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Construction of a Family of Minimal Helicoid Surfaces Passing Through an Isoparametric Helix Curve

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Abstract

This research concerns the construction of a family of minimal surfaces passing through a helical curve. The construction follows the research by Yoon , which employs an approach using the curvature and torsion functions of a given curve. The construction yields a family of generalized helicoids as the minimal surfaces passing through the helical curve.

Keywords: Curvature; Helicoid; Helix; Minimal Surface; Torsion.

1. Introduction

When a closed framework, such as a ring, is immersed in a soap solution, a soap film forms across the frame. Due to surface tension, the naturally formed soap film seeks to minimize its potential energy. This soap film is intrinsically linked to a specific class of surfaces studied in differential geometry known as minimal surfaces [2]. A minimal surface is defined as a surface with zero mean curvature at every point [3]. The theory of minimal surfaces in three-dimensional Euclidean space is grounded in the calculus of variations, first developed by Euler and Lagrange in the 18th century. The study of minimal surfaces was subsequently advanced in the 19th century by mathematicians such as Enneper, Scherk, Schwarz, Riemann, and Weierstrass [4]. Furthermore, minimal surfaces constitute the solution to Plateau's problem, which seeks to find the surface of minimal area spanned by a given fixed boundary curve [5].

The catenoid is an example of a minimal surface that forms naturally from a soap film by positioning two circular frames, which have been coated in soap solution, close to one another. Fig. 1a illustrates this naturally formed soap film catenoid. In addition to the catenoid, another example of a minimal surface is the helicoid, formed via a spiral-shaped framework, as shown in Fig. 1b. Both the catenoid and the helicoid are examples of non-trivial minimal surfaces discovered by Jean Baptiste Meusnier in 1776.

In constructing minimal surfaces, Li, Wang, and Zhu [6] applied an approximation approach involving geodesics and Dirichlet functions that incorporate the constant curvature and torsion of the curve. Li, Wang, and Zhu concluded that the method of Wang, Tang, and Tai [7] is recommended for constructing parametric surfaces from a single geodesic curve using the Dirichlet function approach. Meanwhile, for constructing surfaces involving geodesic pairs, it is suggested to use cubic Hermite polynomials or bicubic surfaces. Kahyaoğlu and Kasap [8] constructed a

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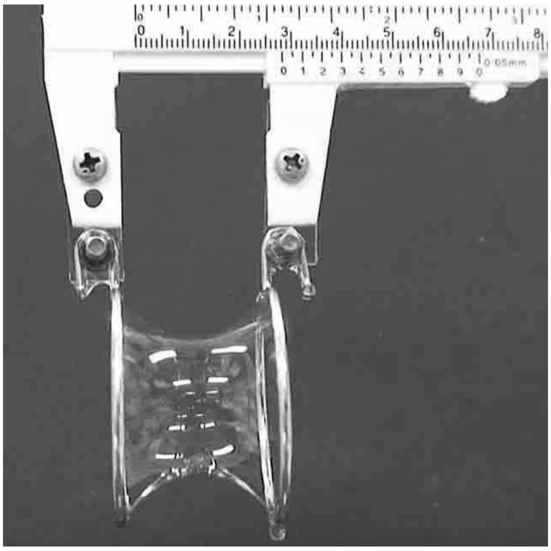


(a) Soap film Catenoid

(b) Soap film Helicoid

family of minimal surfaces involving various surface concepts, including harmonic surfaces and geodesic curves.

Based on the results of previous studies, geodesics, defined as the shortest path between two points on a surface, can be utilized in the construction of minimal surfaces. While this study focuses on the analytical construction of minimal surfaces using differential geometry, the generation of minimal surfaces and geodesics has also been extensively explored through discrete computational methods, such as graph cuts [9]. The study of integrating minimal surfaces and geodesics has emerged as a compelling topic and is applied in numerous fields. Sánchez-Reyes [10] conducted research regarding the construction of minimal surfaces from a geodesic, with applications in clothing, ribbon, and shoe design. Meanwhile, in 2021, González-Quintial and Martín-Pastor applied structural models of minimal surfaces and geodesics in architectural design and construction [11].

Yoon [1] applied a similar method to Kahyaoğlu and Kasap [8] research with slight variations in certain definitions. In that study, Yoon built upon the research of Sánchez-Reyes [10], which outlined a method for constructing minimal surfaces via the geodesic concept, as well as the work of Li, Wang, and Zhu [6]. Yoon 's work yielded theorems concerning the construction of minimal surfaces that pass through circle and helix curves as closed and open curves, with these curves acting as geodesics on the generated minimal surfaces. These results provided a minimal surfaces from both closed and open geodesic curves.

In his research, Yoon [1] constructed minimal surfaces based on curves with constant curvature and constant torsion. However, the resulting surfaces were not classified into specific, well-known families, such as the Catenoid or similar types. This article connects Yoon 's findings regarding constant curvature and torsion to a certain surface, generalized helicoid.

2. Theoretical Background

This section presents the fundamental concepts required for constructing minimal helicoid surfaces. The discussion begins with basic notions of curves and their Frenet frames, followed by generalized helicoid surfaces and minimal surfaces.

2.1. Curves

A curve can be viewed as the path traced by a moving point. Mathematically, a curve is defined as follows.

Definition 1. [3] A parameterized curve in $\mathbb{R}^n$ is a map

$$\boldsymbol{\gamma} : (\alpha, \beta) \to \mathbb{R}^n$$

where (α, β) is an open interval with $(\alpha, \beta) = \{t \in \mathbb{R} | \alpha < t < \beta\}$.

To define the derivative of a parameterized curve, the concept of a smooth function is first introduced. A function $f : (\alpha, \beta) \to \mathbb{R}$ is said to be smooth if the derivative $\frac{d^n f}{dt^n}$ exists for every $n \geq 1$ and for every $t \in (\alpha, \beta)$. Next, several definitions regarding the derivative of a parameterized curve are given, namely:

Definition 2. [3] If $\boldsymbol{\gamma}$ is a parameterized curve, then the first derivative of $\boldsymbol{\gamma}$ with respect to t, denoted by $\dot{\boldsymbol{\gamma}}(t)$, is called the tangent vector of $\boldsymbol{\gamma}$ at the point $\boldsymbol{\gamma}(t)$.

For a parameterized curve, the length of the curve bounded by two points is called the arc length, which is defined as follows.

Definition 3. [3] The arc length of a curve $\boldsymbol{\gamma}$ starting from the point $\boldsymbol{\gamma}(t_0)$, denoted by $s(t)$, is

$$s(t) = \int_{t_0}^{t} ||\dot{\boldsymbol{\gamma}}(u)|| du.$$

The definition of a unit speed curve is given below:

Definition 4. [3] If $\boldsymbol{\gamma} : (\alpha, \beta) \to \mathbb{R}^n$ is a parameterized curve, then the speed of the curve $\boldsymbol{\gamma}$ at the point $\boldsymbol{\gamma}(t)$ is $||\dot{\boldsymbol{\gamma}}(t)||$, and it is called a unit speed curve if $\dot{\boldsymbol{\gamma}}(t)$ is a unit vector for every $t \in (\alpha, \beta)$.

The curvature of a unit speed curve is defined as follows.

Definition 5. [3] If $\boldsymbol{\gamma}$ is a unit speed curve with parameter t, then the curvature of the curve $\boldsymbol{\gamma}$, denoted by $\kappa(t)$ at the point $\boldsymbol{\gamma}(t)$, is defined as $||\ddot{\boldsymbol{\gamma}}(t)||$.

Let $\boldsymbol{\gamma}(s)$ be a unit speed curve in $\mathbb{R}^3$ and let the vector $\mathbf{t}$, given by

$$\mathbf{t} = \dot{\boldsymbol{\gamma}}$$

be the unit tangent vector of the curve $\boldsymbol{\gamma}$. If the curvature $\kappa(s)$ of $\boldsymbol{\gamma}$ is non-zero, then the principal normal vector is defined as

$$\mathbf{n}(s) = \frac{\dot{\mathbf{t}}(s)}{||\dot{\mathbf{t}}(s)||}, \quad \dot{\mathbf{t}} \neq \mathbf{0}.$$

Since $||\dot{\mathbf{t}}|| = \kappa$, $\mathbf{n}$ is a unit vector; thus, the principal normal vector can be referred to as the unit normal vector. Furthermore, the vector $\mathbf{b}(s)$ is defined to be orthogonal to both $\mathbf{t}(s)$ and $\mathbf{n}(s)$, where

$$\mathbf{b} = \mathbf{t} \times \mathbf{n}.$$

The vector $\mathbf{b}(s)$ is called the binormal vector of the curve $\boldsymbol{\gamma}$ at the point $\boldsymbol{\gamma}(s)$.

2.2. Generalized Helicoid Surfaces

Wang, Tang, and Tai [7] provided a definition of a family of parametric surfaces as a set of parametric surfaces formed using the Frenet frame and passing through a space curve. A parametric surface is a surface expressed by

$$\mathbf{X}(s,t) = \boldsymbol{\gamma}(s) + (f(s,t), g(s,t), h(s,t)) \begin{pmatrix} \mathbf{t}(s) \\ \mathbf{n}(s) \\ \mathbf{b}(s) \end{pmatrix}, \tag{1}$$
$$s_1 \leq s \leq s_2, \quad t_1 \leq t \leq t_2, \quad s,t \in \mathbb{R}$$

where s is the arc length parameter, t is the time parameter, and $f(s,t), g(s,t),$ and $h(s,t)$ are smooth functions.

To facilitate the analysis of parametric surfaces, the functions $f(s,t), g(s,t)$ and $h(s,t)$ can be written as

$$f(s,t) = u(t), \quad g(s,t) = v(t), \quad h(s,t) = w(t),$$

where $u(t), v(t), w(t)$ are smooth functions.

A curve lying on a surface is called an isoparametric curve, defined as follows.

Definition 6. [1] A curve $\boldsymbol{\gamma}(s)$ on a parametric surface $X(s,t)$ defined by Eq. (1) is called an isoparametric curve if there exists a time t_0 such that $\boldsymbol{\gamma}(s) = X(s,t_0)$.

A generalized helicoid surface is a surface formed by the rotation and translation of every point about an axis. The definition of a generalized helicoid is given below.

Definition 7. [12] Let $\boldsymbol{\gamma}$ be a space curve with $\boldsymbol{\gamma}(t) = (\alpha(t), \beta(t), \gamma(t))$. Then, a generalized helicoid with base curve $\boldsymbol{\gamma}$ and pitch c is a surface in $\mathbb{R}^3$ with the parameterization

$$\boldsymbol{\sigma}(u,v) = (\alpha(v)\cos u - \beta(v)\sin u, \alpha(v)\sin u + \beta(v)\cos u, cu + \gamma(v)).$$

2.3. Minimal Surfaces

The mean curvature H of a surface $\boldsymbol{\sigma}$ can be formulated as

$$H = \frac{LG - 2MF + NE}{2(EG - F^2)}$$

where

$$E = \boldsymbol{\sigma}_u \cdot \boldsymbol{\sigma}_u, \quad F = \boldsymbol{\sigma}_u \cdot \boldsymbol{\sigma}_v, \quad G = \boldsymbol{\sigma}_v \cdot \boldsymbol{\sigma}_v,$$
$$\mathbf{N} = \frac{\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v}{||\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v||},$$
$$L = \boldsymbol{\sigma}_{uu} \cdot \mathbf{N}, \quad M = \boldsymbol{\sigma}_{uv} \cdot \mathbf{N}, \quad N = \boldsymbol{\sigma}_{vv} \cdot \mathbf{N},$$

and $\mathbf{N}$ is the unit normal vector of the surface [3]. A minimal surface is defined as follows.

Definition 8. [13] A Minimal Surface is a surface whose mean curvature is zero at every point on the surface.

3. Results and Discussion

This section applies the theoretical framework to construct a family of minimal helicoid surfaces passing through an isoparametric helix curve. The discussion begins by defining the circular helix curve used as the base curve before deriving the corresponding minimal surfaces.

3.1. Circular Helix Curves

A helix is a curve with a winding shape resembling a spring, screw, or spiral staircase, which can be found in the model of DNA (*deoxyribonucleic acid*, the genetic material in cells) [14]. A visualization of a helix can be seen in Fig. 2 below.

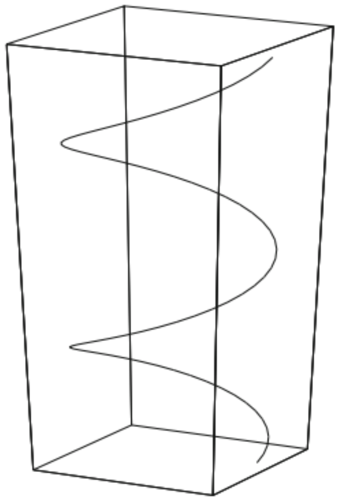


Fig. 2: Helix Curve [2]

Let a unit speed helix curve be given with arc length parameter s, constant curvature $\kappa(s) = \kappa_0$, and constant torsion $\tau(s) = \tau_0$. The parameterization of the given helix curve is

$$\boldsymbol{\gamma}(s) = \left(\frac{\kappa_0}{\kappa_0^2+\tau_0^2}\cos\left(\sqrt{\kappa_0^2+\tau_0^2}\,s\right), \frac{\kappa_0}{\kappa_0^2+\tau_0^2}\sin\left(\sqrt{\kappa_0^2+\tau_0^2}\,s\right), \frac{\tau_0}{\sqrt{\kappa_0^2+\tau_0^2}}s\right). \tag{2}$$

To simplify Eq. (2), let $\mu_0 = \sqrt{\kappa_0^2+\tau_0^2}$, then Eq. (2) can be written as

$$\boldsymbol{\gamma}(s) = \left(\frac{\kappa_0}{\mu_0^2}\cos(\mu_0 s), \frac{\kappa_0}{\mu_0^2}\sin(\mu_0 s), \frac{\tau_0}{\mu_0}s\right). \tag{3}$$

The Frenet frame $\mathbf{t}, \mathbf{n},$ and $\mathbf{b}$ of the helix curve with $\kappa(s) = \kappa_0$ and $\tau(s) = \tau_0$ is

$$\begin{aligned}
\mathbf{t}(s) &= \left(-\frac{\kappa_0}{\mu_0}\sin(\mu_0 s), \frac{\kappa_0}{\mu_0}\cos(\mu_0 s), \frac{\tau_0}{\mu_0}\right).\\
\mathbf{n}(s) &= (-\cos(\mu_0 s), -\sin(\mu_0 s), 0).\\
\mathbf{b}(s) &= \left(\frac{\tau_0}{\mu_0}\sin(\mu_0 s), -\frac{\tau_0}{\mu_0}\cos(\mu_0 s), \frac{\kappa_0}{\mu_0}\right).
\end{aligned} \tag{4}$$

3.2. Construction of Minimal Helicoid Surfaces Passing Through an Isoparametric Helix Curve

Referring to the research of Yoon [1], the initial step in constructing a minimal surface through a curve is to determine the base curve of the surface to be constructed. Before proceeding to the main stages of minimal surface construction, several conditions regarding minimal surfaces are first presented, namely:

> **Definition 9.** [1] If $\mathbf{X}(s,t)$ satisfies $E = G$ and $F = 0$, then the surface $\mathbf{X}(s,t)$ is called an isothermal surface.

Definition 10. [15] If the parametric surface $\mathbf{X}(s,t)$ satisfies $\frac{\partial^2 \mathbf{X}}{\partial s^2} + \frac{\partial^2 \mathbf{X}}{\partial t^2} = \mathbf{0}$, then the surface $\mathbf{X}(s,t)$ is called a harmonic surface.

Based on these conditions, a lemma regarding the requirement for a minimal surface is given:

Lemma 1. *[1] A surface with isothermal parameters is a minimal surface if and only if the surface is a harmonic surface.*

Based on Lemma 1, the following theorem is presented regarding the construction of minimal surfaces as performed by Yoon [1]:

Theorem 1. *Let $\boldsymbol{\gamma}$ be a unit speed isoparametric curve on a surface in $\mathbb{R}^3$. The surface parameterized by*

$$\boldsymbol{X}(s,t) = \boldsymbol{\gamma}(s) + u(t)\boldsymbol{t}(s) + v(t)\boldsymbol{n}(s) + w(t)\boldsymbol{b}(s)$$

is said to be a minimal surface if and only if there exist functions $u, v,$ and w satisfying:

$$\begin{aligned}(1-\kappa(s)v(t))^2 + (\kappa(s)u(t) - \tau(s)w(t))^2 + \tau^2(s)v^2(t)\\ -u'^2(t) - v'^2(t) - w'^2(t) = 0,\end{aligned} \tag{5}$$

$$u'(t)(1-\kappa(s)v(t)) + v'(t)(\kappa(s)u(t) - \tau(s)w(t)) + \tau(s)v(t)w'(t) = 0, \tag{6}$$

$$\kappa'(s)v(t) + \kappa^2(s)u(t) - \kappa(s)\tau(s)w(t) - u''(t) = 0, \tag{7}$$

$$\kappa'(s)u(t) - \tau'(s)w(t) + \kappa(s) - \kappa^2(s)v(t) - \tau^2(s)v(t) + v''(t) = 0, \tag{8}$$

$$\tau'(s)v(t) + \kappa(s)\tau(s)u(t) - \tau^2(s)w(t) + w''(t) = 0, \tag{9}$$

Based on Yoon research, in constructing minimal surface from positive constant curvature and non zero torsion which is a circular helix, this theorem is an extend from Yoon theorem which related to constructing minimal surface from circular helix.

Theorem 2. *Let $\boldsymbol{\gamma}$ be a unit speed isoparametric curve with positive constant curvature and non zero torsion on a surface in $\mathbb{R}^3$. If the surface parameterized by*

$$\boldsymbol{X}(s,t) = \boldsymbol{\gamma}(s) + u(t)\boldsymbol{t}(s) + v(t)\boldsymbol{n}(s) + w(t)\boldsymbol{b}(s)$$

is minimal based on Eqs. (5) until (9), then the surface is part of a generalized Helicoid.

Proof. Using a unit speed isoparametric helix curve with non-zero constant curvature and torsion $\kappa(s) = \kappa_0$ and $\tau(s) = \tau_0$ as the base curve where $\kappa_0 > 0, \tau_0 \neq 0$, then differentiating each with respect to s yields $\kappa'(s) = 0$ and $\tau'(s) = 0$. After substitution, Eqs. (7), (8), and (9) become respectively

$$(1-\kappa_0 v(t))^2 + (\kappa_0 u(t) - \tau_0 w(t))^2 + \tau_0^2 v^2(t) - u'^2(t) - v'^2(t) - w'^2(t) = 0, \tag{10}$$

$$u'(t)(1-\kappa_0 v(t)) + v'(t)(\kappa_0 u(t) - \tau_0 w(t)) + \tau_0 v(t) w'(t) = 0, \tag{11}$$

$$\kappa_0^2 u(t) - \kappa_0\tau_0 w(t) - u''(t) = 0, \tag{12}$$

$$\kappa_0 - \kappa_0^2 v(t) - \tau_0^2 v(t) + v''(t) = 0, \tag{13}$$

$$\kappa_0\tau_0 u(t) - \tau_0^2 w(t) + w''(t) = 0. \tag{14}$$

Note that Eqs. (12) and (14) are interrelated. Eq. (14) can be written as

$$u(t) = \frac{1}{\kappa_0\tau_0}(\tau_0^2 w(t) - w''(t)). \tag{15}$$

By differentiating the function $u(t)$ with respect to t in Eq. (15) and substituting it into Eq. (12), we obtain

$$\kappa_0^2\frac{1}{\kappa_0\tau_0}(\tau_0^2 w(t) - w''(t)) - \kappa_0\tau_0 w(t) - \frac{1}{\kappa_0\tau_0}(\tau_0^2 w''(t) - w^{(4)}(t)) = 0. \tag{16}$$

Simplifying Eq. (16), we obtain

$$w^{(4)}(t) - (\kappa_0^2 + \tau_0^2)w''(t) = 0. \tag{17}$$

To find the solution to the differential equation in Eq. (17), the characteristic equation method is used, yielding

$$w(t) = c_1 e^{\sqrt{\kappa_0^2+\tau_0^2}t} + c_2 e^{-\sqrt{\kappa_0^2+\tau_0^2}t} + c_3 t + c_4. \tag{18}$$

Let $\mu_0 = \sqrt{\kappa_0^2 + \tau_0^2}$, then Eq. (18) can be written as

$$w(t) = c_1 e^{\mu_0 t} + c_2 e^{-\mu_0 t} + c_3 t + c_4.$$

By differentiating the function $w(t)$ with respect to t and substituting it into Eq. (15), we obtain

$$u(t) = \frac{1}{\kappa_0\tau_0}(\tau_0^2(c_1 e^{\mu_0 t} + c_2 e^{-\mu_0 t} + c_3 t + c_4) - (\mu_0^2 c_1 e^{\mu_0 t} + \mu_0^2 c_2 e^{-\mu_0 t})). \tag{19}$$

The function $u(t)$ in Eq. (19) can be simplified to

$$u(t) = \frac{\tau_0}{\kappa_0}c_3 t + \frac{\tau_0}{\kappa_0}c_4 - \frac{\kappa_0}{\tau_0}c_1 e^{\mu_0 t} - \frac{\kappa_0}{\tau_0}c_2 e^{-\mu_0 t}. \tag{20}$$

After finding the solutions to Eqs. (12) and (14), the next step is to find the solution to Eq. (13) using the characteristic equation method, yielding

$$v(t) = d_1 e^{\mu_0 t} + d_2 e^{-\mu_0 t} + \frac{\kappa_0}{\mu_0^2}. \tag{21}$$

Thus, the solutions to Eqs. (12), (13), and (14) are

$$\begin{aligned}
u(t) &= \frac{\tau_0}{\kappa_0}c_3 t + \frac{\tau_0}{\kappa_0}c_4 - \frac{\kappa_0}{\tau_0}c_1 e^{\mu_0 t} - \frac{\kappa_0}{\tau_0}c_2 e^{-\mu_0 t},\\
v(t) &= d_1 e^{\mu_0 t} + d_2 e^{-\mu_0 t} + \frac{\kappa_0}{\mu_0^2},\\
w(t) &= c_1 e^{\mu_0 t} + c_2 e^{-\mu_0 t} + c_3 t + c_4.
\end{aligned} \tag{22}$$

The next step is to substitute the obtained functions $u(t)$, $v(t)$, and $w(t)$, along with their derivatives with respect to t, into Eqs. (10) and (11), yielding

$$\begin{aligned}
&(1 - \kappa_0(d_1 e^{\mu_0 t} + d_2 e^{-\mu_0 t} + \frac{\kappa_0}{\mu_0^2}))^2 + (\kappa_0(\frac{\tau_0}{\kappa_0}c_3 t + \frac{\tau_0}{\kappa_0}c_4 - \frac{\kappa_0}{\tau_0}c_1 e^{\mu_0 t} - \frac{\kappa_0}{\tau_0}c_2 e^{-\mu_0 t})\\
&- \tau_0(c_1 e^{\mu_0 t} + c_2 e^{-\mu_0 t} + c_3 t + c_4))^2 + \tau_0^2(d_1 e^{\mu_0 t} + d_2 e^{-\mu_0 t} + \frac{\kappa_0}{\mu_0^2})^2\\
&- (\frac{\tau_0}{\kappa_0}c_3 - \frac{\kappa_0}{\tau_0}\mu_0 c_1 e^{\mu_0 t} + \frac{\kappa_0}{\tau_0}\mu_0 c_2 e^{-\mu_0 t})^2 - (d_1\mu_0 e^{\mu_0 t} - d_2\mu_0 e^{-\mu_0 t})^2\\
&- (\mu_0 c_1 e^{\mu_0 t} - \mu_0 c_2 e^{-\mu_0 t} + c_3)^2 = 0
\end{aligned} \tag{23}$$

$$\begin{aligned}
&(\frac{\tau_0}{\kappa_0}c_3 - \frac{\kappa_0}{\tau_0}\mu_0 c_1 e^{\mu_0 t} + \frac{\kappa_0}{\tau_0}\mu_0 c_2 e^{-\mu_0 t})(1 - \kappa_0(d_1 e^{\mu_0 t} + d_2 e^{-\mu_0 t} + \frac{\kappa_0}{\mu_0^2}))\\
&+ (d_1\mu_0 e^{\mu_0 t} - d_2\mu_0 e^{-\mu_0 t})(\kappa_0(\frac{\tau_0}{\kappa_0}c_3 t + \frac{\tau_0}{\kappa_0}c_4 - \frac{\kappa_0}{\tau_0}c_1 e^{\mu_0 t} - \frac{\kappa_0}{\tau_0}c_2 e^{-\mu_0 t})\\
&- \tau_0(c_1 e^{\mu_0 t} + c_2 e^{-\mu_0 t} + c_3 t + c_4)) + \tau_0(d_1 e^{\mu_0 t} + d_2 e^{-\mu_0 t} + \frac{\kappa_0}{\mu_0^2})\\
&(\mu_0 c_1 e^{\mu_0 t} - \mu_0 c_2 e^{-\mu_0 t} + c_3) = 0
\end{aligned} \tag{24}$$

Eq. (23) can be written as $A+B+C-D-E-F=0$ with $\mu_0=\sqrt{\kappa_0^2+\tau_0^2}$, where:

$$
\begin{aligned}
A &= \left(1-\kappa_0\left(d_1e^{\mu_0 t}+d_2e^{-\mu_0 t}+\frac{\kappa_0}{\mu_0^2}\right)\right)^2\\
B &= \left(\kappa_0\left(\frac{\tau_0}{\kappa_0}c_3t+\frac{\tau_0}{\kappa_0}c_4-\frac{\kappa_0}{\tau_0}c_1e^{\mu_0 t}-\frac{\kappa_0}{\tau_0}c_2e^{-\mu_0 t}\right)-\tau_0(c_1e^{\mu_0 t}+c_2e^{-\mu_0 t}+c_3t+c_4)\right)^2\\
C &= \tau_0^2\left(d_1e^{\mu_0 t}+d_2e^{-\mu_0 t}+\frac{\kappa_0}{\mu_0^2}\right)^2\\
D &= \left(\frac{\tau_0}{\kappa_0}c_3-\frac{\kappa_0}{\tau_0}\mu_0c_1e^{\mu_0 t}+\frac{\kappa_0}{\tau_0}\mu_0c_2e^{-\mu_0 t}\right)^2\\
E &= (d_1\mu_0e^{\mu_0 t}-d_2\mu_0e^{-\mu_0 t})^2\\
F &= (\mu_0c_1e^{\mu_0 t}-\mu_0c_2e^{-\mu_0 t}+c_3)^2
\end{aligned}
$$

The form of A can be simplified as follows:

$$1-\kappa_0\left(d_1e^{\mu_0 t}+d_2e^{-\mu_0 t}+\frac{\kappa_0}{\mu_0^2}\right)=\frac{\tau_0^2}{\mu_0^2}-\kappa_0d_1e^{\mu_0 t}-\kappa_0d_2e^{-\mu_0 t}.$$

By squaring this expression, we obtain:

$$A=\left(\frac{\tau_0^2}{\mu_0^2}\right)^2+\kappa_0^2d_1^2e^{2\mu_0 t}+\kappa_0^2d_2^2e^{-2\mu_0 t}-\frac{2\kappa_0\tau_0^2}{\mu_0^2}d_1e^{\mu_0 t}-\frac{2\kappa_0\tau_0^2}{\mu_0^2}d_2e^{-\mu_0 t}+2\kappa_0^2d_1d_2.$$

The term $\left(\kappa_0\left(\frac{\tau_0}{\kappa_0}c_3t+\frac{\tau_0}{\kappa_0}c_4-\frac{\kappa_0}{\tau_0}c_1e^{\mu_0 t}-\frac{\kappa_0}{\tau_0}c_2e^{-\mu_0 t}\right)-\tau_0(c_1e^{\mu_0 t}+c_2e^{-\mu_0 t}+c_3t+c_4)\right)$ can be simplified to $-\frac{\mu_0^2}{\tau_0}(c_1e^{\mu_0 t}+c_2e^{-\mu_0 t})$. Squaring this expression yields:

$$B=\frac{\mu_0^4}{\tau_0^2}(c_1^2e^{2\mu_0 t}+c_2^2e^{-2\mu_0 t}+2c_1c_2).$$

The form of C can be simplified to:

$$C=\tau_0^2\left(d_1^2e^{2\mu_0 t}+d_2^2e^{-2\mu_0 t}+\frac{\kappa_0^2}{\mu_0^4}+2d_1d_2+\frac{2\kappa_0}{\mu_0^2}d_1e^{\mu_0 t}+\frac{2\kappa_0}{\mu_0^2}d_2e^{-\mu_0 t}\right).$$

The expression $\left(\frac{\tau_0}{\kappa_0}c_3-\frac{\kappa_0}{\tau_0}\mu_0c_1e^{\mu_0 t}+\frac{\kappa_0}{\tau_0}\mu_0c_2e^{-\mu_0 t}\right)$ can be simplified to $-\frac{\kappa_0\mu_0}{\tau_0}(c_1e^{\mu_0 t}-c_2e^{-\mu_0 t})+\frac{\tau_0c_3}{\kappa_0}$. By squaring this expression, we get:

$$D=\frac{\kappa_0^2\mu_0^2}{\tau_0^2}(c_1^2e^{2\mu_0 t}+c_2^2e^{-2\mu_0 t}-2c_1c_2)-2\mu_0c_3(c_1e^{\mu_0 t}-c_2e^{-\mu_0 t})+\frac{\tau_0^2c_3^2}{\kappa_0^2}.$$

Squaring the term $\mu_0(d_1e^{\mu_0 t}-d_2e^{-\mu_0 t})$ gives:

$$E=\mu_0^2(d_1^2e^{2\mu_0 t}+d_2^2e^{-2\mu_0 t}-2d_1d_2).$$

Squaring the term $\mu_0(c_1e^{\mu_0 t}-c_2e^{-\mu_0 t})+c_3$ gives:

$$F=\mu_0^2(c_1^2e^{2\mu_0 t}+c_2^2e^{-2\mu_0 t}-2c_1c_2)+2\mu_0c_3(c_1e^{\mu_0 t}-c_2e^{-\mu_0 t})+c_3^2.$$

Next, the coefficients are calculated by grouping terms from $A + B + C - D - E - F$. The coefficient of $e^{2\mu_0 t}$ is:

$$\begin{aligned}
&(\kappa_0^2 d_1^2) + \left(\frac{\mu_0^4}{\tau_0^2} c_1^2\right) + (\tau_0^2 d_1^2) - \left(\frac{\kappa_0^2 \mu_0^2}{\tau_0^2} c_1^2\right) - (\mu_0^2 d_1^2) - (\mu_0^2 c_1^2) \\
&= d_1^2(\kappa_0^2 + \tau_0^2 - \mu_0^2) + c_1^2 \left(\frac{\mu_0^4}{\tau_0^2} - \frac{\kappa_0^2 \mu_0^2}{\tau_0^2} - \mu_0^2\right) \\
&= d_1^2(\mu_0^2 - \mu_0^2) + c_1^2 \left(\frac{\mu_0^2(\mu_0^2 - \kappa_0^2)}{\tau_0^2} - \mu_0^2\right) \\
&= 0 + c_1^2 \left(\frac{\mu_0^2 \tau_0^2}{\tau_0^2} - \mu_0^2\right) = 0.
\end{aligned}$$

This result also applies to the coefficient of $e^{-2\mu_0 t}$, so that the coefficient of $e^{-2\mu_0 t}$ is zero. Furthermore, the coefficient of $e^{\mu_0 t}$ is:

$$\begin{aligned}
&= \left(-\frac{2\kappa_0 \tau_0^2}{\mu_0^2} d_1\right) + (0) + \left(\frac{2\kappa_0 \tau_0^2}{\mu_0^2} d_1\right) - (-2\mu_0 c_3 c_1) - (0) - (2\mu_0 c_3 c_1) \\
&= 0
\end{aligned}$$

The same applies to $e^{-\mu_0 t}$, such that the coefficient of $e^{-\mu_0 t}$ is zero. Since the sum of each coefficient for $e^{\mu_0 t}$, $e^{-\mu_0 t}$, $e^{2\mu_0 t}$, and $e^{-2\mu_0 t}$ is zero, Eq. (23) can be written as:

$$\begin{aligned}
&\left(\frac{\tau_0^4}{\mu_0^4} + 2\kappa_0^2 d_1 d_2\right) + \left(\frac{2\mu_0^4}{\tau_0^2} c_1 c_2\right) + \left(\frac{\tau_0^2 \kappa_0^2}{\mu_0^4} + 2\tau_0^2 d_1 d_2\right) \\
&- \left(-\frac{2\kappa_0^2 \mu_0^2}{\tau_0^2} c_1 c_2 + \frac{\tau_0^2 c_3^2}{\kappa_0^2}\right) - (-2\mu_0^2 d_1 d_2) - (-2\mu_0^2 c_1 c_2 + c_3^2) = 0.
\end{aligned}$$

Grouping by constants, we obtain:

$$\left(\frac{\tau_0^4 + \tau_0^2 \kappa_0^2}{\mu_0^4}\right) + d_1 d_2 (2\kappa_0^2 + 2\tau_0^2 + 2\mu_0^2) + c_1 c_2 \left(\frac{2\mu_0^4}{\tau_0^2} + \frac{2\kappa_0^2 \mu_0^2}{\tau_0^2} + 2\mu_0^2\right) - c_3^2 \left(\frac{\tau_0^2}{\kappa_0^2} + 1\right) = 0.$$

Simplifying this form yields:

$$\kappa_0^2 \tau_0^4 + 4\mu_0^4 \kappa_0^2 \tau_0^2 d_1 d_2 + 4\mu_0^6 \kappa_0^2 c_1 c_2 - \mu_0^4 \tau_0^2 c_3^2 = 0$$

Rearranging this form, we get:

$$c_3^2 \mu_0^4 \tau_0^2 - 4 d_1 d_2 \mu_0^4 \kappa_0^2 \tau_0^2 - 4 c_1 c_2 \mu_0^6 \kappa_0^2 - \kappa_0^2 \tau_0^4 = 0$$

Eq. (24) can be written in a simpler form, namely $P_1 + P_2 + P_3 = 0$, where:

$$\begin{aligned}
P_1 &= \left(\frac{\tau_0}{\kappa_0} c_3 - \frac{\kappa_0}{\tau_0} \mu_0 c_1 e^{\mu_0 t} + \frac{\kappa_0}{\tau_0} \mu_0 c_2 e^{-\mu_0 t}\right) \left(1 - \kappa_0 (d_1 e^{\mu_0 t} + d_2 e^{-\mu_0 t} + \frac{\kappa_0}{\mu_0^2})\right), \\
P_2 &= (d_1 \mu_0 e^{\mu_0 t} - d_2 \mu_0 e^{-\mu_0 t}) \left(\kappa_0 \left(\frac{\tau_0}{\kappa_0} c_3 t + \frac{\tau_0}{\kappa_0} c_4 - \frac{\kappa_0}{\tau_0} c_1 e^{\mu_0 t} - \frac{\kappa_0}{\tau_0} c_2 e^{-\mu_0 t}\right)\right. \\
&\qquad \left. - \tau_0 (c_1 e^{\mu_0 t} + c_2 e^{-\mu_0 t} + c_3 t + c_4)\right), \\
P_3 &= \tau_0 \left(d_1 e^{\mu_0 t} + d_2 e^{-\mu_0 t} + \frac{\kappa_0}{\mu_0^2}\right) (\mu_0 c_1 e^{\mu_0 t} - \mu_0 c_2 e^{-\mu_0 t} + c_3).
\end{aligned}$$

The expression for P_1 can be simplified to:

$$\begin{aligned} P_1 &= \left(\frac{\tau_0}{\kappa_0}c_3 - \frac{\kappa_0\mu_0}{\tau_0}c_1 e^{\mu_0 t} + \frac{\kappa_0\mu_0}{\tau_0}c_2 e^{-\mu_0 t}\right)\left(\frac{\tau_0^2}{\mu_0^2} - \kappa_0 d_1 e^{\mu_0 t} - \kappa_0 d_2 e^{-\mu_0 t}\right) \\ &= \frac{\tau_0^3}{\kappa_0\mu_0^2}c_3 - \tau_0 c_3 d_1 e^{\mu_0 t} - \tau_0 d_2 e^{\mu_0 t} - \frac{\kappa_0\tau_0}{\mu_0}c_3 c_1 e^{\mu_0 t} + \frac{\kappa_0^2\mu_0}{\tau_0}c_1 d_1 e^{2\mu_0 t} \\ &\quad + \frac{\kappa_0^2\mu_0}{\tau_0}c_1 d_2 + \frac{\kappa_0\tau_0}{\mu_0}c_2 e^{-\mu_0 t} - \frac{\kappa_0^2\mu_0}{\tau_0}c_2 d_1 - \frac{\kappa_0^2\mu_0}{\tau_0}c_2 d_2 e^{-2\mu_0 t}. \end{aligned}$$

Let the constant term of P_1 be kP_1, then:

$$kP_1 = \frac{\kappa_0^2\mu_0}{\tau_0}(c_1 d_2 - c_2 d_1) + \frac{c_3\tau_0^3}{\kappa_0\mu_0^2}.$$

The expression for P_2 can be simplified to:

$$P_2 = -\frac{\mu_0^3}{\tau_0}(d_1 c_1 e^{2\mu_0 t} + d_1 c_2 - d_2 c_1 - d_2 c_2 e^{-2\mu_0 t})$$

Let the constant term of P_2 be kP_2, then:

$$kP_2 = \frac{\mu_0^3}{\tau_0}(c_1 d_2 - c_2 d_1).$$

The expression for P_3 can be simplified to:

$$\begin{aligned} P_3 = \tau_0(&d_1 c_1 \mu_0 e^{2\mu_0 t} - d_1 c_2 \mu_0 + d_1 c_3 e^{2\mu_0 t} + d_2 c_1 \mu_0 \\ &- d_2 c_2 e^{-2\mu_0 t} + d_2 c_3 e^{-\mu_0 t} + \frac{\kappa_0}{\mu_0}c_1 e^{\mu_0 t} - \frac{\kappa_0}{\mu_0}c_2 e^{-\mu_0 t} + \frac{\kappa_0}{\mu_0^2}c_3). \end{aligned}$$

Let the constant term of P_3 be kP_3, then:

$$kP_3 = \tau_0\mu_0(c_1 d_2 - c_2 d_1) + \frac{c_3\tau_0\kappa_0}{\mu_0^2}.$$

Next, all constant terms $kP_1 + kP_2 + kP_3$ are summed and set to zero:

$$\left(\frac{\kappa_0^2\mu_0}{\tau_0}(c_1 d_2 - c_2 d_1) + \frac{c_3\tau_0^3}{\kappa_0\mu_0^2}\right) + \left(\frac{\mu_0^3}{\tau_0}(c_1 d_2 - c_2 d_1)\right) + \left(\tau_0\mu_0(c_1 d_2 - c_2 d_1) + \frac{c_3\tau_0\kappa_0}{\mu_0^2}\right) = 0.$$

Grouping terms with common factors, we get:

$$\begin{aligned} &(c_1 d_2 - c_2 d_1)\left[\frac{\kappa_0^2\mu_0}{\tau_0} + \frac{\mu_0^3}{\tau_0} + \tau_0\mu_0\right] + c_3\left[\frac{\tau_0^3}{\kappa_0\mu_0^2} + \frac{\tau_0\kappa_0}{\mu_0^2}\right] \\ &= (c_1 d_2 - c_2 d_1)\frac{2\mu_0^3}{\tau_0} + c_3\frac{\tau_0}{\kappa_0}. \end{aligned}$$

By combining the simplified results and setting them to zero, the final equations are obtained as:

$$(c_1 d_2 - c_2 d_1)2\mu_0^3\kappa_0 + c_3\tau_0^2 = 0.$$

The simplified forms of Eqs. (23) and (24) are obtained respectively as:

$$\begin{aligned} c_3^2\mu_0^4\tau_0^2 - 4d_1 d_2\mu_0^4\kappa_0^2\tau_0^2 - 4c_1 c_2\mu_0^6\kappa_0^2 - \kappa_0^2\tau_0^4 &= 0, \\ (c_1 d_2 - c_2 d_1)2\mu_0^3\kappa_0 + c_3\tau_0^2 &= 0. \end{aligned}$$

The next step is to substitute $\boldsymbol{\gamma}(s)$ given by Eq. (3), $\mathbf{t}(s)$, $\mathbf{n}(s)$, and $\mathbf{b}(s)$ given by Eq. (4), and the functions $u(t)$, $v(t)$, and $w(t)$ given by Eq. (22) into the parametric surface equation

$$\mathbf{X}(s,t) = \boldsymbol{\gamma}(s) + u(t)\mathbf{t}(s) + v(t)\mathbf{n}(s) + w(t)\mathbf{b}(s),$$

obtaining

$$\begin{aligned}\mathbf{X}(s,t) = (&\frac{\mu_0}{\tau_0}\sin(\mu_0 s)\left[c_1 e^{\mu_0 t} + c_2 e^{-\mu_0 t}\right] - \cos(\mu_0 s)\left[d_1 e^{\mu_0 t} + d_2 e^{-\mu_0 t}\right],\\ &-\frac{\mu_0}{\tau_0}\left[c_1 e^{\mu_0 t} + c_2 e^{-\mu_0 t}\right]\cos(\mu_0 s) - \left[d_1 e^{\mu_0 t} + d_2 e^{-\mu_0 t}\right]\sin(\mu_0 s),\\ &\frac{\tau_0}{\mu_0}s + \frac{\mu_0}{\kappa_0}(c_3 t + c_4)),\end{aligned} \tag{25}$$

with coefficients $c_1, c_2, c_3, d_1,$ and d_2 satisfying

$$c_3^2\mu_0^4\tau_0^2 - 4d_1d_2\mu_0^4\kappa_0^2\tau_0^2 - 4c_1c_2\mu_0^6\kappa_0^2 - \kappa_0^2\tau_0^4 = 0, \tag{26}$$

$$(c_1d_2 - c_2d_1)2\mu_0^3\kappa_0 + c_3\tau_0^2 = 0, \tag{27}$$

where $s_1 \leq s \leq s_2$, $t_1 \leq t \leq t_2$, $\mu_0 = \sqrt{\kappa_0^2 + \tau_0^2}$, and $s, t, c_1, c_2, c_3, c_4, d_1, d_2 \in \mathbb{R}$. Note that the helix curve serving as the base curve is an isoparametric curve. This means that the helix curve is a curve lying on the surface. Based on Definition 6, to satisfy the condition $\mathbf{X}(s, t_0) = \boldsymbol{\gamma}(s)$, we obtain

$$u(t_0) = v(t_0) = w(t_0) = 0.$$

Based on the functions u, v, w obtained in Eq. (22), then

$$\frac{\tau_0}{\kappa_0}c_3t_0 + \frac{\tau_0}{\kappa_0}c_4 - \frac{\kappa_0}{\tau_0}c_1e^{\mu_0t_0} - \frac{\kappa_0}{\tau_0}c_2e^{-\mu_0t_0} = 0, \tag{28}$$

$$d_1e^{\mu_0t_0} + d_2e^{-\mu_0t_0} + \frac{\kappa_0}{\mu_0^2} = 0, \tag{29}$$

$$c_1e^{\mu_0t_0} + c_2e^{-\mu_0t_0} + c_3t_0 + c_4 = 0. \tag{30}$$

From Eqs. (28) to (30), we obtain

$$c_2 = -c_1e^{2\mu_0t_0}, \quad c_4 = -c_3t_0, \quad d_2 = -d_1e^{2\mu_0t_0} - \frac{\kappa_0}{\mu_0^2}e^{\mu_0t_0}. \tag{31}$$

Substituting the coefficients obtained in Eq. (31) into Eqs.(26) to (27), the parametric surface equation is obtained as

$$c_1^2\mu_0^4 + \tau_0^4\left[\left(d_1 + \frac{\kappa_0}{2\mu_0^2}\right)^2 - \left(\frac{\kappa_0}{2\mu_0^2}\right)^2\right] = \frac{\tau_0^6}{4e^{2\mu_0t_0}\mu_0^4} \tag{32}$$

Eq. (32) can be simplified to

$$c_1 = \pm\frac{\tau_0^2}{\mu_0^2}\sqrt{\frac{\tau_0^2}{4e^{2\mu_0t_0}\mu_0^4} - \left(d_1 + \frac{\kappa_0}{\mu_0^2}\right)d_1}. \tag{33}$$

where

$$\frac{\tau_0^2}{4e^{2\mu_0t_0}\mu_0^4} - \left(d_1 + \frac{\kappa_0}{\mu_0^2}\right)d_1 \geq 0 \tag{34}$$

$$d_1^2 + \frac{\kappa_0}{\mu_0^2}d_1 - \frac{\tau_0^2}{4e^{2\mu_0t_0}\mu_0^4} \leq 0. \tag{35}$$

By applying the quadratic formula, it yields

$$d_1 = -\frac{\kappa_0 \pm \sqrt{\kappa_0^2 + \tau_0^2 e^{-2\mu_0 t_0}}}{2\mu_0^2}.$$

From equation sign in Eq. (35), we obtain

$$-\frac{\kappa_0 - \sqrt{\kappa_0^2 + \tau_0^2 e^{-2\mu_0 t_0}}}{2\mu_0^2} \leq d_1 \leq -\frac{\kappa_0 + \sqrt{\kappa_0^2 + \tau_0^2 e^{-2\mu_0 t_0}}}{2\mu_0^2} \tag{36}$$

Notice that, $\kappa_0^2 + \frac{\tau_0^2}{e^{2\mu_0 t_0}}$ is always real positive since $\kappa_0 > 0, \tau_0 \neq 0, \mu_0 = \sqrt{\kappa_0^2 + \tau_0^2} > 0$. Hence, solution for d_1 is always real number.
Thus, by substituting Eq. (31) and Eq. (33), the variation of minimal surfaces passing through a unit speed isoparametric helix curve with the parameterization given by Eq. (25) become

$$\begin{aligned}\mathbf{X}(s,t) = \Bigg(&\frac{\mu_0}{\tau_0}\sin(\mu_0 s)\left[\left(\frac{\tau_0^2}{\mu_0^2}\sqrt{\frac{\tau_0^2}{4e^{2\mu_0 t_0}\mu_0^4} - \left(d_1 + \frac{\kappa_0}{\mu_0^2}\right)d_1}\right)\left(e^{\mu_0 t} - e^{2\mu_0 t_0}e^{-\mu_0 t}\right)\right] \\ &- \cos(\mu_0 s)\left[d_1 e^{\mu_0 t} + (-d_1 e^{2\mu_0 t_0} - \frac{\kappa_0}{\mu_0^2}e^{\mu_0 t_0})e^{-\mu_0 t}\right], \\ &- \frac{\mu_0}{\tau_0}\left[\left(\frac{\tau_0^2}{\mu_0^2}\sqrt{\frac{\tau_0^2}{4e^{2\mu_0 t_0}\mu_0^4} - \left(d_1 + \frac{\kappa_0}{\mu_0^2}\right)d_1}\right)\left(e^{\mu_0 t} - e^{2\mu_0 t_0}e^{-\mu_0 t}\right)\right]\cos(\mu_0 s) \\ &- \left[d_1 e^{\mu_0 t} + (-d_1 e^{2\mu_0 t_0} - \frac{\kappa_0}{\mu_0^2}e^{\mu_0 t_0})e^{-\mu_0 t}\right]\sin(\mu_0 s), \frac{\tau_0}{\mu_0}s + \\ &\frac{2\left(\frac{\tau_0^2}{\mu_0^2}\sqrt{\frac{\tau_0^2}{4e^{2\mu_0 t_0}\mu_0^4} - \left(d_1 + \frac{\kappa_0}{\mu_0^2}\right)d_1}\right)e^{\mu_0 t_0}\mu_0^2\kappa_0}{\tau_0^2}(t - t_0)\Bigg),\end{aligned} \tag{37}$$

and

$$\begin{aligned}\mathbf{X}(s,t) = \Bigg(&\frac{\mu_0}{\tau_0}\sin(\mu_0 s)\left[\left(-\frac{\tau_0^2}{\mu_0^2}\sqrt{\frac{\tau_0^2}{4e^{2\mu_0 t_0}\mu_0^4} - \left(d_1 + \frac{\kappa_0}{\mu_0^2}\right)d_1}\right)\left(e^{\mu_0 t} - e^{2\mu_0 t_0}e^{-\mu_0 t}\right)\right] \\ &- \cos(\mu_0 s)\left[d_1 e^{\mu_0 t} + (-d_1 e^{2\mu_0 t_0} - \frac{\kappa_0}{\mu_0^2}e^{\mu_0 t_0})e^{-\mu_0 t}\right], \\ &- \frac{\mu_0}{\tau_0}\left[\left(-\frac{\tau_0^2}{\mu_0^2}\sqrt{\frac{\tau_0^2}{4e^{2\mu_0 t_0}\mu_0^4} - \left(d_1 + \frac{\kappa_0}{\mu_0^2}\right)d_1}\right)\left(e^{\mu_0 t} - e^{2\mu_0 t_0}e^{-\mu_0 t}\right)\right]\cos(\mu_0 s) \\ &- \left[d_1 e^{\mu_0 t} + (-d_1 e^{2\mu_0 t_0} - \frac{\kappa_0}{\mu_0^2}e^{\mu_0 t_0})e^{-\mu_0 t}\right]\sin(\mu_0 s), \frac{\tau_0}{\mu_0}s + \\ &\frac{2\left(-\frac{\tau_0^2}{\mu_0^2}\sqrt{\frac{\tau_0^2}{4e^{2\mu_0 t_0}\mu_0^4} - \left(d_1 + \frac{\kappa_0}{\mu_0^2}\right)d_1}\right)e^{\mu_0 t_0}\mu_0^2\kappa_0}{\tau_0^2}(t - t_0)\Bigg),\end{aligned} \tag{38}$$

with

$$-\frac{\kappa_0}{\mu_0^2} - \frac{1}{\mu_0^2}\sqrt{\kappa_0^2 + \frac{\tau_0^2}{e^{2\mu_0 t_0}}} \leq d_1 \leq -\frac{\kappa_0}{\mu_0^2} + \frac{1}{\mu_0^2}\sqrt{\kappa_0^2 + \frac{\tau_0^2}{e^{2\mu_0 t_0}}}$$

Notice that parametric equation in Eqs. (37) and (38) which is minimal surface similar to generalized helicoid surface definition in Definition 7, where

$$\begin{aligned}
\alpha(v) &= -d_1 e^{\mu_0 t} + (-d_1 e^{2\mu_0 t_0} - \frac{\kappa_0}{\mu_0^2} e^{\mu_0 t_0}) e^{-\mu_0 t}, \\
\beta(v) &= \mp \frac{\mu_0}{\tau_0} \left(\frac{\tau_0^2}{\mu_0^2} \sqrt{\frac{\tau_0^2}{4 e^{2\mu_0 t_0} \mu_0^4} - \left(d_1 + \frac{\kappa_0}{\mu_0^2} \right) d_1} \right) \left(e^{\mu_0 t} - e^{2\mu_0 t_0} e^{-\mu_0 t} \right), \\
c &= \frac{\tau_0}{\mu_0}, \\
\gamma(v) &= \frac{2 \left(\frac{\tau_0^2}{\mu_0^2} \sqrt{\frac{\tau_0^2}{4 e^{2\mu_0 t_0} \mu_0^4} - \left(d_1 + \frac{\kappa_0}{\mu_0^2} \right) d_1} \right) e^{\mu_0 t_0} \mu_0^2 \kappa_0}{\tau_0^2} (t - t_0) \Bigg).
\end{aligned}$$

Hence, the minimal surface in Eqs. (37) and (38) is part of Generalized Helicoid Surface. □

The following is an example of a generalized helicoid minimal surface passing through an isoparametric unit speed helix curve.

Example 1. Let there be a helix curve with $\kappa_0 = \sqrt{2}$ and $\tau_0 = \sqrt{2}$. Based on Eq. (2), i.e., $\boldsymbol{\gamma}(s) = (\frac{\kappa_0}{\mu_0^2} \cos(\mu_0 s), \frac{\kappa_0}{\mu_0^2} \sin(\mu_0 s), \frac{\tau_0}{\mu_0} s)$ where $\mu_0 = \sqrt{\kappa_0^2 + \tau_0^2}$, the obtained parameterization is

$$\boldsymbol{\gamma}(s) = (\frac{\sqrt{2}}{4} \cos(2s), \frac{\sqrt{2}}{4} \sin(2s), \frac{\sqrt{2}}{2} s).$$

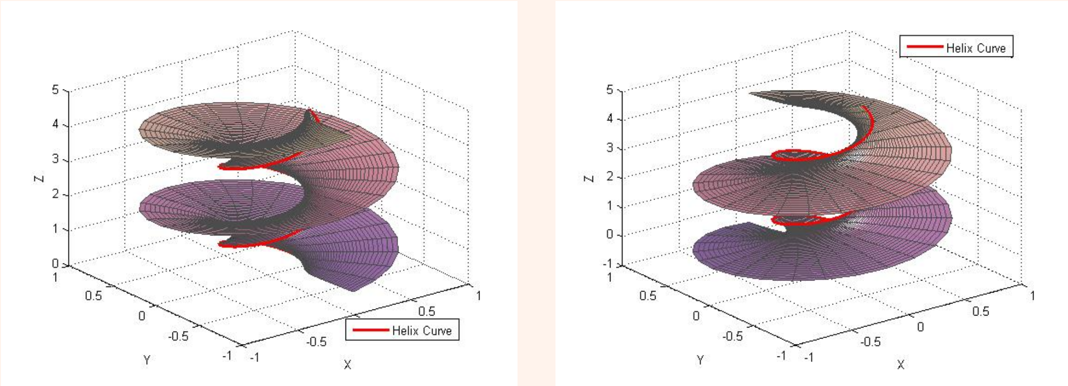
Next, by assuming $t_0 = 0$, to choose the value of d_1, we should look at the interval for d_1 by referring to Eq. (36), which results in $-6.828427 \leq d1 \leq 1.171573$. Therefore, for example, if we choose $d_1 = 0$, and substitute it along with the previously chosen κ_0, τ_0, and t_0 into parametric Eqs. (37) and (38), yielding

$$\begin{aligned}
\mathbf{X}(s,t) = \Bigg(&\frac{1}{4} \sin(2s) \sinh(2t) + \frac{\sqrt{2}}{4} \cos(2s) e^{-2t}, \\
&- \frac{1}{4} \cos(2s) \sinh(2t) + \frac{\sqrt{2}}{4} \sin(2s) e^{-2t}, \frac{\sqrt{2}}{2} s + \frac{1}{2} t \Bigg)
\end{aligned} \tag{39}$$

and

$$\begin{aligned}
\mathbf{X}(s,t) = \Bigg(&- \frac{1}{4} \sin(2s) \sinh(2t) + \frac{\sqrt{2}}{4} \cos(2s) e^{-2t}, \\
&\frac{1}{4} \cos(2s) \sinh(2t) + \frac{\sqrt{2}}{4} \sin(2s) e^{-2t}, \frac{\sqrt{2}}{2} s - \frac{1}{2} t \Bigg)
\end{aligned} \tag{40}$$

The visualization of the helicoid parameterization given by Eqs.(39) and (40) can be seen in Fig. 3a and Fig. 3b below where $0 \leq s \leq 2\pi$ and $0 \leq t \leq 1$.



(a) Helicoid for Eq. (39) **(b)** Helicoid for Eq. (40)

Example 2. Now, let a helix curve with $\kappa_0 = \sqrt{2}$ and $\tau_0 = -\sqrt{2}$. Based on Eq. (2), i.e., $\boldsymbol{\gamma}(s) = (\frac{\kappa_0}{\mu_0^2}\cos(\mu_0 s), \frac{\kappa_0}{\mu_0^2}\sin(\mu_0 s), \frac{\tau_0}{\mu_0}s)$ where $\mu_0 = \sqrt{\kappa_0^2 + \tau_0^2}$, the obtained parameterization is

$$\boldsymbol{\gamma}(s) = (\frac{\sqrt{2}}{4}\cos(2s), \frac{\sqrt{2}}{4}\sin(2s), -\frac{\sqrt{2}}{2}s).$$

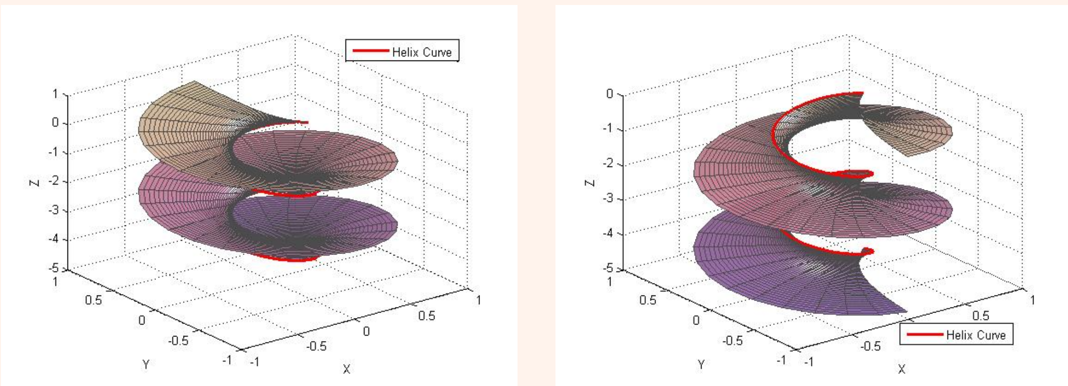
Next, by assuming $t_0 = 0$, to choose the value of d_1, we should look at the interval for d_1 by referring to Eq. (36), which results in $-6.828427 \leq d1 \leq 1.171573$. Therefore, for example, if we choose $d_1 = 0$, and substitute it along with the previously chosen κ_0, τ_0, and t_0 into parametric Eqs. (37) and (38), yielding

$$\begin{aligned}\mathbf{X}(s,t) =& \Big(-\frac{1}{4}\sin(2s)\sinh(2t) + \frac{\sqrt{2}}{4}\cos(2s)e^{-2t}, \\ & \frac{1}{4}\cos(2s)\sinh(2t) + \frac{\sqrt{2}}{4}\sin(2s)e^{-2t}, -\frac{\sqrt{2}}{2}s + \frac{1}{2}t\Big)\end{aligned} \tag{41}$$

and

$$\begin{aligned}\mathbf{X}(s,t) =& \Big(\frac{1}{4}\sin(2s)\sinh(2t) + \frac{\sqrt{2}}{4}\cos(2s)e^{-2t}, \\ & -\frac{1}{4}\cos(2s)\sinh(2t) + \frac{\sqrt{2}}{4}\sin(2s)e^{-2t}, -\frac{\sqrt{2}}{2}s - \frac{1}{2}t\Big)\end{aligned} \tag{42}$$

The visualization of the helicoid parameterization given by Eqs. (39) and (40) can be seen in Fig. 4a and Fig. 4b below where $0 \leq s \leq 2\pi$ and $0 \leq t \leq 1$.



(a) Helicoid for Eq. (41) **(b)** Helicoid for Eq. (42)

Example 3. Let there be a helix curve with $\kappa_0 = \sqrt{2}$ and $\tau_0 = \sqrt{2}$. Based on Eq. (2), i.e., $\gamma(s) = (\frac{\kappa_0}{\mu_0^2}\cos(\mu_0 s), \frac{\kappa_0}{\mu_0^2}\sin(\mu_0 s), \frac{\tau_0}{\mu_0}s)$ where $\mu_0 = \sqrt{\kappa_0^2 + \tau_0^2}$, the obtained parameterization is

$$\gamma(s) = (\frac{\sqrt{2}}{4}\cos(2s), \frac{\sqrt{2}}{4}\sin(2s), \frac{\sqrt{2}}{2}s).$$

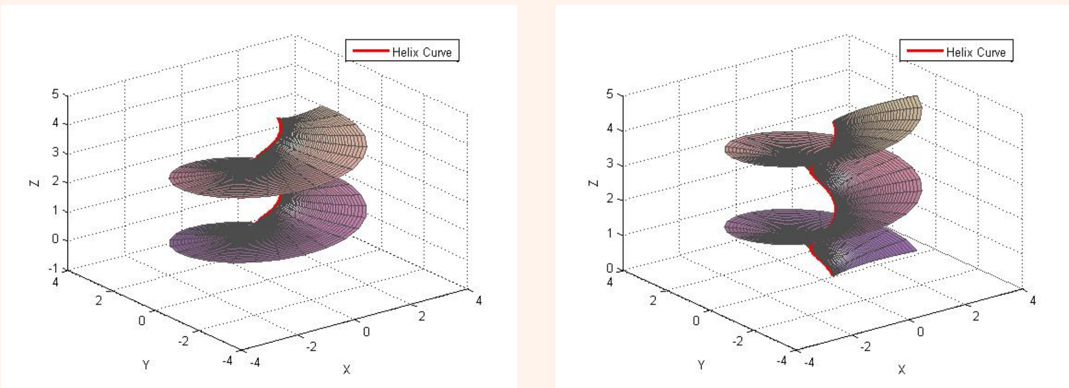
Next, by assuming $t_0 = 1$, to choose the value of d_1, we should look at the interval for d_1 by referring to Eq. (36), which results in $-5.682639 \leq d1 \leq 0.025785$. Therefore, for example, if we choose $d_1 = 0$, and substitute it along with the previously chosen κ_0, τ_0, and t_0 into parametric Eqs. (37) and (38), yielding

$$\begin{aligned}\mathbf{X}(s,t) = \Big(&\frac{1}{4}\sin(2s)\sinh(2(t-1)) + \frac{\sqrt{2}}{4}\cos(2s)e^{-2(t-1)}, -\frac{1}{4}\cos(2s)\sinh(2(t-1)) \\ &+ \frac{\sqrt{2}}{4}\sin(2s)e^{-2(t-1)}, \frac{\sqrt{2}}{2}s + \frac{1}{2}(t-1)\Big)\end{aligned} \tag{43}$$

and

$$\begin{aligned}\mathbf{X}(s,t) = \Big(&-\frac{1}{4}\sin(2s)\sinh(2(t-1)) + \frac{\sqrt{2}}{4}\cos(2s)e^{-2(t-1)}, \frac{1}{4}\cos(2s)\sinh(2(t-1)) \\ &+ \frac{\sqrt{2}}{4}\sin(2s)e^{-2(t-1)}, \frac{\sqrt{2}}{2}s - \frac{1}{2}(t-1)\Big)\end{aligned} \tag{44}$$

The visualization of the helicoid parameterization given by Eqs. (39) and (40) can be seen in Fig. 5a and Fig. 5b below where $0 \leq s \leq 2\pi$ and $0 \leq t \leq 1$.



(a) Helicoid for Eq. (43) **(b)** Helicoid for Eq. (44)

4. Conclusion

Therefore, based on the obtained results, it is proven that the family of minimal surfaces passing through an isoparametric unit-speed circular helix curve is a family of generalized helicoid minimal surfaces. By adding the isoparametric condition, we ensure that the selected base curve lies precisely on the surface. Also, for some parametric chosen like τ_0 when its value is negative, the direction of surface is opposite to τ_0 when it has a positive value.

CRediT Authorship Contribution Statement

Ridwan Prawira Yunaldi: Conceptualization, Methodology, Formal Analysis, Writing–Original Draft. **Haripamyu:** Supervision, Project Administration, Validation, Writing–Review & Editing.

Efendi: Validation, Writing–Review & Editing.

Declaration of Generative AI and AI-assisted technologies

Generative AI tools (specifically Google Gemini 3 Pro) were utilised exclusively for language refinement and grammar editing during the preparation of this work. The analysis, interpretation, and core content were not generated by AI. All substantive results and discussions were entirely developed by the authors themselves.

Declaration of Competing Interest

The authors declare no competing interests

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Data and Code Availability

MATLAB code for constructing minimal surface is publicly available in GitHub[1].

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