) + \mu} < 1$, and noting that $S^{(1)} \le \frac{\alpha}{\mu} = S^{(0)}$ and $P^{(1)} > 0$, we have

$$r_8 \le (\delta\xi_1 + m(1-\delta) + \mu)(R_{01} - 1) - \delta\zeta_1 P^{(1)} < 0$$

Therefore, when $R_{01} < 1$, we obtain $(\delta\xi_1 + m(1-\delta) + \mu)(R_{01} - 1) < 0$, and moreover $-\delta\zeta_1 P^{(1)} < 0$ further decreases r_8, ensuring that $r_8 < 0$. It's implying that $R_{01} < 1$ is one of the conditions for endemic equilibrium $E^{(1)}$ to be locally asymptotically stable.

The remaining eigenvalues are determined using equation

$$r^4 + Y_1 r^3 + Y_2 r^2 + Y_3 r + Y_4 = 0 \tag{8}$$

Eq. (8) will have roots with negative real parts if and only if the following Routh-Hurwitz conditions are satisfied.

$$Y_1, Y_2, Y_3, Y_4 > 0, \qquad Y_1 Y_2 - Y_3 > 0, \qquad Y_1 Y_2 Y_3 - Y_3^2 - Y_1^2 Y_4 > 0.$$

However, the Eq. (8) are highly complex result, making analytical verification of the Routh-Hurwitz condition impractical. Therefore, numerical simulations are performed in the next subsection to confirm the stability condition.

3.3.3. Endemic Equilibrium Point of Conventional Smokers ($E^{(2)}$)

The Jacobian matrix of the system evaluated at $(E^{(2)})$ is obtained as follows.

$$J\left(E^{(2)}\right) = \begin{pmatrix} c_{11} & c_{12} & -\eta_1 S^{(2)} & \theta & 0 & 0 \\ c_{21} & c_{22} & 0 & 0 & -\delta\zeta_1 I_K^{(2)} & 0 \\ 0 & 0 & c_{33} & 0 & 0 & 0 \\ 0 & c_{42} & c_{43} & c_{44} & \delta\zeta_1 I_K^{(2)} & 0 \\ 0 & c_{52} & c_{53} & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & \lambda & -\mu \end{pmatrix} \tag{9}$$

where $c_{11} = -\frac{\eta_1 I_K^{(2)}}{1+\beta_1 I_K^{(2)}} - \mu, c_{12} = -\frac{\eta_1 S^{(2)}}{(1+\beta_1 I_K^{(2)})^2}, c_{21} = \frac{\eta_1 I_K^{(2)}}{1+\beta_1 I_K^{(2)}}, c_{22} = \frac{\eta_1 S^{(2)}}{(1+\beta_1 I_K^{(2)})^2} - \delta(\xi_1 + \zeta_1 P^{(2)}) - m(1-\delta) - \mu, c_{33} = \eta_2 S^{(2)} - \epsilon(\xi_2 + \zeta_2 P^{(2)}) - n(1-\epsilon) - \mu, c_{42} = \delta(\xi_1 + \zeta_1 P^{(2)}),$
$c_{43} = \epsilon(\xi_2 + \zeta_2 P^{(2)}), c_{44} = -\theta - \mu, c_{52} = m(1-\delta), c_{53} = n(1-\epsilon), c_{55} = -\lambda - \mu - \mu_p.$

From matrix Eq. (9), by solving $|J(E^{(2)}) - rI| = 0$ the characteristic equation can be obtained, as follows.

$$(-\mu - r)\left(\eta_2 S^{(2)} - \epsilon(\xi_2 + \zeta_2 P^{(2)}) - n(1-\epsilon) - \mu - r\right)(r^4 + Z_1 r^3 + Z_2 r^2 + Z_3 r + Z_4) = 0 \quad (10)$$

where

$$\begin{aligned}
Z_1 &= -\left(c_{11} + c_{22} + c_{44} + c_{55}\right),\\
Z_2 &= c_{52}\delta I_K^{(2)}\zeta_1 + c_{11}c_{22} + c_{11}c_{44} + c_{11}c_{55} - c_{12}c_{21} + c_{22}c_{44} + c_{22}c_{55} + c_{44}c_{55},\\
Z_3 &= -c_{11}c_{52}\delta I_K^{(2)}\zeta_1 - c_{44}c_{52}\delta I_K^{(2)}\zeta_1 - c_{11}c_{22}c_{44} - c_{11}c_{22}c_{55} - c_{11}c_{44}c_{55} + c_{12}c_{21}c_{44} +\\
&\quad c_{12}c_{21}c_{55} - c_{21}c_{42}\theta - c_{22}c_{44}c_{55},\\
Z_4 &= c_{11}c_{44}c_{52}\delta I_K^{(2)}\zeta_1 - c_{21}c_{52}\delta I_K^{(2)}\theta\zeta_1 + c_{11}c_{22}c_{44}c_{55} - c_{12}c_{21}c_{44}c_{55} + c_{21}c_{42}c_{55}\theta.
\end{aligned}$$

We obtain two eigenvalues, $r_9 = -\mu$ which is negative and

$$r_{10} = \eta_2 S^{(2)} - \epsilon(\xi_2 + \zeta_2 P^{(2)}) - n(1-\epsilon) - \mu \iff \eta_2 < \frac{\epsilon(\xi_2 + \zeta_2 P^{(2)}) + n(1-\epsilon) + \mu}{S^{(2)}}$$

Using the definition of the basic reproduction number $R_{02} = \frac{\eta_2 S^{(0)}}{\delta\xi_2 + n(1-\epsilon) + \mu} < 1$, and noting that $S^{(2)} \leq \frac{\alpha}{\mu} = S^{(0)}$ and $P^{(2)} > 0$, we have

$$r_{10} \leq (\epsilon\xi_2 + n(1-\epsilon) + \mu)(R_{02} - 1) - \epsilon\zeta_2 P^{(2)} < 0$$

Therefore, when $R_{02} < 1$, we obtain $(\epsilon\xi_2 + m(1-\epsilon) + \mu)(R_{02} - 1) < 0$, and moreover $-\epsilon\zeta_2 P^{(2)} < 0$ further decreases r_{10}, ensuring that $r_{10} < 0$. It's implying that $R_{02} < 1$ is one of the conditions for endemic equilibrium $E^{(2)}$ to be locally asymptotically stable.

The remaining eigenvalues are determined using equation

$$r^4 + Z_1 r^3 + Z_2 r^2 + Z_3 r + Z_4 = 0 \quad (11)$$

Eq. (11) will have roots with negative real parts if and only if the following Routh-Hurwitz conditions are satisfied.

$$Z_1, Z_2, Z_3, Z_4 > 0, \qquad Z_1 Z_2 - Z_3 > 0, \qquad Z_1 Z_2 Z_3 - Z_3^2 - Z_1^2 Z_4 > 0.$$

However, the Eq. (11) are highly complex result, making analytical verification of the Routh-Hurwitz condition impractical. Therefore, numerical simulations are performed in the next subsection to confirm the stability condition.

3.3.4. Endemic Equilibrium Point of Conventional and E-cigarette Smokers ($E^{(3)}$)

Due to the analytical complexity of determining the stability of the endemic equilibrium point of conventional and e-cigarette smokers, we investigate its stability through numerical methods in the next subsection.

3.4. Parameter Sensitivity Analysis

Parameter sensitivity analysis was performed to assess the stability robustness of the model with respect to uncertainty in parameter values, particularly for parameters that were assumed due to limited empirical data. This approach allows for the identification of the parameters that most influence the stability of the model.

Table 2: Parameter Values

Parameter	Value	Unit	Source
α	1000	individual/day	Assumed
μ	0.031	day^{-1}	[17]
δ (proportion)	0.74	-	[25]
β_1	0.0005	individual^{-1}	Assumed
ξ_1	0.01	day^{-1}	[18]
ζ_1	0.0002	(individual $\cdot$ day)$^{-1}$	Assumed
ϵ (proportion)	0.98232	-	[26]
β_2	0.0005	individual^{-1}	Assumed
ξ_2	0.01	day^{-1}	[18]
ζ_2	0.0002	(individual $\cdot$ day)$^{-1}$	Assumed
m	0.02	day^{-1}	[17]
n	0.00136	day^{-1}	[26]
λ	0.2	day^{-1}	[27]
μ_p	0.8	day^{-1}	[27]
θ	0.4	day^{-1}	Assumed

To quantify the influence of each model parameter on the basic reproduction numbers R_{01} and R_{02}, the following formula is employed for given parameter ϕ in the Table 2.

$$C_{\phi}^{R_0} = \frac{\partial R_0}{\partial \phi} \times \frac{\phi}{R_0}$$

It should be noted that the larger the sensitivity index, the more influential the corresponding parameter. The positive or negative sign of the sensitivity index indicates the direction of the relationship between the parameter and R_0. Positive sign implies that an increase in the parameter increases R_0, whereas a negative sign indicates that an increase in the parameter decreases R_0. Parameters with zero sensitivity indices have no impact on the reproduction number and therefore do not affect the threshold behavior or the local stability.

Sensitivity indices were computed for all parameters. The results are summarized in the Table 3 and Table 4.

Table 3: Sensitivity indices with respect to R_{01}

Parameter	Value	Sensitivity Index	Impact on R_{01}
α	1000	1.00000	Increasing R_{01}
η_1	0.0000007	1.00000	Increasing R_{01}
μ	0.031	-1.71101	Decreasing R_{01}
δ	0.74	0.16972	Increasing R_{01}
ξ_1	0.01	-0.16972	Decreasing R_{01}
m	0.02	-0.11927	Decreasing R_{01}

Based on the sensitivity index results presented in the Table 3 and Table 4, it can be observed that the parameters with the greatest influence on R_{01} and R_{02} are the recruitment rate (α), the transmission rate of potential smokers to conventional smokers (η_1), the transmission rate of potential smokers to e-cigarette smokers (η_2), and the natural death rate (μ). The transmission rate (η_1), (η_2), and the recruitment rate (α) have sensitivity indices equal to one. This indicates

Table 4: Sensitivity indices with respect to R_{02}

Parameter	Value	Sensitivity Index	Impact on R_{02}
α	1000	1.00000	Increasing R_{02}
η_2	0.0000005	1.00000	Increasing R_{02}
μ	0.031	-1.75893	Decreasing R_{02}
ϵ	0.98232	-0.20778	Decreasing R_{02}
ξ_2	0.01	-0.24049	Decreasing R_{02}
n	0.00136	-0.00059	Decreasing R_{02}

a linear influence on the reproduction numbers, meaning that increases in these parameters significantly enhance the initial smoking potential and its spread. In contrast, the natural death rate (μ) has a negative sensitivity index, implying a reducing effect on R_{01} and R_{02}.

The remain parameter including $\beta_1, \beta_2, \zeta_1, \zeta_2, \lambda, \mu_p$, and θ have zero sensitivity indices. Consequently, variations in these parameters across a wide range of values do not alter the reproduction number or local stability. These results confirm that this stability is robust and does not depend on the specific values assumed for these parameters.

From the sensitivity analysis, it is evident that the reproduction number is primarily driven by the recruitment rate (α), the transmission rate of potential smokers to conventional smokers (η_1), the transmission rate of potential smokers to e-cigarette smokers (η_2), and the natural death rate (μ). This suggests that the spread of smoking behavior is mainly governed by these parameters.

3.5. Numerical Analysis

In this subsection we investigate the stability of equilibrium points numerically. This simulation uses the parameter values from Table 2. However, the transmission parameters η_1 and η_2 are varied based on the analytical conditions for the existence and stability.

3.5.1. Smoker-Free Equilibrium Point

Based on Eq. (3.3.1) and (3.3.1), if the system satisfies $\eta_1 < 0.0000013516$, equivalent to $R_{01} < 1$, and $\eta_2 < 0.0000012663$, equivalent to $R_{02} < 1$, then the system is asymptotically stable at the smoker-free equilibrium point. Using the parameter values $\eta_1 = 0.0000007$ and $\eta_2 = 0.0000005$, the reproduction number and eigenvalues were obtained as $R_{01} = 0.518, R_{02} = 0.395$ and eigenvalues $r_1 = -0.031, r_2 = -0.021, r_3 = -0.025, r_4 = -0.431, r_5 = -1.031, r_6 = -0.031$. Based on eigenvalues and basic reproduction number, smoker-free equilibrium point is asymptotically stable at $E^{(0)} = (32\,202, 0, 0, 0, 0, 0)$.

3.5.2. Endemic Equilibrium Point of E-cigarette Smokers ($E^{(1)}$)

$R_{02} > 1$ guarantees the existence of this equilibrium. $R_{02} > 1$ is equivalent to $\eta_2 > 0.0000012663$. Thus, we use $\eta_2 = 0.000005$ to analyze this equilibrium point. Based on the eigenvalue r_8 we obtained from the Jacobian matrix, $\eta_1 = 0.0000007$ was chosen.

Consequently, by solving the system with the given parameter values, the basic reproduction numbers were obtained as $R_{01} = 0.518$ and $R_{02} = 3.949$. Furthermore, this equilibrium point is obtained as follows.

$$E^{(1)} = \left(S^{(1)}, I_K^{(1)}, I_E^{(1)}, R_1^{(1)}, P^{(1)}, R_2^{(1)}\right) = (27\,434.06, 0, 4712.71, 107.65, 0.11, 0.71)$$

Next, we investigate the stability through the remaining eigenvalues of Eq. (3.3.2). The remaining eigenvalues are determined using equation Eq. (8). The Routh-Hurwitz conditions are obtained by substituting the parameter values given in Table 2 into the coefficients Y_1, Y_2, Y_3 and Y_4. We obtain the following coefficient values:

$$Y_1 = 1.5389, Y_2 = 0.5582, Y_3 = 0.0363, Y_4 = 0.0006.$$

It satisfies the conditions $Y_1, Y_2, Y_3, Y_4 > 0$ and $Y_1Y_2 - Y_3 > 0, Y_1Y_2Y_3 - Y_3^2 - Y_1^2Y_4 > 0$. All the Routh-Hurwitz conditions are satisfied. Therefore, all roots of Eq. (8) have negative real parts. Consequently, the endemic equilibrium point of e-cigarette smokers ($E^{(3)}$) is asymptotically stable.

3.5.3. Endemic Equilibrium Point of Conventional Smokers ($E^{(2)}$)

Furthermore, the parameter values were modified to $\eta_1 = 0.000007$ since it guarantees the existence of this equilibrium, namely when $R_{01} > 1$ equivalent to $\eta_1 > 0.0000013516$. The transmission parameter $\eta_2 = 0.0000005$ was chosen since it satisfies the condition of the eigenvalue r_{10}.

Consequently, by solving the system with the given parameter values, the basic reproduction numbers were obtained as $R_{01} = 5.179$ and $R_{02} = 0.395$. Furthermore, this equilibrium point is obtained as follows.

$$E^{(2)} = \left(S^{(2)}, I_K^{(2)}, I_E^{(2)}, R_1^{(2)}, P^{(2)}, R_2^{(2)}\right) = (25\,656.93, 5526.36, 0, 147.78, 27.87, 179.83)$$

Next, we investigate the remaining eigenvalues of Eq. (10). The remaining eigenvalues are determined using equation Eq. (11). The Routh-Hurwitz conditions are obtained by substituting the parameter values into the coefficients Z_1, Z_2, Z_3 and Z_4. We obtain the remaining eigenvalues, which are determined using equation Eq. (11). The Routh-Hurwitz conditions are obtained by substituting the parameter values into the coefficients Z_1, Z_2, Z_3 and Z_4. We obtain the following coefficient values:

$$Z_1 = 1.5287, Z_2 = 0.5431, Z_3 = 0.0313, Z_4 = 0.0005.$$

It satisfies the conditions $Z_1, Z_2, Z_3, Z_4 > 0$ and $Z_1Z_2 - Z_3 > 0, Z_1Z_2Z_3 - Z_3^2 - Z_1^2Z_4 > 0$. Hence, all the Routh-Hurwitz conditions are satisfied. Therefore, all roots of Eq. (11) have negative real parts. Consequently, the endemic equilibrium point of conventional smokers ($E^{(2)}$) is asymptotically stable.

3.5.4. Endemic Equilibrium Point of Conventional and E-cigarette Smokers ($E^{(3)}$)

This equilibrium point was obtained numerically using *fsolve* in Maple software after substituting the parameter values into the system. Using $\eta_1 = 0.000007$ which is equivalent to $R_{01} > 1$ and $\eta_2 = 0.000005$ which is equivalent to $R_{02} > 1$, since it guarantees the existence of this equilibrium point. The procedure was performed using the following initial guesses: $S^{(3)} = 100\,000, I_K^{(3)} = 10\,000, I_E^{(3)} = 10\,000, R_1^{(3)} = 10\,000, P^{(3)} = 10\,000, R_2^{(3)} = 10\,000$, and numerical tolerance was set to ensure the accuracy of the calculated solution to 10 digits. The results are given as follows.

$$E^{(3)} = \left(S^{(3)}, I_K^{(3)}, I_E^{(3)}, R_1^{(3)}, P^{(3)}, R_2^{(3)}\right) = (23\,237.26, 4884.17, 3084.62, 230.34, 24.71, 159.39)$$

The validity of the equilibrium point is confirmed by substituting the obtained solution back into the model equations, which provides ODE residuals on the order of 10^{-7} to 10^{-8}. It is verifying the equilibrium's validity.

Subsequently, we investigate its stability with the Jacobian matrix through a numerical method. Using Maple software, the eigenvalues are given as follows, $r_{11} = -1.027, r_{12} = -0.050 + 0.017i, r_{13} = -0.050 - 0.017i, r_{14} = -0.028, r_{15} = -0.033, r_{16} = -0.031$. All eigenvalues have negative real parts. Therefore, these numerical results show that the endemic equilibrium point of conventional and e-cigarette smokers ($E^{(3)}$) is asymptotically stable.

To obtain more accurate results, a numerical analysis of the Jacobian matrix was performed. The transmission parameters η_1 and η_2 were intentionally excluded from this substitution, as they serve as threshold parameters that determine the values of R_{01} and R_{02}. As a result, we obtain an eigenvalue, $r_{16} = -0.031$. The remaining eigenvalues are determined using Routh-Hurwitz conditions. From this Routh-Hurwitz condition we obtain that $\eta_2 \leq 0.0000078$.

Thus, if we set $\eta_2 = 0.0000078$, the characteristic equation is given as follows: $1000000000000000r^5 + (-54233282200000000\eta_1 + 1207317700000000)r^4 + (-619544624000000000\eta_1 + 196756790700000)r^3 + (-598694049600000000\eta_1 + 12339386900000)r^2 + (-3113342497000000\eta_1 + 336698020900)r - 60918253780000\eta_1 + 3354863761 = 0$ and we derive it using Routh-Hurwitz conditions.

Based on the Routh-Hurwitz conditions, we obtain $\eta_1 < 0.000055$. Both η_1 and η_2 satisfy the $R_{01} > 1$ and $R_{02} > 1$ conditions. Hence, this equilibrium point is asymptotically stable if both conditions are satisfied. Conversely, if both conditions are not satisfied, then at least one eigenvalue becomes positive. For instance, if we set $\eta_1 = 0.00006$ and $\eta_2 = 0.000009$ then one of the eigenvalues gives a positive number: $r = 0.0062$. However, if we set $\eta_1 = 0.00004$ and $\eta_2 = 0.000005$, then all the eigenvalues are negative.

3.6. Numerical Simulation

3.6.1. Time Series

To verify the analytical results, numerical simulations were conducted. By varying the value of η_1 and η_2, as explained in the previous subsection, we aim to further support the stability results. Numerical simulations are generated using the parameter values in Table 2 and the initial values: $S(0) = 1520, I_K(0) = 1600, I_E(0) = 1400, R_1(0) = 1150, P(0) = 110, R_2(0) = 15$, using MATLAB Online with ODE45.

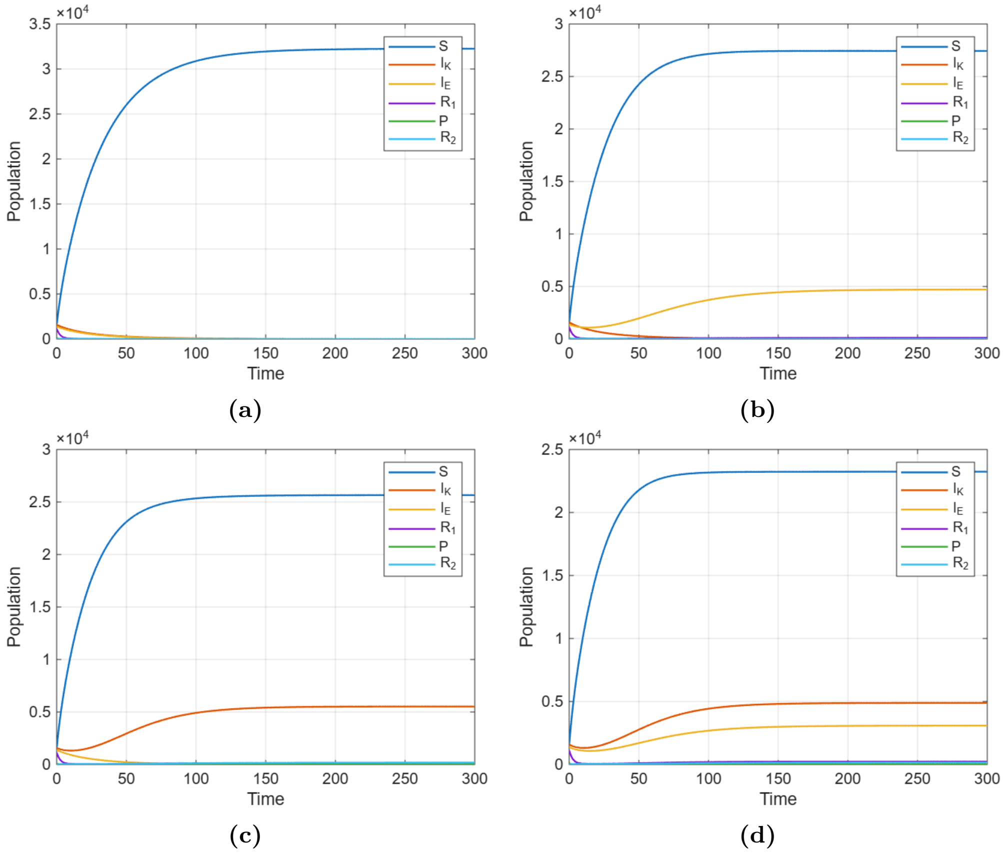


Fig. 2: Numerical Simulation of (a) Smoker-Free Equilibrium Point, (b) Endemic Equilibrium Point of E-Cigarette Smoker, (c) Endemic Equilibrium Point of Conventional Smoker, (d) Endemic Equilibrium Point of Conventional and E-cigarette Smokers

Fig. 2a shows that the populations of conventional smokers and e-cigarette smokers decrease to 0, indicating no smokers remain in the population. Only the susceptible or potential smokers population remains stable at value 32 202. This result is consistent with the analytical result and confirms the stability of smoker-free equilibrium point. All compartments will face extinction except susceptible or potential smokers.

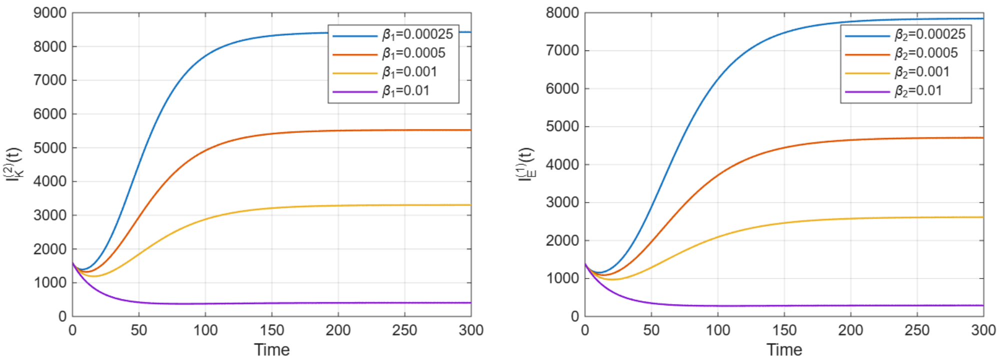
Furthermore, as simulated in Fig. 2b, the population of conventional smokers converges to 0, while the e-cigarette smokers stabilize at 4712.71. The remaining compartments persist in the population. It confirms the stability of the endemic equilibrium point of e-cigarette smokers we obtained in the previous subsection.

Next, in Fig. 2c, the population of conventional smokers is stable at 5526.36, while the population of e-cigarette smokers tends to 0. The susceptible or potential smokers exist in the population. The remaining, such as individuals who have quit smoking, smokers with lung cancer, and smokers who have recovered from lung cancer, remain in the population, although at relatively low levels. It confirms the stability of the endemic equilibrium point of conventional smokers we obtained.

Based on Fig. 2d, it is shown that both conventional smokers and e-cigarette smokers persist within the human population. None of the compartments tend to 0, which indicates that smoking behavior can be sustained in the human population over time. This simulation further supports the conclusion that the endemic equilibrium of conventional smokers is asymptotically stable, consistent with the results we obtained in the previous subsection.

Numerical simulation conducted in Fig. 2 illustrates the local stability of the four equilibrium points obtained from the analytical results. The simulation focused primarily on variations in the transmission rates η_1 and η_2. However, although transmission parameters determine the initial intensity of smoking, the simulation did not reflect the effect of saturation in the spread of smoking.

To determine the influence of the saturation effect on the spread of conventional smokers, variations of the parameter β_1 were introduced. The simulations are shown in the Fig. 3a. It demonstrates that varying the value of β_1 while keeping β_2 constant in the endemic equilibrium point of conventional smokers leads to changes in the conventional smokers population. When β_1 is small, the conventional smokers population reaches relatively high levels. As β_1 increases, the population gradually decreases. This behavior indicates that the saturation effect of potential smokers toward conventional smoking (β_1) plays an important role in reducing the prevalence of conventional smokers within the human population.

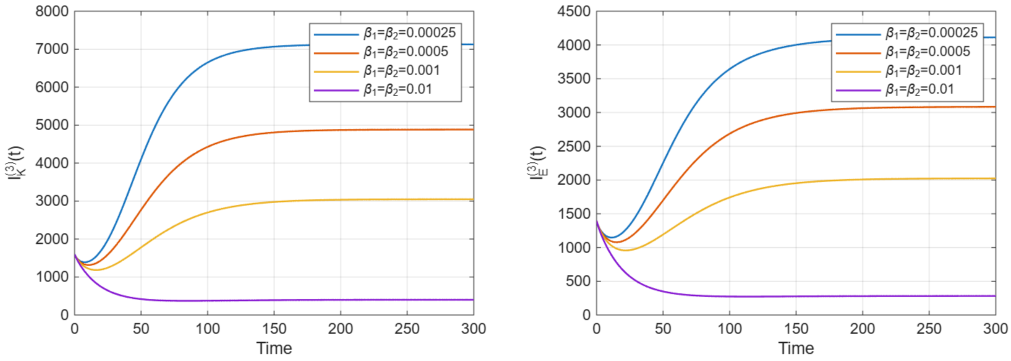


(a) Behavior of $I_K^2(t)$ under variations in β_1 **(b)** Behavior of $I_E^1(t)$ under variations in β_2

Fig. 3: Effect of saturation parameters on smoker populations.

Furthermore, to determine the extent of the saturation effect on the spread of e-cigarette smokers, variations of the parameter β_2 were introduced. From Fig. 3b, it can be concluded that varying the value of β_2 while keeping β_1 constant in the endemic equilibrium point of e-cigarette smokers also leads to changes in the e-cigarette smokers population. As the value of β_2 increases, the population of e-cigarette smokers declines. Therefore, the saturation effect of potential smokers toward e-cigarette smoking (β_2) plays an important role in reducing the population of e-cigarette smokers in the human population.

The saturation effect also influences the behavior of I_K and I_E under the endemic equilibrium point of conventional and e-cigarette smokers conditions. We use the same value to investigate the combined impact of saturation effects on the overall transmission dynamics under these conditions. While keeping the other parameters fixed and varying β_1 and β_2 simultaneously, it can be observed at the Fig. 4 that when β_1 and β_2 increase, both e-cigarette and conventional smokers populations decrease significantly.



(a) **(b)**

Fig. 4: Behavior of (a) $I^3_K(t)$, (b) $I^3_E(t)$ under variations in β_1 and β_2

3.6.2. Multi-scenario Simulations

A range analysis (multi-scenario simulations) were used to conduct an extra range analysis in order to address the uncertainty of assumed parameter values and confirm the robustness of the qualitative stability. The assumed parameters $\beta_1, \beta_2, \zeta_1, \zeta_2, \theta$ were changed within $\pm 20\%$ of their baseline values, while the other parameters remained constant. The analysis was conducted under conditions where both I_K and I_E had positive values, as this case represented the dynamics of interaction covering all parameters assumed in the system.

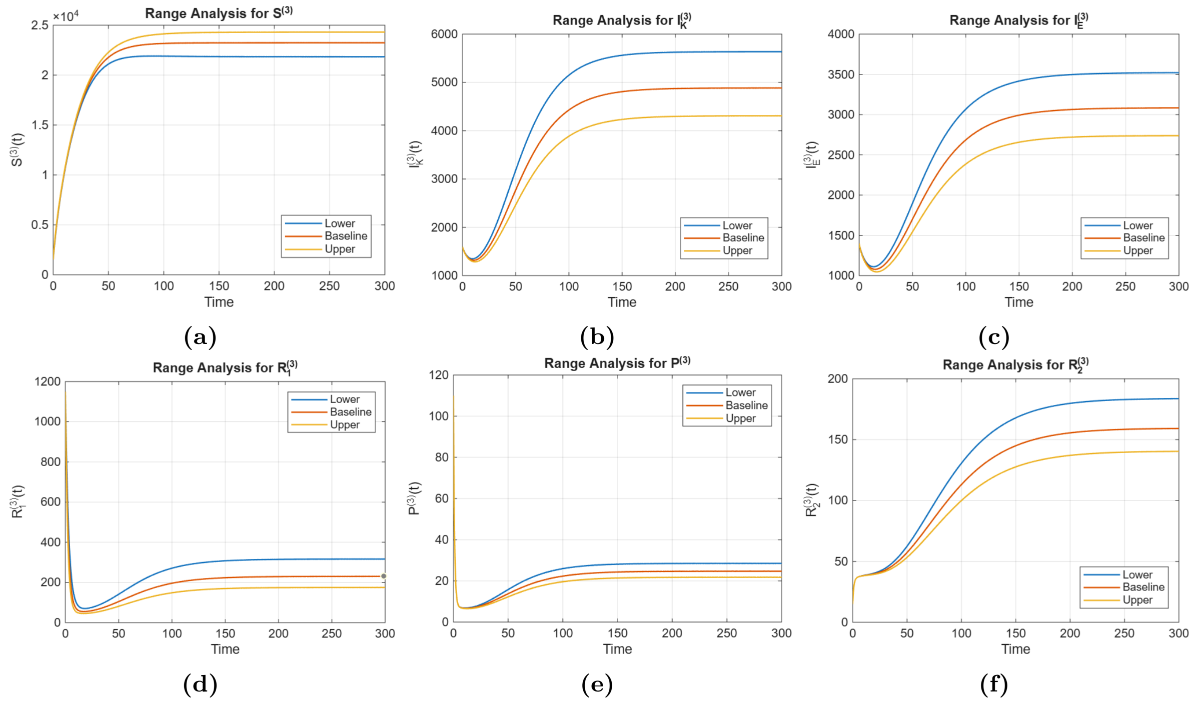


Fig. 5: Range Analysis for (a) $S^{(3)}$, (b) $I^{(3)}_K$, (c) $I^{(3)}_E$, (d) $R^{(3)}_1$, (e) $P^{(3)}$, (f) $R^{(3)}_2$.

By varying the assumed parameters by $\pm 20\%$ from the baseline value, a time series simulation is produced in Fig. 5. Although the numerical values change as the baseline parameters are varied, the qualitative stability behavior remains unchanged. There are no oscillations or divergent patterns. I_K and I_E remain in the population and exhibit the same qualitative behavior as in the baseline scenario. Therefore, the qualitative stability of the system is robust with respect to variations in the assumed parameters.

4. Conclusion

This study aims to develop and analyze a mathematical model of smoking behavior that classifies between conventional smokers and e-cigarette smokers, incorporates interaction with lung cancer patients, and considers the saturation effect on potential smokers as the number of smokers in the population increases. This model has four equilibrium points. Its local stability is determined using eigenvalue analysis and the basic reproduction number R_0 as a threshold parameter. The results show that when $R_{01} < 1, R_{02} < 1$, the smoker-free equilibrium point is asymptotically stable, indicating that smoking behavior gradually disappears from the population. When $R_{01} < 1, R_{02} > 1$, the endemic equilibrium point of e-cigarette smokers becomes asymptotically stable, implying the persistence of e-cigarette smoking behavior. When $R_{01} > 1$ and $R_{02} < 1$, the endemic equilibrium point of conventional smokers becomes asymptotically stable, implying the persistence of conventional smoking behavior. Meanwhile, when $R_{01} > 1$ and $R_{02} > 1$, the endemic equilibrium point of conventional and e-cigarette smokers becomes asymptotically stable, implying the persistence of smoking behavior that always exists over time.

Numerical simulations confirm the analytical findings and show that the intensity of smoking transmission affects the long term dynamics of the system. When transmission rates from both conventional and e-cigarette smokers are lower, the population transitions faster toward a smoker-free population. The saturation effect towards conventional smokers and e-cigarette smokers plays an important role in reducing the conventional and e-cigarette smokers in the human population.

Since this study classifies conventional cigarette smokers and e-cigarette users into two mutually exclusive groups, transitions between smoking products (dual use) are not included in this study. Allowing the transition between conventional smokers and e-cigarette smokers would introduce coupling between the two smoking compartments. This coupling may influence the overall threshold condition and lead to changes in equilibrium existence and stability. Therefore, further research can be conducted for further analysis.

CRediT Authorship Contribution Statement

Binti Mu'alafi Suryantini: Conceptualization, Methodology, Formal Analysis, Software, Writing–Original Draft. **Budi Priyo Prawoto:** Conceptualization, Methodology, Writing–Review, Validation.

Declaration of Generative AI and AI-assisted technologies

The authors acknowledge the use of generative artificial intelligence (AI) and AI-assisted technologies in the preparation of this manuscript. Specifically, Consensus AI was used to search for relevant scientific literature, QuillBot and ChatGPT version 5.2 were employed to assist with paraphrasing and improving grammatical quality. All intellectual contributions, interpretations, and final decisions regarding the content of the manuscript were made by the authors. The use of these technologies complies with ethical publication standards, and all AI-assisted content was edited to ensure accuracy, originality, and integrity.

Declaration of Competing Interest

The authors declare no competing interests.

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Data and Code Availability

All data and source code used in this research are freely accessible through public repositories.

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