CAUCHY – Jurnal Matematika Murni dan Aplikasi
Volume 11 (1) (2026), Pages 970-986
p-ISSN: 2086-0382; e-ISSN: 2477-3344

Total Edge Irregularity Strength of Cycle Snake Graphs

Stenly Pranata*, Vira Hari Krisnawati, and Darmajid

Department of Mathematics, Faculty of Mathematics and Natural Science,
University of Brawijaya, Indonesia

Abstract

Let G be a simple undirected graph with vertex and edge sets. A total labeling that assigns integers from 1 to k to the union of the vertex and edge sets is called a k-total labeling. The weight of an edge uv, denoted by $w(uv)$, is defined as the sum of the labels of the two vertices of u, v and the label of edge uv itself. A k-total labeling is called an edge irregular total k-labeling of G if the weights of all distinct edges are different. The minimum k for which every edge of G has a distinct weight is called the total edge irregularity strength of G, denoted by $tes(G)$. A cycle snake graph $CS_{m,n}$ is obtained from the path graph P_n with $n+1$ vertices and n edges by replacing each edge with a cycle graph C_m, where $m \geq 3$ and $n \geq 2$. In this paper, we study the graphs $CS_{3,n}$ and $CS_{m,n}$ and determine their total edge irregularity strength. The case $m = 3$ is considered separately because the structure of $CS_{3,n}$ gives a different representation of the vertex and edge sets than for $m \geq 4$, requiring a different labeling construction.

Keywords: cycle snake graph; edge irregular total k-labeling; total edge irregularity strength.

1. Introduction

Graph labeling is one of the fundamental topics in graph theory. A labeling of a graph G is defined as a function from the elements of G to a set of positive integers. Depending on the domain of the mapping, graph labelings are typically classified into three categories: vertex labelings, edge labelings, and total labelings, where labels are assigned to vertices, edges, or both, respectively [1]. Graph labeling has been widely studied due to its applications in various fields, including vehicle routing [2], coding theory [3], electrical networks [4], and brain tumor [5].

Total labeling has received considerable attention from researchers because it involves the assignment of labels to both vertices and edges. Let $G = (V, E)$ be a simple undirected graph with vertex set $V(G)$ and edge set $E(G)$, and let ∂ be a labeling from $V(G) \cup E(G)$ to the integer set $\{1, 2, \ldots, k\}$, called a k-total labeling. The weight of a vertex u, denoted by $w(u)$, is defined as $w(u) = \partial(u) + \sum_{uv \in E(G)} \partial(uv)$. The weight of an edge uv, denoted by $w(uv)$, is defined as $w(uv) = \partial(u) + \partial(v) + \partial(uv)$ [1]. One of the well-studied types of total labeling is total irregular labeling, introduced by M. Bača et al. in 2007 [6]. A k-total labeling of G is said to induce vertex irregularity if $w(u) \neq w(v)$ for all distinct $u, v \in V(G)$. Similarly, it induces edge irregularity if $w(uv) \neq w(ef)$ for all distinct $uv, ef \in E(G)$. The total vertex irregularity strength of a graph G, denoted by $tvs(G)$, is defined as $tvs(G) = \min\{k \in \mathbb{N} : w(u) \neq w(v) \text{ for all } u \neq v \in V(G)\}$. Analogously, the total edge irregularity strength of a graph G, denoted by $tes(G)$, is defined

---

*__Corresponding author__. E-mail: stenlypranata@gmail.com

Submitted: March 12, 2026 Reviewed: March 20, 2026 Accepted: April 14, 2026
DOI: https://doi.org/10.18860/cauchy.v11i1.41552

as $tes(G) = \min\{k \in \mathbb{N} : w(uv) \neq w(ef) \text{ for all } uv \neq ef \in E(G)\}$. The total vertex and edge irregularity strengths for several simple graph families, including paths, cycles, and stars, have been established [6]. This concept provides a fundamental basis for further research on the total vertex irregularity strength of various graph classes, as seen in [7–9]. It also underpins studies on the total edge irregularity strength of several other classes, such as complete and complete bipartite graphs [10], generalized prism graphs [11], and uniform centralized theta graphs [12].

The concepts of total vertex irregularity strength and total edge irregularity strength were further developed for classes of snake graph. A snake graph is defined as a path or cycle graph in which each edge is replaced by a graph from a specific class. The total edge irregularity strength of quintet snake graphs constructed by replacing each edge of the path graph P_n (with $n+1$ vertices and n edges) with a cycle graph C_5, denoted by $CS_{5,n}$ has been determined in 2021 by Salama [13]. In 2021, Salama and Elanin examined the total edge irregularity strength of several specific types of uniform theta snake graphs [14]. The total edge irregularity strength of heptagonal snake graphs, denoted by $CS_{7,n}$ has also been analyzed in 2022 by Salama [15]. More recently, the total edge irregularity strength for a general form of snake graphs derived from star graphs was established in 2025 by Attiya et al [16].

Based on previous studies [13, 15], the value of total edge irregularity strength has been investigated for cycle snake graphs constructed from the cycles C_5 and C_7. In these graphs, the number of vertices and edges in each cycle is fixed, allowing the construction of labeling patterns specifically tailored to the corresponding graph structures. However, when the cycle is generalized to C_m, the graph structure becomes more complex. This structural change causes the labeling patterns used for specific cycles to not be directly extendable to the more general case. Therefore, determining the total edge irregularity strength of the cycle snake graph $CS_{m,n}$ becomes a more mathematically challenging problem, since it requires constructing a labeling that produces distinct edge weights consistently for various values of m and n. Based on this motivation, this study aims to determine the total edge irregularity strength of the graph $CS_{3,n}$ and to obtain a general formula for the graph $CS_{m,n}$.

2. Methods

To facilitate the labeling process, first we need to consider the theorems the bounds on the total edge irregularity strength of graphs.

Theorem 1. *[6] Let $G = (V, E)$ be a graph with vertex set $V(G)$ and a non-empty edge set $E(G)$, then*

$$\left\lceil \frac{|E(G)| + 2}{3} \right\rceil \leq tes(G) \leq |E(G)|.$$

Theorem 2. *[6] Let $G = (V, E)$ be a graph with maximum degree $\Delta = \Delta(G)$, then*

1. $tes(G) \geq \left\lceil \frac{\Delta + 1}{2} \right\rceil$, *and*
2. $tes(G) \leq |E(G)| - \Delta$ *if* $\Delta \leq \frac{|E(G)| - 1}{2}$.

Next, we review the definition of the cycle snake graph $CS_{m,n}$, which is related to the path graph P_n with $n+1$ vertices and n edges, and the cycle graph C_m.

Definition 1. A cycle snake graph, denoted by $CS_{m,n}$, is constructed from a path graph P_n by replacing every edge with a copy of the cycle graph C_m, for $m \geq 3$ and $n \geq 2$.

An illustration of the snake graph $CS_{m,n}$ is presented in Fig. 1 to clarify the structure of the graph considered in this study. The cycle snake graph $CS_{m,n}$ is constructed from a base path consisting of $n+1$ vertices, $x_1, x_2, \ldots, x_{n+1}$. Each edge $x_i x_{i+1}$ with $1 \le i \le n$ of the base path is replaced by a cycle C_m consisting of m vertices and m edges, which becomes the i-th cycle along the base path. Each cycle comprises two additional sets of vertices: the top vertices $y_1^i, \ldots, y^i_{\lceil \frac{m-2}{2} \rceil}$ and the bottom vertices $z_1^i, \ldots, z^i_{\lceil \lfloor \frac{m-2}{2} \rfloor \rceil}$. The endpoints x_i and x_{i+1} are also included in the cycle. Edges are formed by connecting x_i to the first vertices of both the top and bottom sets. Consecutive vertices within each set are connected, and the last vertices of each set are connected to x_{i+1}. This construction results in a *snake*-like graph composed of n cycles, each with m vertices and m edges, connected consecutively along the base path.

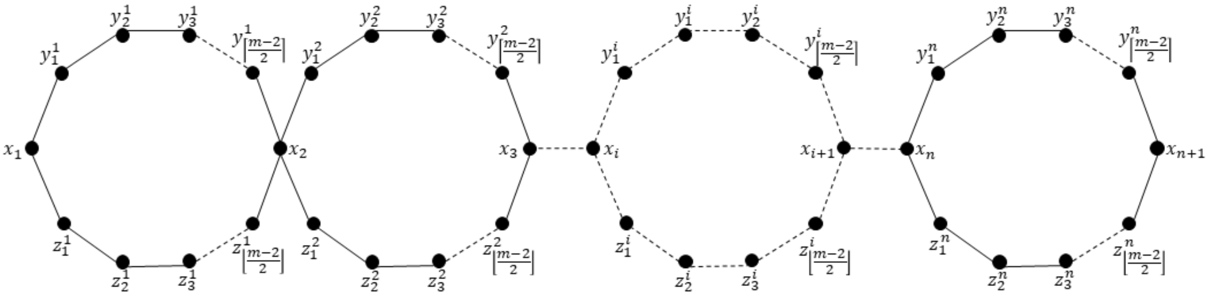

Fig. 1: Cycle snake graph $CS_{m,n}$

This study employs a theoretical and constructive approach to determine the total edge irregularity strength of the cycle snake graphs $CS_{3,n}$ and $CS_{m,n}$ for $m \ge 4$. The study is conducted through the following steps:

i. The structure of the graphs $CS_{3,n}$ and $CS_{m,n}$ is defined constructively, with each edge of the path graph P_n replaced by a cycle C_3 or C_m. Each new vertex in the cycle is assigned a pair of indices i and j, where the first index i indicates the i-th cycle in the path graph, and the second index j indicates the position of the vertex within the cycle.

ii. Based on the maximum value of the lower bounds in Theorem 1 and Theorem 2, the lower bound for the total edge irregularity strength of the cycle snake graphs is determined using the general formula $tes(G) \ge \left\lceil \frac{|E(G)|+2}{3} \right\rceil$. This value is then explicitly calculated according to the number of edges in the graphs $CS_{3,n}$ and $CS_{m,n}$.

iii. A total edge irregular labeling is constructed following a systematic pattern that depends on the position of each cycle in the path graph. For the first edge ($i=1$), the vertices and the edge are each assigned a label of 1, resulting in the smallest weight of 3. The labels on the subsequent vertices and edges gradually increase in a coordinated manner, which facilitates the derivation of explicit formulas for vertex labels, edge labels, and edge weights. Through this approach, each resulting weight is unique and it forms a systematic sequence.

iv. The edge weights are verified mathematically for each edge by calculating the weight values as given in poin iii and checking that each edge weight is unique. Consequently, the set of weights forms a sequence of consecutive integers, thereby providing an upper bound for $tes(CS_{3,n})$ and $tes(CS_{m,n})$ which satisfies $tes(G) \le \left\lceil \frac{|E(G)|+2}{3} \right\rceil$.

v. By using the constructive labeling and the mathematical verification in poin iv, it follows that the lower and upper bounds coincide. Therefore, the exact values $tes(CS_{3,n})$ and $tes(CS_{m,n})$ can be determined mathematically.

The findings of this research advance graph labeling theory by extending the concept of total edge irregularity strength to cycle snake graphs.

3. Results and Discussion

In this section, the main results on the total edge irregularity strength of cycle snake graphs are presented. Before discussing the general result for the graph $CS_{m,n}$, the special case $m = 3$ is first considered. This separation is necessary because the structure of the graph $CS_{3,n}$ has a different partition of the vertex and edge sets compared to the general case $m \geq 4$; therefore, a different labeling construction is required.

Lemma 1. *Let* $CS_{3,n}$ *be a cycle snake graph with* $n \geq 2$*, then*

$$tes(CS_{3,n}) = n + 1$$

Proof. The cycle snake graph $CS_{3,n}$ has order $2n + 1$ vertices, size $3n$ edges, and maximum degree $\Delta(CS_{3,n}) = 4$. According to Theorem 1, we have

$$tes(CS_{3,n}) \geq n + 1.$$

Let the vertex and edge set of $CS_{3,n}$ be defined as follows:

$$\begin{aligned} V(CS_{3,n}) &= \{x_i : 1 \leq i \leq n+1\} \cup \{y_i : 1 \leq i \leq n\}, \\ E(CS_{3,n}) &= \{x_i x_{i+1} : 1 \leq i \leq n\} \cup \{x_i y_i : 1 \leq i \leq n\} \cup \{x_{i+1} y_i : 1 \leq i \leq n\}, \end{aligned}$$

and define $\partial : V(CS_{3,n}) \cup E(CS_3, n) \rightarrow \{1, 2, ..., k\}$ as a total k-labeling given by

$$\begin{aligned} \partial(x_i) &= i, \quad \text{for } 1 \leq i \leq n+1. \\ \partial(y_i) = \partial(x_i y_i) = \partial(x_{i+1} y_i) &= i, \quad \text{for } 1 \leq i \leq n. \\ \partial(x_i x_{i+1}) &= i + 1, \quad \text{for } 1 \leq i \leq n. \end{aligned}$$

The edge weights of $CS_{3,n}$ are given as follows:

$$\begin{aligned} w(x_i y_i) &= 3i, \quad \text{for } 1 \leq i \leq n. \\ w(x_{i+1} y_i) &= 3i + 1, \quad \text{for } 1 \leq i \leq n. \\ w(x_i x_{i+1}) &= 3i + 2, \quad \text{for } 1 \leq i \leq n. \end{aligned}$$

It can be easily seen that by assigning labels to adjacent vertices and their edges, each edge weight is distinct and form a sequence of consecutive integers $3, 4, \ldots, 3n + 2$. Based on the definition of the labeling provided, it follows that $tes(CS_{3,n}) \leq n + 1$. The cycle snake graph $CS_{3,n}$ exhibits coinciding lower and upper bounds of the total edge irregularity strength, resulting in the exact value

$$tes(CS_{3,n}) = n + 1.$$

The constructed labeling thus attains the optimal value. □

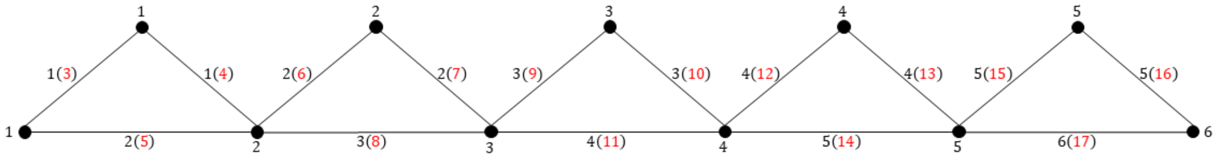

Fig. 2: Cycle snake graph $CS_{3,5}$ with $tes = 6$

Example 1. Fig. 2 illustrates the total labeling of $CS_{3,5}$ based on Lemma 1. The maximum vertex and edge labels are 6, and the edge weights are distinct and form the set $\{3,4,\dots,17\}$.

We now extend the previous results on snake cycle graphs with three vertices per cycle to a general case, in which each cycle contains $m \geq 4$ and $n \geq 2$. The following theorem presents the total edge irregularity strength for this broader class of snake cycle graphs.

Theorem 3. *Let $CS_{m,n}$ be a cycle snake graph with $m \geq 4$, then the following relation holds.*

$$tes(CS_{m,n}) = \left\lceil \frac{mn+2}{3} \right\rceil$$

Proof. The cycle snake graph has order $n(m-1)+1$ vertices, size mn edges, and maximum degree $\Delta(CS_{m,n}) = 4$. According to Theorem 1, we obtain the inequality

$$tes(CS_{m,n}) \geq \left\lceil \frac{mn+2}{3} \right\rceil.$$

Let the vertex and edge set of $CS_{m,n}$ be defined as follows:

$$\begin{aligned} V(CS_{m,n}) =& \{x_i : 1 \leq i \leq n+1\} \cup \left\{ y_j^i : 1 \leq i \leq n, 1 \leq j \leq \left\lceil \frac{m-2}{2} \right\rceil \right\} \\ & \cup \left\{ z_j^i : 1 \leq i \leq n, 1 \leq j \leq \left\lfloor \frac{m-2}{2} \right\rfloor \right\}, \\ E(CS_{m,n}) =& \{x_i y_j^i, x_i z_j^i : 1 \leq i \leq n, j = 1\} \cup \left\{ y_j^i y_{j+1}^i : 1 \leq i \leq n, 1 \leq j \leq \left\lceil \frac{m-2}{2} \right\rceil - 1 \right\} \\ & \cup \left\{ z_j^i z_{j+1}^i : 1 \leq i \leq n, 1 \leq j \leq \left\lfloor \frac{m-2}{2} \right\rfloor - 1 \right\} \\ & \cup \left\{ x_{i+1} y_j^i : 1 \leq i \leq n, j = \left\lceil \frac{m-2}{2} \right\rceil \right\} \\ & \cup \left\{ x_{i+1} z_j^i : 1 \leq i \leq n, j = \left\lfloor \frac{m-2}{2} \right\rfloor \right\}, \end{aligned}$$

and define $\partial : V(CS_{m,n}) \cup E(CS_{m,n}) \to \{1,2,\dots,k\}$ be a total k-labeling defined by the following cases.
Case 1: $m \equiv 0 (\mathrm{mod}\ 3)$. A total edge irregular labeling is defined as follows:

$$\partial(x_i) = \partial(x_i y_1^i) = \partial(x_i z_1^i) = (i-1)\left(\frac{m}{3}\right) + 1, \quad \text{for } 1 \leq i \leq n.$$

$$\partial(x_{n+1}) = n\left(\frac{m}{3}\right) + 1.$$

$$\partial(y_j^i) = (i-1)\left(\frac{m}{3}\right) + j - \left\lfloor \frac{j-1}{3} \right\rfloor, \quad \text{for } 1 \leq i \leq n, 1 \leq j \leq \left\lceil \frac{m-2}{2} \right\rceil.$$

$$\partial(z_j^i) = (i-1)\left(\frac{m}{3}\right) + j - \left\lceil \frac{j-1}{3} \right\rceil + 1, \quad \text{for } 1 \leq i \leq n, 1 \leq j \leq \left\lfloor \frac{m-2}{2} \right\rfloor.$$

$$\partial(y_j^i y_{j+1}^i) = (i-1)\left(\frac{m}{3}\right) + j - \left\lceil \frac{j-1}{3} \right\rceil + 1, \quad \text{for } 1 \leq i \leq n, 1 \leq j \leq \left\lceil \frac{m-2}{2} \right\rceil - 1.$$

$$\partial(z_j^i z_{j+1}^i) = (i-1)\left(\frac{m}{3}\right) + j - \left\lfloor \frac{j}{3} \right\rfloor + 1, \quad \text{for } 1 \leq i \leq n, 1 \leq j \leq \left\lfloor \frac{m-2}{2} \right\rfloor - 1.$$

For $x_{i+1}y_j^i$ with $1 \leq i \leq n$ and $j = \left\lceil \frac{m-2}{2} \right\rceil$,

$$\partial(x_{i+1}y_j^i) = \begin{cases} \frac{im}{3}, & \text{for } m \equiv 0(\text{mod } 2), \\ \frac{im}{3} + 1, & \text{for } m \equiv 1(\text{mod } 2). \end{cases}$$

For $x_{i+1}z_j^i$ with $1 \leq i \leq n$ and $j = \left\lfloor \frac{m-2}{2} \right\rfloor$,

$$\partial(x_{i+1}z_j^i) = \begin{cases} \frac{im}{3} + 1, & \text{for } m \equiv 0(\text{mod } 2), \\ \frac{im}{3}, & \text{for } m \equiv 1(\text{mod } 2). \end{cases}$$

The edge weights of $CS_{m,n}$ are given as follows:

$$w(x_iy_1^i) = m(i-1) + 3, \quad \text{for } 1 \leq i \leq n.$$
$$w(x_iz_1^i) = m(i-1) + 4, \quad \text{for } 1 \leq i \leq n.$$

For $y_j^iy_{j+1}^i$ with $1 \leq i \leq n$,

$$w(y_j^iy_{j+1}^i) = \begin{cases} m(i-1) + 3\left(j - \left\lceil \frac{j-1}{3} \right\rceil + 1\right), & \text{for } j \equiv 0(\text{mod } 3), \\ m(i-1) + 3\left(j - \left\lceil \frac{j-1}{3} \right\rceil\right) + 2, & \text{for } j \equiv 1(\text{mod } 3), \\ m(i-1) + 3\left(j - \left\lceil \frac{j-1}{3} \right\rceil\right) + 4, & \text{for } j \equiv 2(\text{mod } 3). \end{cases}$$

For $z_j^iz_{j+1}^i$ with $1 \leq i \leq n$,

$$w(z_j^iz_{j+1}^i) = \begin{cases} m(i-1) + 3\left(j - \left\lfloor \frac{j}{3} \right\rfloor\right) + 4, & \text{for } j \equiv 0(\text{mod } 3), \\ m(i-1) + 3\left(j - \left\lfloor \frac{j}{3} \right\rfloor + 1\right), & \text{for } j \equiv 1(\text{mod } 3), \\ m(i-1) + 3\left(j - \left\lfloor \frac{j}{3} \right\rfloor\right) + 2, & \text{for } j \equiv 2(\text{mod } 3). \end{cases}$$

For $x_{i+1}y_j^i$ with $1 \leq i \leq n$ and $j = \left\lceil \frac{m-2}{2} \right\rceil$,

$$w(x_{i+1}y_j^i) = \begin{cases} im + 1, & \text{for } m \equiv 0(\text{mod } 2), \\ im + 2, & \text{for } m \equiv 1(\text{mod } 2). \end{cases}$$

For $x_{i+1}z_j^i$ with $1 \leq i \leq n$ and $j = \left\lfloor \frac{m-2}{2} \right\rfloor$

$$w(x_{i+1}z_j^i) = \begin{cases} im + 2, & \text{for } m \equiv 0(\text{mod } 2), \\ im + 1, & \text{for } m \equiv 1(\text{mod } 2). \end{cases}$$

Case 2: $m \equiv 1(\text{mod } 3)$. Define the total edge irregular labeling as follows:

$$\partial(x_i) = (i-1)\left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{i}{3} \right\rfloor + 1, \quad \text{for } 1 \leq i \leq n+1.$$

$$\partial(x_i y_1^i) = \begin{cases} (i-1)\left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{i}{3} \right\rfloor, & \text{for } i \equiv 0 (\text{mod } 3), \\ (i-1)\left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{i}{3} \right\rfloor + 1, & \text{for } i \equiv 1 (\text{mod } 3) \text{ or } i \equiv 2 (\text{mod } 3). \end{cases}$$

$$\partial(x_i z_1^i) = \begin{cases} (i-1)\left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{i}{3} \right\rfloor + 1, & \text{for } i \equiv 0 (\text{mod } 3) \text{ or } i \equiv 1 (\text{mod } 3), \\ (i-1)\left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{i}{3} \right\rfloor + 2, & \text{for } i \equiv 2 (\text{mod } 3). \end{cases}$$

For y_j^i with $1 \le j \le \left\lceil \frac{m-2}{2} \right\rceil$,

$$\partial(y_j^i) = \begin{cases} (i-1)\left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{i}{3} \right\rfloor - \left\lfloor \frac{j}{3} \right\rfloor + j, & \text{for } i \equiv 0 (\text{mod } 3), \\ (i-1)\left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{i}{3} \right\rfloor - \left\lfloor \frac{j-1}{3} \right\rfloor + j, & \text{for } i \equiv 1 (\text{mod } 3), \\ (i-1)\left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{i}{3} \right\rfloor - \left\lfloor \frac{j+1}{3} \right\rfloor + j + 1, & \text{for } i \equiv 2 (\text{mod } 3). \end{cases}$$

For z_j^i with $1 \le j \le \left\lfloor \frac{m-2}{2} \right\rfloor$,

$$\partial(z_j^i) = \begin{cases} (i-1)\left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{i}{3} \right\rfloor - \left\lfloor \frac{j-1}{3} \right\rfloor + j, & \text{for } i \equiv 0 (\text{mod } 3), \\ (i-1)\left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{i}{3} \right\rfloor - \left\lfloor \frac{j+1}{3} \right\rfloor + j + 1, & \text{for } i \equiv 1 (\text{mod } 3), \\ (i-1)\left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{i+1}{3} \right\rfloor - \left\lfloor \frac{j}{3} \right\rfloor + j, & \text{for } i \equiv 2 (\text{mod } 3). \end{cases}$$

For $y_j^i y_{j+1}^i$ with $1 \le j \le \left\lceil \frac{m-2}{2} \right\rceil - 1$,

$$\partial(y_j^i y_{j+1}^i) = \begin{cases} (i-1)\left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{i}{3} \right\rfloor - \left\lfloor \frac{j-1}{3} \right\rfloor + j, & \text{for } i \equiv 0 (\text{mod } 3), \\ (i-1)\left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{i}{3} \right\rfloor - \left\lfloor \frac{j+1}{3} \right\rfloor + j + 1, & \text{for } i \equiv 1 (\text{mod } 3), \\ (i-1)\left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{i+1}{3} \right\rfloor - \left\lfloor \frac{j}{3} \right\rfloor + j, & \text{for } i \equiv 2 (\text{mod } 3). \end{cases}$$

For $z_j^i z_{j+1}^i$ with $1 \le j \le \left\lfloor \frac{m-2}{2} \right\rfloor - 1$,

$$\partial(z_j^i z_{j+1}^i) = \begin{cases} (i-1)\left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{i}{3} \right\rfloor - \left\lfloor \frac{j+1}{3} \right\rfloor + j + 1, & \text{for } i \equiv 0 (\text{mod } 3), \\ (i-1)\left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{i+2}{3} \right\rfloor - \left\lfloor \frac{j}{3} \right\rfloor + j, & \text{for } i \equiv 1 (\text{mod } 3), \\ (i-1)\left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{i+1}{3} \right\rfloor - \left\lfloor \frac{j-1}{3} \right\rfloor + j, & \text{for } i \equiv 2 (\text{mod } 3). \end{cases}$$

For $x_{i+1} y_j^i$ with $j = \left\lceil \frac{m-2}{2} \right\rceil$:
If $m \equiv 0 (\text{mod } 2)$, then

$$\partial(x_{i+1} y_j^i) = \begin{cases} i\left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{i}{3} \right\rfloor, & \text{for } i \equiv 0 (\text{mod } 3) \text{ or } i \equiv 2 (\text{mod } 3), \\ i\left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{i}{3} \right\rfloor + 1, & \text{for } i \equiv 1 (\text{mod } 3). \end{cases}$$

If $m \equiv 1 (\mathrm{mod}\ 2)$, then

$$\partial(x_{i+1}y_j^i) = i\left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{i}{3} \right\rfloor + 1, \quad \text{for } 1 \leq i \leq n.$$

For $x_{i+1}z_j^i$ with $j = \left\lfloor \frac{m-2}{2} \right\rfloor$:
If $m \equiv 0 (\mathrm{mod}\ 2)$, then

$$\partial(x_{i+1}z_j^i) = i\left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{i}{3} \right\rfloor + 1, \quad \text{for } 1 \leq i \leq n.$$

If $m \equiv 1 (\mathrm{mod}\ 2)$, then

$$\partial(x_{i+1}z_j^i) = \begin{cases} i\left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{i}{3} \right\rfloor, & \text{for } i \equiv 0 (\mathrm{mod}\ 3) \text{ or } i \equiv 2 (\mathrm{mod}\ 3), \\ i\left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{i}{3} \right\rfloor + 1, & \text{for } i \equiv 1 (\mathrm{mod}\ 3). \end{cases}$$

The edge weights of $CS_{m,n}$ are given as follows:

$$w(x_i y_1^i) = \begin{cases} 3\Big((i-1)\left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{i}{3} \right\rfloor\Big) + 2, & \text{for } i \equiv 0 (\mathrm{mod}\ 3), \\ 3\Big((i-1)\left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{i}{3} \right\rfloor + 1\Big), & \text{for } i \equiv 1 (\mathrm{mod}\ 3), \\ 3\Big((i-1)\left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{i}{3} \right\rfloor\Big) + 4, & \text{for } i \equiv 2 (\mathrm{mod}\ 3). \end{cases}$$

$$w(x_i z_1^i) = \begin{cases} 3\Big((i-1)\left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{i}{3} \right\rfloor + 1\Big), & \text{for } i \equiv 0 (\mathrm{mod}\ 3), \\ 3\Big((i-1)\left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{i}{3} \right\rfloor\Big) + 4, & \text{for } i \equiv 1 (\mathrm{mod}\ 3), \\ 3\Big((i-1)\left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{i}{3} \right\rfloor\Big) + 5, & \text{for } i \equiv 2 (\mathrm{mod}\ 3). \end{cases}$$

For $y_j^i y_{j+1}^i$:
If $i \equiv 0 (\mathrm{mod}\ 3)$, then

$$w(y_j^i y_{j+1}^i) = \begin{cases} 3\Big((i-1)\left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{i}{3} \right\rfloor - \left\lfloor \frac{j}{3} \right\rfloor + j\Big) + 2, & \text{for } j \equiv 0 (\mathrm{mod}\ 3), \\ 3\Big((i-1)\left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{i}{3} \right\rfloor - \left\lfloor \frac{j}{3} \right\rfloor + j\Big) + 1, & \text{for } j \equiv 1 (\mathrm{mod}\ 3), \\ 3\Big((i-1)\left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{i-1}{3} \right\rfloor - \left\lfloor \frac{j}{3} \right\rfloor + j\Big) + 3, & \text{for } j \equiv 2 (\mathrm{mod}\ 3). \end{cases}$$

If $i \equiv 1 (\mathrm{mod}\ 3)$, then

$$w(y_j^i y_{j+1}^i) = \begin{cases} 3\Big((i-1)\left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{i}{3} \right\rfloor - \left\lfloor \frac{j}{3} \right\rfloor + j\Big) + 3, & \text{for } j \equiv 0 (\mathrm{mod}\ 3), \\ 3\Big((i-1)\left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{i}{3} \right\rfloor - \left\lfloor \frac{j}{3} \right\rfloor + j\Big) + 2, & \text{for } j \equiv 1 (\mathrm{mod}\ 3), \\ 3\Big((i-1)\left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{i}{3} \right\rfloor - \left\lfloor \frac{j}{3} \right\rfloor + j\Big) + 1, & \text{for } j \equiv 2 (\mathrm{mod}\ 3). \end{cases}$$

If $i \equiv 2 (\mathrm{mod}\ 3)$, then

$$w(y_j^i y_{j+1}^i) = \begin{cases} 3\Big((i-1)\left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{i}{3} \right\rfloor - \left\lfloor \frac{j-1}{3} \right\rfloor + j\Big) + 1, & \text{for } j \equiv 0 (\mathrm{mod}\ 3), \\ 3\Big((i-1)\left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{i}{3} \right\rfloor - \left\lfloor \frac{j}{3} \right\rfloor + j\Big) + 3, & \text{for } j \equiv 1 (\mathrm{mod}\ 3), \\ 3\Big((i-1)\left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{i}{3} \right\rfloor - \left\lfloor \frac{j}{3} \right\rfloor + j\Big) + 2, & \text{for } j \equiv 2 (\mathrm{mod}\ 3). \end{cases}$$

For $z_j^i z_{j+1}^i$:
If $i \equiv 0 (\mathrm{mod}\ 3)$, then

$$w(z_j^i z_{j+1}^i) = \begin{cases} 3\Big((i-1)\left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{i+3}{3} \right\rfloor - \left\lfloor \frac{j}{3} \right\rfloor + j\Big), & \text{for } j \equiv 0 (\mathrm{mod}\ 3), \\ 3\Big((i-1)\left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{i}{3} \right\rfloor - \left\lfloor \frac{j}{3} \right\rfloor + j\Big) + 2, & \text{for } j \equiv 1 (\mathrm{mod}\ 3), \\ 3\Big((i-1)\left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{i}{3} \right\rfloor - \left\lfloor \frac{j}{3} \right\rfloor + j\Big) + 1, & \text{for } j \equiv 2 (\mathrm{mod}\ 3). \end{cases}$$

If $i \equiv 1 (\mathrm{mod}\ 3)$, then

$$w(z_j^i z_{j+1}^i) = \begin{cases} 3\Big((i-1)\left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{i+2}{3} \right\rfloor - \left\lfloor \frac{j}{3} \right\rfloor + j\Big) + 1, & \text{for } j \equiv 0 (\mathrm{mod}\ 3), \\ 3\Big((i-1)\left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{i+2}{3} \right\rfloor - \left\lfloor \frac{j}{3} \right\rfloor + j\Big), & \text{for } j \equiv 1 (\mathrm{mod}\ 3), \\ 3\Big((i-1)\left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{i}{3} \right\rfloor - \left\lfloor \frac{j}{3} \right\rfloor + j\Big) + 2, & \text{for } j \equiv 2 (\mathrm{mod}\ 3). \end{cases}$$

If $i \equiv 2 (\mathrm{mod}\ 3)$, then

$$w(z_j^i z_{j+1}^i) = \begin{cases} 3\Big((i-1)\left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{i}{3} \right\rfloor - \left\lfloor \frac{j-1}{3} \right\rfloor + j\Big) + 2, & \text{for } j \equiv 0 (\mathrm{mod}\ 3), \\ 3\Big((i-1)\left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{i+1}{3} \right\rfloor - \left\lfloor \frac{j}{3} \right\rfloor + j\Big) + 1, & \text{for } j \equiv 0 (\mathrm{mod}\ 3), \\ 3\Big((i-1)\left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{i+1}{3} \right\rfloor - \left\lfloor \frac{j}{3} \right\rfloor + j\Big), & \text{for } j \equiv 0 (\mathrm{mod}\ 3). \end{cases}$$

For $x_{i+1} y_j^i$ with $j = \left\lceil \frac{m-2}{2} \right\rceil$:
If $m \equiv 0 (\mathrm{mod}\ 2)$, then

$$w(x_{i+1} y_j^i) = \begin{cases} 3\Big(i \left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{i}{3} \right\rfloor\Big) + 1, & \text{for } i \equiv 0 (\mathrm{mod}\ 3), \\ 3\Big(i \left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{i}{3} \right\rfloor\Big) + 2, & \text{for } i \equiv 1 (\mathrm{mod}\ 3), \\ 3\Big(i \left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{i}{3} \right\rfloor + 1\Big), & \text{for } i \equiv 2 (\mathrm{mod}\ 3). \end{cases}$$

If $m \equiv 1(\text{mod } 2)$, then

$$w(x_{i+1}y_j^i) = \begin{cases} 3\left(i\left\lfloor \frac{m}{3}\right\rfloor + \left\lfloor \frac{i}{3}\right\rfloor\right) + 2, & \text{for } i \equiv 0(\text{mod } 3), \\ 3\left(i\left\lfloor \frac{m}{3}\right\rfloor + \left\lfloor \frac{i}{3}\right\rfloor + 1\right), & \text{for } i \equiv 1(\text{mod } 3) \\ 3\left(i\left\lfloor \frac{m}{3}\right\rfloor + \left\lfloor \frac{i}{3}\right\rfloor\right) + 4, & \text{for } i \equiv 2(\text{mod } 3). \end{cases}$$

For $x_{i+1}z_j^i$ with $j = \left\lfloor \frac{m-2}{2} \right\rfloor$:
If $m \equiv 0(\text{mod } 2)$, then

$$w(x_{i+1}z_j^i) = \begin{cases} 3\left(i\left\lfloor \frac{m}{3}\right\rfloor + \left\lfloor \frac{i}{3}\right\rfloor\right) + 2, & \text{for } i \equiv 0(\text{mod } 3), \\ 3\left(i\left\lfloor \frac{m}{3}\right\rfloor + \left\lfloor \frac{i}{3}\right\rfloor + 1\right), & \text{for } i \equiv 1(\text{mod } 3), \\ 3\left(i\left\lfloor \frac{m}{3}\right\rfloor + \left\lfloor \frac{i}{3}\right\rfloor\right) + 4, & \text{for } i \equiv 2(\text{mod } 3). \end{cases}$$

If $m \equiv 1(\text{mod } 2)$, then

$$w(x_{i+1}z_j^i) = \begin{cases} 3\left(i\left\lfloor \frac{m}{3}\right\rfloor + \left\lfloor \frac{i}{3}\right\rfloor\right) + 1, & \text{for } i \equiv 0(\text{mod } 3), \\ 3\left(i\left\lfloor \frac{m}{3}\right\rfloor + \left\lfloor \frac{i}{3}\right\rfloor\right) + 2, & \text{for } i \equiv 1(\text{mod } 3), \\ 3\left(i\left\lfloor \frac{m}{3}\right\rfloor + \left\lfloor \frac{i}{3}\right\rfloor + 1\right), & \text{for } i \equiv 2(\text{mod } 3). \end{cases}$$

Case 3: $m \equiv 2(\text{mod } 3)$. The total edge irregular labeling is defined as follows.

$$\partial(x_i) = (i-1)\left\lfloor \frac{m+3}{3} \right\rfloor - \left\lfloor \frac{i}{3} \right\rfloor + 1, \quad \text{for } 1 \le i \le n+1.$$

$$\partial(x_iy_1^i) = \begin{cases} (i-1)\left\lfloor \frac{m+3}{3} \right\rfloor - \left\lfloor \frac{i}{3} \right\rfloor + 1, & \text{for } i \equiv 0(\text{mod } 3) \text{ or } i \equiv 1(\text{mod } 3), \\ (i-1)\left\lfloor \frac{m+3}{3} \right\rfloor - \left\lfloor \frac{i}{3} \right\rfloor, & \text{for } i \equiv 2(\text{mod } 3). \end{cases}$$

$$\partial(x_iz_1^i) = \begin{cases} (i-1)\left\lfloor \frac{m+3}{3} \right\rfloor - \left\lfloor \frac{i}{3} \right\rfloor + 2, & \text{for } i \equiv 0(\text{mod } 3), \\ (i-1)\left\lfloor \frac{m+3}{3} \right\rfloor - \left\lfloor \frac{i}{3} \right\rfloor + 1, & \text{for } i \equiv 1(\text{mod } 3) \text{ or } i \equiv 2(\text{mod } 3). \end{cases}$$

For y_j^i with $1 \le j \le \left\lceil \frac{m-2}{2} \right\rceil$,

$$\partial(y_j^i) = \begin{cases} (i-1)\left\lfloor \frac{m+3}{3} \right\rfloor - \left\lfloor \frac{i}{3} \right\rfloor - \left\lfloor \frac{j+1}{3} \right\rfloor + j + 1, & \text{for } i \equiv 0(\text{mod } 3), \\ (i-1)\left\lfloor \frac{m+3}{3} \right\rfloor - \left\lfloor \frac{i}{3} \right\rfloor - \left\lfloor \frac{j-1}{3} \right\rfloor + j, & \text{for } i \equiv 1(\text{mod } 3), \\ (i-1)\left\lfloor \frac{m+3}{3} \right\rfloor - \left\lfloor \frac{i}{3} \right\rfloor - \left\lfloor \frac{j}{3} \right\rfloor + j, & \text{for } i \equiv 2(\text{mod } 3). \end{cases}$$

For z_j^i with $1 \leq j \leq \left\lfloor \frac{m-2}{2} \right\rfloor$,

$$\partial(z_j^i) = \begin{cases} (i-1)\left\lfloor \frac{m+3}{3} \right\rfloor - \left\lfloor \frac{i}{3} \right\rfloor - \left\lfloor \frac{j}{3} \right\rfloor + j + 1, & \text{for } i \equiv 0 (\text{mod } 3), \\ (i-1)\left\lfloor \frac{m+3}{3} \right\rfloor - \left\lfloor \frac{i}{3} \right\rfloor - \left\lfloor \frac{j+1}{3} \right\rfloor + j + 1, & \text{for } i \equiv 1 (\text{mod } 3), \\ (i-1)\left\lfloor \frac{m+3}{3} \right\rfloor - \left\lfloor \frac{i}{3} \right\rfloor - \left\lfloor \frac{j-1}{3} \right\rfloor + j, & \text{for } i \equiv 2 (\text{mod } 3). \end{cases}$$

For $y_j^i y_{j+1}^i$ with $1 \leq j \leq \left\lceil \frac{m-2}{2} \right\rceil - 1$,

$$\partial(y_j^i y_{j+1}^i) = \begin{cases} (i-1)\left\lfloor \frac{m+3}{3} \right\rfloor - \left\lfloor \frac{i}{3} \right\rfloor - \left\lfloor \frac{j}{3} \right\rfloor + j + 1, & \text{for } i \equiv 0 (\text{mod } 3), \\ (i-1)\left\lfloor \frac{m+3}{3} \right\rfloor - \left\lfloor \frac{i}{3} \right\rfloor - \left\lfloor \frac{j+1}{3} \right\rfloor + j + 1, & \text{for } i \equiv 1 (\text{mod } 3), \\ (i-1)\left\lfloor \frac{m+3}{3} \right\rfloor - \left\lfloor \frac{i}{3} \right\rfloor - \left\lfloor \frac{j-1}{3} \right\rfloor + j, & \text{for } i \equiv 2 (\text{mod } 3). \end{cases}$$

For $z_j^i z_{j+1}^i$ with $1 \leq j \leq \left\lfloor \frac{m-2}{2} \right\rfloor - 1$,

$$\partial(z_j^i z_{j+1}^i) = \begin{cases} (i-1)\left\lfloor \frac{m+3}{3} \right\rfloor - \left\lfloor \frac{i}{3} \right\rfloor - \left\lfloor \frac{j-1}{3} \right\rfloor + j + 1, & \text{for } i \equiv 0 (\text{mod } 3), \\ (i-1)\left\lfloor \frac{m+3}{3} \right\rfloor - \left\lfloor \frac{i}{3} \right\rfloor - \left\lfloor \frac{j}{3} \right\rfloor + j + 1, & \text{for } i \equiv 1 (\text{mod } 3), \\ (i-1)\left\lfloor \frac{m+3}{3} \right\rfloor - \left\lfloor \frac{i}{3} \right\rfloor - \left\lfloor \frac{j+1}{3} \right\rfloor + j + 1, & \text{for } i \equiv 2 (\text{mod } 3). \end{cases}$$

For $x_{i+1} y_j^i$ with $j = \left\lceil \frac{m-2}{2} \right\rceil$:
If $m \equiv 0 (\text{mod } 2)$, then

$$\partial(x_{i+1} y_j^i) = \begin{cases} i\left\lfloor \frac{m+3}{3} \right\rfloor - \left\lfloor \frac{i-1}{3} \right\rfloor - 1, & \text{for } i \equiv 0 (\text{mod } 3) \text{ or } i \equiv 1 (\text{mod } 3), \\ i\left\lfloor \frac{m+3}{3} \right\rfloor - \left\lfloor \frac{i-1}{3} \right\rfloor, & \text{for } i \equiv 2 (\text{mod } 3). \end{cases}$$

If $m \equiv 1 (\text{mod } 2)$, then

$$\partial(x_{i+1} y_j^i) = i\left\lfloor \frac{m+3}{3} \right\rfloor - \left\lfloor \frac{i-1}{3} \right\rfloor, \quad \text{for } 1 \leq i \leq n.$$

For $x_{i+1} z_j^i$ with $j = \left\lfloor \frac{m-2}{2} \right\rfloor$:
If $m \equiv 0 (\text{mod } 2)$, then

$$\partial(x_{i+1} z_j^i) = i\left\lfloor \frac{m+3}{3} \right\rfloor - \left\lfloor \frac{i-1}{3} \right\rfloor, \quad \text{for } 1 \leq i \leq n.$$

If $m \equiv 1 (\text{mod } 2)$, then

$$\partial(x_{i+1} z_j^i) = \begin{cases} i\left\lfloor \frac{m+3}{3} \right\rfloor - \left\lfloor \frac{i-1}{3} \right\rfloor - 1, & \text{for } i \equiv 0 (\text{mod } 3) \text{ or } i \equiv 1 (\text{mod } 3), \\ i\left\lfloor \frac{m+3}{3} \right\rfloor - \left\lfloor \frac{i-1}{3} \right\rfloor, & \text{for } i \equiv 2 (\text{mod } 3). \end{cases}$$

The edge weights of $CS_{m,n}$ are given as follows.

$$w(x_iy_1^i) = \begin{cases} 3\Big((i-1)\left\lfloor \frac{m+3}{3} \right\rfloor - \left\lfloor \frac{i}{3} \right\rfloor\Big) + 4, & \text{for } i \equiv 0 (\text{mod } 3), \\ 3\Big((i-1)\left\lfloor \frac{m+3}{3} \right\rfloor - \left\lfloor \frac{i}{3} \right\rfloor + 1\Big), & \text{for } i \equiv 1 (\text{mod } 3), \\ 3\Big((i-1)\left\lfloor \frac{m+3}{3} \right\rfloor - \left\lfloor \frac{i}{3} \right\rfloor\Big) + 2, & \text{for } i \equiv 2 (\text{mod } 3). \end{cases}$$

$$w(x_iz_1^i) = \begin{cases} 3\Big((i-1)\left\lfloor \frac{m+3}{3} \right\rfloor - \left\lfloor \frac{i}{3} \right\rfloor\Big) + 5, & \text{for } i \equiv 0 (\text{mod } 3), \\ 3\Big((i-1)\left\lfloor \frac{m+3}{3} \right\rfloor - \left\lfloor \frac{i}{3} \right\rfloor\Big) + 4, & \text{for } i \equiv 1 (\text{mod } 3), \\ 3\Big((i-1)\left\lfloor \frac{m+3}{3} \right\rfloor - \left\lfloor \frac{i}{3} \right\rfloor + 1\Big), & \text{for } i \equiv 2 (\text{mod } 3). \end{cases}$$

For $y_j^iy_{j+1}^i$:
If $i \equiv 0 (\text{mod } 3)$, then

$$w(y_j^iy_{j+1}^i) = \begin{cases} 3\Big((i-1)\left\lfloor \frac{m+3}{3} \right\rfloor - \left\lfloor \frac{i}{3} \right\rfloor - \left\lfloor \frac{j}{3} \right\rfloor + j\Big) + 4, & \text{for } j \equiv 0 (\text{mod } 3), \\ 3\Big((i-1)\left\lfloor \frac{m+3}{3} \right\rfloor - \left\lfloor \frac{i}{3} \right\rfloor - \left\lfloor \frac{j}{3} \right\rfloor + j + 1\Big), & \text{for } j \equiv 1 (\text{mod } 3), \\ 3\Big((i-1)\left\lfloor \frac{m+3}{3} \right\rfloor - \left\lfloor \frac{i}{3} \right\rfloor - \left\lfloor \frac{j}{3} \right\rfloor + j\Big) + 2, & \text{for } j \equiv 2 (\text{mod } 3). \end{cases}$$

If $i \equiv 1 (\text{mod } 3)$, then

$$w(y_j^iy_{j+1}^i) = \begin{cases} 3\Big((i-1)\left\lfloor \frac{m+3}{3} \right\rfloor - \left\lfloor \frac{i}{3} \right\rfloor - \left\lfloor \frac{j}{3} \right\rfloor + j + 1\Big), & \text{for } j \equiv 0 (\text{mod } 3), \\ 3\Big((i-1)\left\lfloor \frac{m+3}{3} \right\rfloor - \left\lfloor \frac{i}{3} \right\rfloor - \left\lfloor \frac{j}{3} \right\rfloor + j\Big) + 2, & \text{for } j \equiv 1 (\text{mod } 3), \\ 3\Big((i-1)\left\lfloor \frac{m+3}{3} \right\rfloor - \left\lfloor \frac{i}{3} \right\rfloor - \left\lfloor \frac{j+1}{3} \right\rfloor + j\Big) + 4, & \text{for } j \equiv 2 (\text{mod } 3). \end{cases}$$

If $i \equiv 2 (\text{mod } 3)$, then

$$w(y_j^iy_{j+1}^i) = \begin{cases} 3\Big((i-1)\left\lfloor \frac{m+3}{3} \right\rfloor - \left\lfloor \frac{i}{3} \right\rfloor - \left\lfloor \frac{j}{3} \right\rfloor + j\Big) + 2, & \text{for } j \equiv 0 (\text{mod } 3), \\ 3\Big((i-1)\left\lfloor \frac{m+3}{3} \right\rfloor - \left\lfloor \frac{i}{3} \right\rfloor - \left\lfloor \frac{j+2}{3} \right\rfloor + j\Big) + 4, & \text{for } j \equiv 1 (\text{mod } 3), \\ 3\Big((i-1)\left\lfloor \frac{m+3}{3} \right\rfloor - \left\lfloor \frac{i}{3} \right\rfloor - \left\lfloor \frac{j+1}{3} \right\rfloor + j + 1\Big), & \text{for } j \equiv 2 (\text{mod } 3). \end{cases}$$

For $z_j^iz_{j+1}^i$:

If $i \equiv 0 (\text{mod } 3)$, then

$$w(z_j^i z_{j+1}^i) = \begin{cases} 3\Big((i-1)\left\lfloor \frac{m+3}{3} \right\rfloor - \left\lfloor \frac{i}{3} \right\rfloor - \left\lfloor \frac{j-1}{3} \right\rfloor + j\Big) + 2, & \text{for } j \equiv 0 (\text{mod } 3), \\ 3\Big((i-1)\left\lfloor \frac{m+3}{3} \right\rfloor - \left\lfloor \frac{i}{3} \right\rfloor - \left\lfloor \frac{j}{3} \right\rfloor + j\Big) + 4, & \text{for } j \equiv 1 (\text{mod } 3), \\ 3\Big((i-1)\left\lfloor \frac{m+3}{3} \right\rfloor - \left\lfloor \frac{i}{3} \right\rfloor - \left\lfloor \frac{j}{3} \right\rfloor + j + 1\Big), & \text{for } j \equiv 2 (\text{mod } 3). \end{cases}$$

If $i \equiv 1 (\text{mod } 3)$, then

$$w(z_j^i z_{j+1}^i) = \begin{cases} 3\Big((i-1)\left\lfloor \frac{m+3}{3} \right\rfloor - \left\lfloor \frac{i}{3} \right\rfloor - \left\lfloor \frac{j}{3} \right\rfloor + j\Big) + 4, & \text{for } j \equiv 0 (\text{mod } 3), \\ 3\Big((i-1)\left\lfloor \frac{m+3}{3} \right\rfloor - \left\lfloor \frac{i}{3} \right\rfloor - \left\lfloor \frac{j}{3} \right\rfloor + j + 1\Big), & \text{for } j \equiv 1 (\text{mod } 3), \\ 3\Big((i-1)\left\lfloor \frac{m+3}{3} \right\rfloor - \left\lfloor \frac{i}{3} \right\rfloor - \left\lfloor \frac{j}{3} \right\rfloor + j\Big) + 2, & \text{for } j \equiv 2 (\text{mod } 3). \end{cases}$$

If $i \equiv 2 (\text{mod } 3)$, then

$$w(z_j^i z_{j+1}^i) = \begin{cases} 3\Big((i-1)\left\lfloor \frac{m+3}{3} \right\rfloor - \left\lfloor \frac{i}{3} \right\rfloor - \left\lfloor \frac{j}{3} \right\rfloor + j + 1\Big), & \text{for } j \equiv 0 (\text{mod } 3), \\ 3\Big((i-1)\left\lfloor \frac{m+3}{3} \right\rfloor - \left\lfloor \frac{i}{3} \right\rfloor - \left\lfloor \frac{j}{3} \right\rfloor + j\Big) + 2, & \text{for } j \equiv 1 (\text{mod } 3), \\ 3\Big((i-1)\left\lfloor \frac{m+3}{3} \right\rfloor - \left\lfloor \frac{i}{3} \right\rfloor - \left\lfloor \frac{j+1}{3} \right\rfloor + j\Big) + 4, & \text{for } j \equiv 2 (\text{mod } 3). \end{cases}$$

For $x_{i+1}y_j^i$ with $j = \left\lceil \frac{m-2}{2} \right\rceil$:
If $m \equiv 0 (\text{mod } 2)$, then

$$w(x_{i+1}y_j^i) = \begin{cases} 3\Big(i\left\lfloor \frac{m+3}{3} \right\rfloor - \left\lfloor \frac{i}{3} \right\rfloor\Big) + 1, & \text{for } i \equiv 0 (\text{mod } 3), \\ 3\Big(i\left\lfloor \frac{m+3}{3} \right\rfloor - \left\lfloor \frac{i}{3} \right\rfloor\Big), & \text{for } i \equiv 1 (\text{mod } 3), \\ 3\Big(i\left\lfloor \frac{m+3}{3} \right\rfloor - \left\lfloor \frac{i}{3} \right\rfloor\Big) - 1, & \text{for } i \equiv 2 (\text{mod } 3). \end{cases}$$

If $m \equiv 1 (\text{mod } 2)$, then

$$w(x_{i+1}y_j^i) = \begin{cases} 3\Big(i\left\lfloor \frac{m+3}{3} \right\rfloor - \left\lfloor \frac{i}{3} \right\rfloor\Big) + 2, & \text{for } i \equiv 0 (\text{mod } 3), \\ 3\Big(i\left\lfloor \frac{m+3}{3} \right\rfloor - \left\lfloor \frac{i}{3} \right\rfloor\Big) + 1, & \text{for } i \equiv 1 (\text{mod } 3), \\ 3\Big(i\left\lfloor \frac{m+3}{3} \right\rfloor - \left\lfloor \frac{i}{3} \right\rfloor\Big), & \text{for } i \equiv 2 (\text{mod } 3). \end{cases}$$

For $x_{i+1}z_j^i$ with $j = \left\lfloor \frac{m-2}{2} \right\rfloor$:

If $m \equiv 0 (\text{mod } 2)$, then

$$w(x_{i+1}z_j^i) = \begin{cases} 3\left(i\left\lfloor\frac{m+3}{3}\right\rfloor - \left\lfloor\frac{i-1}{3}\right\rfloor\right) - 1, & \text{for } i \equiv 0 (\text{mod } 3), \\ 3\left(i\left\lfloor\frac{m+3}{3}\right\rfloor - \left\lfloor\frac{i}{3}\right\rfloor\right) + 1, & \text{for } i \equiv 1 (\text{mod } 3), \\ 3\left(i\left\lfloor\frac{m+3}{3}\right\rfloor - \left\lfloor\frac{i}{3}\right\rfloor\right), & \text{for } i \equiv 2 (\text{mod } 3). \end{cases}$$

If $m \equiv 1 (\text{mod } 2)$, then

$$w(x_{i+1}z_j^i) = \begin{cases} 3\left(i\left\lfloor\frac{m+3}{3}\right\rfloor - \left\lfloor\frac{i-1}{3}\right\rfloor\right) - 2, & \text{for } i \equiv 0 (\text{mod } 3), \\ 3\left(i\left\lfloor\frac{m+3}{3}\right\rfloor - \left\lfloor\frac{i}{3}\right\rfloor\right), & \text{for } i \equiv 1 (\text{mod } 3), \\ 3\left(i\left\lfloor\frac{m+3}{3}\right\rfloor - \left\lfloor\frac{i}{3}\right\rfloor\right) - 1, & \text{for } i \equiv 2 (\text{mod } 3). \end{cases}$$

Based on the labeling rule, the resulting edge weights depend on the cycle index i and the position of the vertex within the cycle, denoted by the index j. The smallest edge weight is 3, which is obtained from the first edge in the cycle with $i = 1$, corresponding to the sum of the minimum labels assigned to two adjacent vertices and the label of the edge connecting them. From the weight formulas derived for each type of edge, it can be observed that the weight values increase systematically and consecutively as the indices i and j increase throughout the graph construction. Furthermore, the largest edge weight is $mn + 2$, which is obtained from the last edge in the cycle with $i = n$. Each edge in the graph is assigned a distinct weight, since the formulas defining the edge weights yield increasing sequences of integers for each edge type. When all edges in the graph are considered together, these values combine to from the consecutive set of integers $\{3, 4, 5, \ldots, mn + 2\}$. Since the number of these weights is equal to the number of edges in the graph $CS_{m,n}$, the constructed labeling yields all required edge weights exactly once. Using the determined labeling pattern, it can be concluded that $tes(CS_{m,n}) \leq \left\lceil\frac{mn+2}{3}\right\rceil$. This indicates that the lower and upper bounds of the total edge irregularity strength of the cycle snake graph $CS_{m,n}$ coincide, yielding the exact value

$$tes(CS_{m,n}) = \left\lceil\frac{mn+2}{3}\right\rceil.$$

This indicates that the constructed labeling achieves the optimal value. This result is consistent with previous studies on cycle snake graphs based on C_5 and C_7, which also attain optimal values of total edge irregularity strength. Therefore, this study extends those results by providing a general formula for $tes(CS_{m,n})$ for all m. □

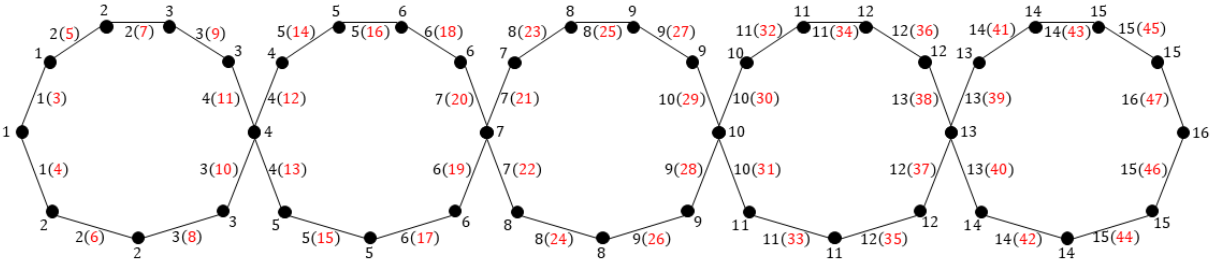

Fig. 3: Cycle snake graph $CS_{9,5}$ with $tes = 16$

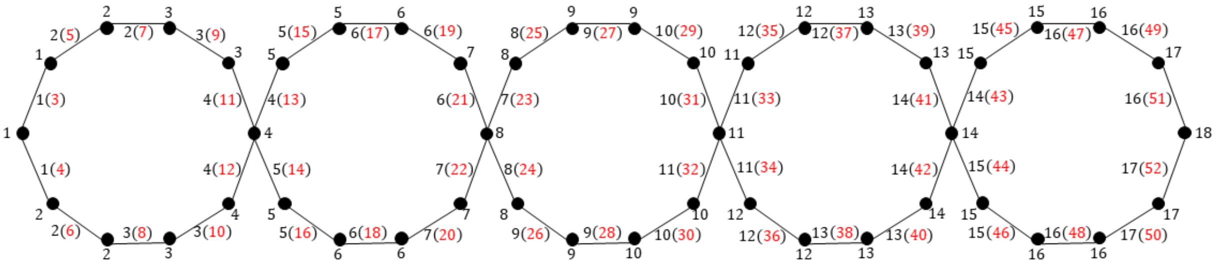

Fig. 4: Cycle snake graph $CS_{10,5}$ with $tes = 18$

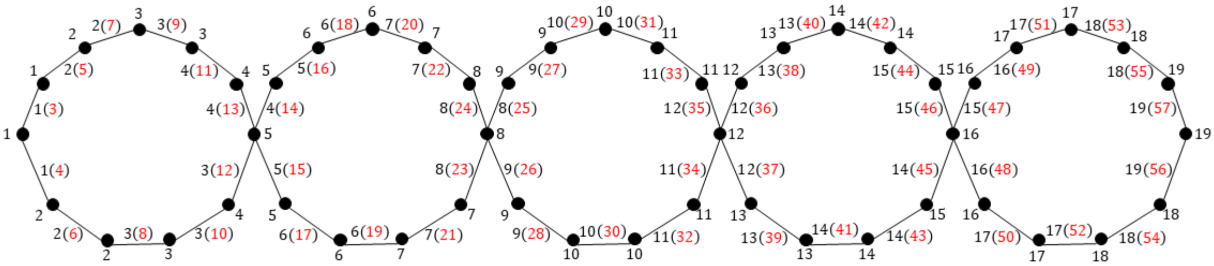

Fig. 5: Cycle snake graph $CS_{11,5}$ with $tes = 19$

Example 2. Fig. 3, Fig. 4, and Fig. 5 show the total labeling for the cases of Theorem 3. Fig. 3 depicts $CS_{9,5}$ as an example of Case 1 ($m \equiv 0 (\text{mod } 3)$), whose distinct edge weights form the set $\{3, 4, \ldots, 47\}$. An example of Case 2 ($m \equiv 1(\text{mod } 3)$) is presented in Fig. 4 using $CS_{10,5}$, whose distinct edge weights form the set $\{3, 4, \ldots, 52\}$. Case 3 ($m \equiv 2(\text{mod } 3)$) is showing in Fig. 5 using $CS_{11,5}$, with distinct edge weights forming the set $\{3, 4, \ldots, 57\}$.

4. Conclusion

In this article, the exact values of the total edge irregularity strength of the cycle snake graphs $CS_{3,n}$ and $CS_{m,n}$ are obtained mathematically as follows:

$$tes(CS_{3,n}) = n + 1 \quad \text{and} \quad tes(CS_{m,n}) = \left\lceil \frac{mn + 2}{3} \right\rceil.$$

These results are achieved through systematic constructive labeling and mathematical verification, which ensure that all edge weights are distinct and form a consecutive sequence of integers. This approach show that the lower and upper bounds of the total edge irregularity strength coincide, thereby confirming that the labeling attains the optimal value. The findings are consistent with previous studies on cycle snake graphs based on C_5 and C_7, which also attained the optimal total edge irregularity strength. Therefore, this study extends those results by providing a general formula for $tes(CS_{m,n})$ for all m, thus contributing to the development of graph labeling theory. These findings also suggest directions for future research, such as determining the total edge irregularity strength of snake graphs constructed from other simple graphs or investigating the total vertex irregularity strength of cycle snake graphs.

CRediT Authorship Contribution Statement

Stenly Pranata: Conceptualization, Methodology, Writing–Original Draft. **Vira Hari Krisnawati:** Supervision, validation, Writing–Review & Editing. **Darmajid:** Supervision, validation, Writing–Review & Editing.

Declaration of Generative AI and AI-assisted technologies

The authors declare that no generative AI or AI-assisted technologies were used to generate or modify the results of this research. AI tools were used solely for grammatical and language editing purposes.

Declaration of Competing Interest

The authors declare no competing interests.

Funding and Acknowledgments

This research received no external funding.

References

[1] W. D. Wallis. *Magic Graphs*. Birkhäuser, 2001. DOI: 10.1007/978-1-4612-0123-6.

[2] Ruslan Sadykov, Artur Pessoa, and Eduardo Uchoa. "A Bucket Graph Based Labelling Algorithm for Vehicle Routing Pricing". In: *POC Autumn School on Advanced BCP Tools*. Paris, France, Nov. 2019. https://hal.science/hal-02378624/.

[3] Ghaffar Raeisi and Mohammad Gholami. "Edge coloring of graphs with applications in coding theory". In: *China Communications* 18.1 (2021), pp. 181–195. DOI: 10.23919/JCC.2021.01.016.

[4] P. Mariaraja, Hussein Z. Almngoshi, C. M. Hilda Jerlin, S. Muthuperumal, K. Amarendra, Sudhakar Sengan, and Pankaj Dadheech. "Lucky edge geometric mean labeling of graphs and applications in electrical networks". In: *Journal of Discrete Mathematical Sciences and Cryptography* 27.7 (2024), pp. 2133–2141. DOI: 10.47974/JDMSC-2086.

[5] C. Yogalakshmi and B. J. Balamurugan. "Graph coloring–driven topological indices for QSPR modeling and MCDM prioritization of brain tumor drugs". In: *Results in Engineering* 29 (2026), p. 109470. DOI: 10.1016/j.rineng.2026.109470.

[6] Martin Bača, Stanislav Jendrol', Mirka Miller, and Joseph Ryan. "On irregular total labellings". In: *Discrete Mathematics* 307.11–12 (2007), pp. 1378–1388. DOI: 10.1016/j.disc.2005.11.075.

[7] Nurdin, E. T. Baskoro, A. N. M. Salman, and N. N. Gaos. "On the total vertex irregularity strength of trees". In: *Discrete Mathematics* 310.21 (2010), pp. 3043–3048. DOI: 10.1016/j.disc.2010.06.041.

[8] Ali Ahmad, E. T. Baskoro, and M. Imran. "Total vertex irregularity strength of disjoint union of helm graphs". In: *Discussiones Mathematicae Graph Theory* 32.3 (2012), pp. 427–434. DOI: 10.7151/dmgt.1619.

[9] Nurdin Hinding, Hye Kyung Kim, Nurtiti Sunusi, and Riskawati Mise. "On Total Vertex Irregularity Strength of Hexagonal Cluster Graphs". In: *International Journal of Mathematics and Mathematical Sciences* 2021 (2021), p. 2743858. DOI: 10.1155/2021/2743858.

[10] Stanislav Jendrol', Jozef Miškuf, and Roman Soták. "Total edge irregularity strength of complete graphs and complete bipartite graphs". In: *Discrete Mathematics* 310.3 (2010), pp. 400–407. DOI: 10.1016/j.disc.2009.03.006.

[11] Martin Bača and Muhammad Kamran Siddiqui. "Total edge irregularity strength of generalized prism". In: *Applied Mathematics and Computation* 235 (2014), pp. 168–173. DOI: 10.1016/j.amc.2014.03.001.

[12] Riyan Wicaksana Putra and Yeni Susanti. "On total edge irregularity strength of centralized uniform theta graphs". In: *AKCE International Journal of Graphs and Combinatorics* 15.1 (2018), pp. 7–13. DOI: 10.1016/j.akcej.2018.02.002.

[13] F. Salama. "Computing the total edge irregularity strength for quintet snake graph and related graphs". In: *Journal of Discrete Mathematical Sciences and Cryptography* 25.8 (2022), pp. 2491–2504. DOI: 10.1080/09720529.2021.1878627.

[14] Fatma Salama and Randa M. Abo Elanin. "On total edge irregularity strength for some special types of uniform theta snake graphs". In: *AIMS Mathematics* 6.8 (2021), pp. 8127–8148. DOI: 10.3934/math.2021471.

[15] F. Salama. "Computing total edge irregularity strength for heptagonal snake graph and related graphs". In: *Soft Computing* 26.1 (2022), pp. 155–164. DOI: 10.1007/s00500-021-06364-2.

[16] Hala Attiya, Nasr Ahmed, and Fatema Salama. "Total Edge Irregularity Strength of Star Snake Graphs". In: *European Journal of Pure and Applied Mathematics* 18.2 (2025), p. 5673. DOI: 10.29020/nybg.ejpam.v18i2.5673.