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Properties of Clear Graph of Ring $\mathbb{Z}_p$

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Abstract

Let $\mathbb{Z}_p$ be the ring of integers modulo a prime $p > 3$. Clear graph of $\mathbb{Z}_p$, denoted by $\mathrm{Cr}_2(\mathbb{Z}_p)$, is graph whose vertices are ordered pairs (x, u) with a nonzero regular unit x and a unit u of $\mathbb{Z}_p$ and two vertices (x, u) and (y, v) are adjacent if either $xy = yx = 0$ or $uv = vu = 1$. This work extends previous research on clear graphs, which established the basic structure of $\mathrm{Cr}_2(R)$ for certain rings, including aspects of isomorphism, connectedness, and other structural properties. In this paper, we focus on the prime ring $\mathbb{Z}_p$ and analyze several fundamental graph-theoretic properties of $\mathrm{Cr}_2(\mathbb{Z}_p)$. Specifically, we show that this graph has order $(p-1)^2$, size $\frac{1}{2}(p^2-2p-1)(p-1)$, diameter ∞, radius at most 2, independence number $\frac{1}{2}(p^2-4p+7)$, and clique, chromatic, and domination numbers each equal to $p-1$. The results provide a deeper understanding of how algebraic properties of $\mathbb{Z}_p$ influence the combinatorial structure of its associated clear graph.

Keywords: Clear Graph; Graph-Theoretic Properties; Ring.

1. Introduction

Graph theory is a fundamental branch of discrete mathematics that studies relational structures consisting of vertices and edges, and it has become an essential tool in modeling and analyzing complex systems arising in mathematics, computer science, engineering, and the natural and social sciences. In recent decades, the interaction between graph theory and abstract algebra has led to the development of algebraic graph theory. This interplay allows algebraic properties to be translated into combinatorial features, providing powerful methods for investigating symmetry, connectivity, coloring, and spectral properties of graphs [1]. Algebraically defined graphs, including graph of a ring have found important applications in areas such as cryptography [2, 3] and coding theory [4–6].

There has been significant research interest in constructing graphs from algebraic structures, particularly focusing on the zero-divisor graph of a commutative ring. This graph was first introduced by Beck [7], and later Anderson and Livingston [8] reformulated the definition of the zero-divisor graph of a commutative ring. The interaction between ring theory and graph theory in the study of zero-divisor graphs has led to deeper insights into algebraic properties of rings, particularly the structure of their sets of zero divisors. Akbari et al. [9] introduced the idempotent graph, defined as the graph whose vertex set consists of the non-trivial idempotents of a ring, with an edge between two vertices x and y if and only if $xy = yx = 0$. Combining the

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concept of clean rings with idempotent graphs, Habibi et al. [10] introduced the notion of the clean graph, whose vertices are pairs consisting of an idempotent and a unit, and where two vertices are adjacent if either the product of the idempotents is zero or the product of the units is one. Building on the clean graph, Mir et al. [11] proposed a more general construction called the clear graph, whose vertex set consists of pairs of a non-zero regular unit and a unit, with adjacency defined by either the product of the regular units being zero or the product of the units being one. In that study, they examined the properties of clear graphs, including connectedness, planarity, and genus. In their research, they also discussed the isomorphism of clear graphs over a specific ring with other classes of graphs, one of which is clear graphs over the ring $\mathbb{Z}_p$.

The ring $\mathbb{Z}_p$ offers a rich area of exploration from both algebraic and graph-theoretic standpoints. Among its notable algebraic and combinatorial characteristics are the fact that every nonzero element serves as a unit, each element possesses a unique inverse, and inverse pairs arrange themselves in a regular combinatorial structure, yielding a graph that admits a particularly elegant decomposition. Building on this foundation, and motivated by the work of Mir et al. [11], the study of clear graphs over the ring $\mathbb{Z}_p$ with $p > 3$ and p prime reveals a structure that is isomorphic to a union of several complete graphs and complete bipartite graphs. However, the fundamental properties of this graph, such as its order, chromatic number, clique number, and other basic invariants, have not yet been explicitly investigated. Therefore, this paper aims to extend the results of Mir et al. [11] by focusing on a detailed study of these graph-theoretic properties. We observe the order, size, degree, distance, diameter, radius, clique number, chromatic number, domination number, and independence number of graph $\mathrm{Cr}_2(\mathbb{Z}_p)$ with $p > 3$ prime.

2. Method

This section presents the basic concepts, notation, and preliminary results used throughout this paper. We refer to the following basic concepts from [12–15]. A simple graph $G = (V, E)$ is defined by a vertex set $V(G)$ and an edge set $E(G)$, with order and size denoted as $|V(G)|$ and $|E(G)|$, respectively. The degree of a vertex $u \in V(G)$, denoted by $\deg(u)$, is the number of vertices adjacent to u. The distance between two vertices $u, v \in V(G)$, denoted by $d(u, v)$, is the length of the shortest path between u and v in graph G. For a connected graph G, the eccentricity of a vertex v is the largest distance from v to any other vertex in G. The diameter $\mathrm{diam}(G)$ and radius of G are the largest and smallest eccentricities among all vertices of G, respectively. The complement of a graph G, denoted $\overline{G}$, is defined as the graph with vertex set $V(G)$ such that two distinct vertices $u, v \in V(\overline{G})$ are adjacent if and only if $uv \notin E(G)$. Notation $\omega(G)$, $\chi(G)$, $\alpha(G)$, and $\gamma(G)$ denote the clique number, chromatic number, independence number and domination number of graph G, respectively. Furthermore, K_n and $K_{m,n}$ refer to complete graphs and complete bipartite graphs, respectively.

The symbol $\mathbb{Z}$ denotes the set of all integers, whereas $\mathbb{Z}_n$ represents the ring of integers modulo n. An element $r \in R$ is said to be a unit if there exists $s \in R$ such that $sr = rs = 1$. Moreover, an element $x \in R$ is called unit regular if there exists a unit $u \in R$ satisfying $x = xux$. We denote by $U(R)$ and $U_{\mathrm{reg}}(R)$ the sets of units and unit regular elements of R, respectively. Let $U(\mathbb{Z}_p)$ denote the group of units of $\mathbb{Z}_p$, and define the following subsets

$$U'(\mathbb{Z}_p) = \{u \in U(\mathbb{Z}_p) \mid u^2 \equiv 1 \pmod{p}\},$$

$$U''(\mathbb{Z}_p) = \{u \in U(\mathbb{Z}_p) \mid u^2 \not\equiv 1 \pmod{p}\}.$$

It is straightforward to verify that $U'(\mathbb{Z}_p) = \{1, p-1\}$. Furthermore, the set $U''(\mathbb{Z}_p)$ can be partitioned into $\frac{p-3}{2}$ inverse pairs $\{u_j, v_j\}$ where $v_j = u_j^{-1}$.

Definition 1. [11] Let R be a ring with identity. The *clear graph* of R is a graph whose vertices are ordered pairs (x, u), where x is a non-zero regular unit and u is a unit of R and two distinct vertices (x, u) and (y, v) are adjacent if either

$$xy = yx = 0 \quad \text{or} \quad uv = vu = 1.$$

The clear graph of R is denoted by $\mathrm{Cr}(R)$, and its vertex set is denoted by $V(\mathrm{Cr}(R))$.

Observe that every vertex of the form $(0, u)$ is adjacent to every vertex in $\mathrm{Cr}(R)$. We now construct two disjoint subgraphs $\mathrm{Cr}_1(R)$ and $\mathrm{Cr}_2(R)$ of $\mathrm{Cr}(R)$ such that

$$V(\mathrm{Cr}(R)) = V(\mathrm{Cr}_1(R)) \cup V(\mathrm{Cr}_2(R)),$$

where

$$V(\mathrm{Cr}_1(R)) = \{(0, u) \mid u \in U(R)\}$$

and

$$V(\mathrm{Cr}_2(R)) = \{(x, u) \mid x \in U_{\mathrm{reg}}(R) \setminus \{0\},\ u \in U(R)\}.$$

In this paper, we focus on studying the basic properties of the graph $\mathrm{Cr}_2(\mathbb{Z}_p)$ for $p > 3$ and p prime. The following proposition serves as the structural foundation from which we develop the further analysis and results presented in this paper.

Proposition 1. *[11] For $p > 3$ and p prime, the clear graph $\mathrm{Cr}_2(\mathbb{Z}_p)$ is isomorphic to*

$$2K_{p-1} \cup \frac{p-3}{2} K_{p-1,p-1}.$$

However, previous studies have not explicitly examined these properties for the graph $\mathrm{Cr}_2(\mathbb{Z}_p)$. This paper addresses this gap by investigating the graph from a structural perspective, specifically through vertex set partitioning. Furthermore, this paper serves to support the results established in earlier literature.

Example 1. Let $\mathbb{Z}_5$ be the ring of integers modulo 5. We identify the vertex set of $\mathrm{Cr}_2(\mathbb{Z}_5)$ by determining the units and regular units of $\mathbb{Z}_5$ as follows:

- $U(\mathbb{Z}_5) = \{1, 2, 3, 4\}$, partitioned into:

$$U'(\mathbb{Z}_5) = \{1, 4\} \begin{cases} 1 \cdot 1 = 1 & \mod 5 \\ 4 \cdot 4 = 1 & \mod 5 \end{cases}$$

$$U''(\mathbb{Z}_5) = \{2, 3\} \left\{ 2 \cdot 3 = 1 \quad \mod 5. \right.$$

- $U_{reg}(\mathbb{Z}_5) \setminus \{0\} = \{1, 2, 3, 4\}$.

The vertex set is obtained as follows:

$$\begin{aligned} V(\mathrm{Cr}_2(\mathbb{Z}_5)) &= \{U_{reg}(\mathbb{Z}_5) \setminus \{0\}\} \times U(\mathbb{Z}_5) \\ &= \{(1,1), (2,1), (3,1), (4,1), (1,4), (2,4), (3,4), (4,4), \\ &\quad (1,2), (2,2), (3,2), (4,2), (1,3), (2,3), (3,3), (4,3)\}. \end{aligned}$$

Based on Definition 1, the graph decomposes into disjoint components as follows:

- For $u \in U'(\mathbb{Z}_5)$, the vertices form complete graphs K_4.
- For $u \in U''(\mathbb{Z}_5)$, the vertices form a complete bipartite graph $K_{4,4}$.

Thus, the structure of $\text{Cr}_2(\mathbb{Z}_5)$ can be expressed as $\text{Cr}_2(\mathbb{Z}_5) \cong 2K_4 \cup K_{4,4}$ and as illustrated in Fig. 1.

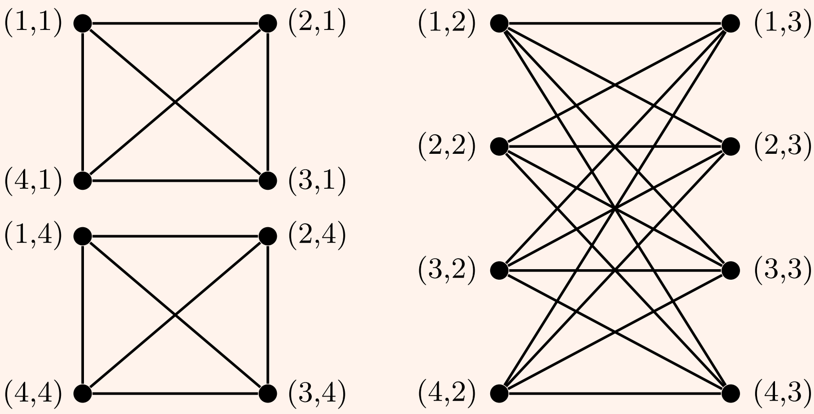

Fig. 1: $\text{Cr}_2(\mathbb{Z}_5)$

The following lemma is presented as a known result that utilized to facilitate the proofs of the primary theorems later in this paper.

Lemma 1. *[14] A complete graph K_n has chromatic number n, whereas a bipartite graph $K_{m,n}$ has chromatic number 2.*

Lemma 2. *[14] If G is a disconnected graph, then is $\overline{G}$ connected.*

This study uses a theoretical approach in algebraic graph theory to investigate the clear graph $\text{Cr}_2(\mathbb{Z}_p)$ for $p > 3$ with p prime. This research begins by partitioning the set of vertices of graph G into several subsets based on the characteristics of the ring elements with respect to the resulting graph. Next, the properties of graph $\text{Cr}_2(\mathbb{Z}_p)$ including order, size, degree, distance, diameter, and radius are identified from these partitions. Additionally, Proposition 1 is used to determine other properties such as the clique number, chromatic number, domination, and independence of graph $\text{Cr}_2(\mathbb{Z}_p)$.

3. Results and Discussion

In this section, we present the main results of this study concerning the structural and combinatorial properties of the clear graph $\text{Cr}_2(\mathbb{Z}_p)$ for $p > 3$ with p prime. The results are formulated in the form of theorems along with their corresponding proofs.

Theorem 1. *Let $\text{Cr}_2(\mathbb{Z}_p)$ be the clear graph with $p > 3$ and p prime. Then the order and size of the graph $\text{Cr}_2(\mathbb{Z}_p)$ is $(p-1)^2$ and $\frac{(p^2-2p-1)(p-1)}{2}$, respectively.*

Proof. Based on Definition 1, we first determine the sets $U(\mathbb{Z}_p)$ and $U_{\text{reg}}(\mathbb{Z}_p)\setminus\{0\}$. We obtain $U(\mathbb{Z}_p) = U_{\text{reg}}(\mathbb{Z}_p) \setminus \{0\} = \{1, 2, ..., p-1\}$. Observe that $|U(\mathbb{Z}_p)| = |U_{\text{reg}}(\mathbb{Z}_p) \setminus \{0\}| = p - 1$. Hence, the order of $\text{Cr}_2(\mathbb{Z}_p)$ is

$$|V(\text{Cr}_2(\mathbb{Z}_p))| = |U_{\text{reg}}(\mathbb{Z}_p) \setminus \{0\}| \cdot |U(\mathbb{Z}_p)| = (p-1)(p-1) = (p-1)^2.$$

Next, let $V(\mathrm{Cr}_2(\mathbb{Z}_p)) = A_1 \cup A_2 \cup B_1 \cup \cdots \cup B_{\frac{p-3}{2}}$ where

$$A_i = \{(x, u_i) \mid x \in U_{\mathrm{reg}}(\mathbb{Z}_p) \setminus \{0\},\ u_i \in U'(\mathbb{Z}_p)\}, \quad i = 1, 2,$$

and

$$B_{j,1} = \{(x, u_j) \mid x \in U_{\mathrm{reg}}(\mathbb{Z}_p) \setminus \{0\}\}, \quad B_{j,2} = \{(x, v_j) \mid x \in U_{\mathrm{reg}}(\mathbb{Z}_p) \setminus \{0\}\}$$

for every pair $\{u_j, v_j\}$ where $B_j = B_{j,1} \cup B_{j,2}$, $j = 1, 2, ..., \frac{p-3}{2}$.
In this case $A_1 = \{(x, 1) \mid x \in U_{\mathrm{reg}}(\mathbb{Z}_p) \setminus \{0\}\}$ and $A_2 = \{(x, p-1) \mid x \in U_{\mathrm{reg}}(\mathbb{Z}_p) \setminus \{0\}\}$.
By Definition 1, every two vertices in A_i are adjacent for $i = 1, 2$, since $1 \cdot 1 \equiv 1 \pmod p$ and $(p-1)(p-1) \equiv 1 \pmod p$. Hence the subgraph induced by A_i is a complete graph K_{p-1} with $\frac{(p-1)(p-2)}{2}$ edges.
Next, any two vertices in $B_{j,1}$ is not adjacent since $u_j^2 \not\equiv 1 \pmod p$, and the same holds for $B_{j,2}$. Moreover, each vertex in $B_{j,1}$ is adjacent to every vertex in $B_{j,2}$ because $u_j v_j \equiv 1 \pmod p$. Therefore, the subgraph induced by B_j is a complete bipartite graph $K_{p-1,p-1}$ with $(p-1)^2$ edges.
Consequently, the size of $\mathrm{Cr}_2(\mathbb{Z}_p)$ is

$$2\frac{(p-1)(p-2)}{2} + \frac{p-3}{2}(p-1)^2 = \frac{(p^2-2p-1)(p-1)}{2}.$$ □

Theorem 2. *The degree of a vertex (x, u) in $\mathrm{Cr}_2(\mathbb{Z}_p)$ is given by*

$$\deg((x,u)) = \begin{cases} p-2, & \textit{if } u \in U'(\mathbb{Z}_p), \\ p-1, & \textit{if } u \in U''(\mathbb{Z}_p). \end{cases}$$

Proof. Let the sets A_i and B_j be as defined in Theorem 1. Thus, the vertices of $\mathrm{Cr}_2(\mathbb{Z}_p)$ can be classified according to whether their second component belongs to $U'(\mathbb{Z}_p)$ or to $U''(\mathbb{Z}_p)$. From the proof of Theorem 1, every two vertices in A_i are adjacent for $i = 1, 2$, so the subgraphs induced by A_i are complete. Hence each vertex $(x, u) \in A_i$ is adjacent to all other $p-2$ vertices in the same set, and therefore has degree $p-2$ when $u \in U'(\mathbb{Z}_p)$.
On the other hand, for each $B_j = B_{j,1} \cup B_{j,2}$, vertices within $B_{j,1}$ or within $B_{j,2}$ are mutually nonadjacent, while every vertex of $B_{j,1}$ is adjacent to every vertex of $B_{j,2}$. Thus, each vertex in B_j is adjacent to all $p-1$ vertices in the opposite part, and hence has degree $p-1$ when $u \in U''(\mathbb{Z}_p)$. □

Corollary 1. *The graph $\mathrm{Cr}_2(\mathbb{Z}_p)$ is not regular but each component is a regular.*

Proof. By Theorem 2, it is clear that $\mathrm{Cr}_2(\mathbb{Z}_p)$ is not regular graph. Furthermore, since the components of $\mathrm{Cr}_2(\mathbb{Z}_p)$ are K_{p-1} and $K_{p-1,p-1}$, which are $(p-2)$-regular and $(p-1)$-regular, respectively, it follows that every component of $\mathrm{Cr}_2(\mathbb{Z}_p)$ is a regular graph. □

Theorem 3. *For any two vertices (x, u) and (y, v) in $\mathrm{Cr}_2(\mathbb{Z}_p)$, the distance between them* *is given by*

$$d((x,u),(y,v)) = \begin{cases} 0, & \textit{if } (x,u),(y,v) \in V(\mathrm{Cr}_2(\mathbb{Z}_p)), (x,u) = (y,v), \\ 1, & \textit{if } u,v \in U'(\mathbb{Z}_p), u = v \textit{ or } u,v \in U''(\mathbb{Z}_p), u = v^{-1}, \\ 2, & \textit{if } u,v \in U''(\mathbb{Z}_p), u = v, \\ \infty, & \textit{otherwise.} \end{cases}$$

Proof. Take arbitrary vertices $(x,u),(y,v) \in V(\mathrm{Cr}_2(\mathbb{Z}_p))$ and consider the following cases.

(1) For $u,v \in U'(\mathbb{Z}_p)$. If $u \neq v$, then (x,u) and (y,v) are disconnected since $uv \not\equiv 1 \pmod p$ and $xy \not\equiv 0 \pmod p$. If $u = v$, then (x,u) and (y,u) are adjacent because $u^2 \equiv 1 \pmod p$, and hence $d((x,u),(y,u)) = 1$.

(2) For $u,v \in U''(\mathbb{Z}_p)$. If $u \neq v$ and $u = v^{-1}$, then (x,u) and (y,v) are adjacent since $uv \equiv vu \equiv 1 \pmod p$, so $d((x,u),(y,v)) = 1$. If $u \neq v$ and $u \neq v^{-1}$, then (x,u) and (y,v) are disconnected since $uv \not\equiv 1 \pmod p$ and $xy \not\equiv 0 \pmod p$. If $u = v$, then (x,u) and (y,u) are not adjacent, but they are connected via a path $(x,u)-(x,v)-(y,u)$, where $v = u^{-1}$. Hence $d((x,u),(y,u)) = 2$.

(3) For $u \in U'(\mathbb{Z}_p)$ and $v \in U''(\mathbb{Z}_p)$. The vertices (x,u) and (y,v) are disconnected so the distance is ∞.

Therefore, the distance between any two vertices (x,u) and (y,v) in $\mathrm{Cr}_2(\mathbb{Z}_p)$ is as stated. □

Corollary 2. *The number of connected components of* $\mathrm{Cr}_2(\mathbb{Z}_p)$ *is* $\frac{p+1}{2}$.

Proof. By Proposition 1, it is clear that the number of components of $\mathrm{Cr}_2(\mathbb{Z}_p)$ is

$$2 + \frac{p-3}{2} = \frac{p+1}{2}.$$ □

Corollary 3. *Let* $\mathrm{Cr}_2(\mathbb{Z}_p)$ *be the clear graph with* $p > 3$ *prime. Then* $\mathrm{diam}(\mathrm{Cr}_2(\mathbb{Z}_p)) = \infty$ *and every connected component has radius at most* 2.

Proof. This is a direct consequence of Theorem 3, which states that $\mathrm{diam}(\mathrm{Cr}_2(\mathbb{Z}_p)) = \infty$. Furthermore, we know that the radius of K_{p-1} is 1 and the radius of $K_{p-1,p-1}$ is 2. Therefore, it is clear that the radius of each connected component in $\mathrm{Cr}_2(\mathbb{Z}_p)$ is at most 2. □

Corollary 4. *Let* $\mathrm{Cr}_2(\mathbb{Z}_p)$ *be the clear graph with* $p > 3$ *prime. Then the complement graph of* $\mathrm{Cr}_2(\mathbb{Z}_p)$ *is connected.*

Proof. Since $\mathrm{Cr}_2(\mathbb{Z}_p)$ is disconnected graph, by Lemma 2, the complement of the graph $\mathrm{Cr}_2(\mathbb{Z}_p)$ is a connected graph. □

Theorem 4. *The* $\mathrm{Cr}_2(\mathbb{Z}_p)$ *has* $\frac{(p-1)(p-2)(p-3)}{3}$ *triangles.*

Proof. Since every set of three vertices in complete graph K_{p-1} forms a triangle, while complete bipartite graph $K_{p-1,p-1}$ contains none and $\mathrm{Cr}_2(\mathbb{Z}_p)$ has 2 K_{p-1}. Therefore, the number of triangles in graph $\mathrm{Cr}_2(\mathbb{Z}_p)$ is

$$2\binom{p-1}{3} = 2\frac{(p-1)(p-2)(p-3)}{6} = \frac{(p-1)(p-2)(p-3)}{3}. \qquad \square$$

Example 2. Let $\mathbb{Z}_7$ be the ring of integers modulo 7. Hence, we have $\mathrm{Cr}_2(\mathbb{Z}_7)$ as shown in Fig. 2.

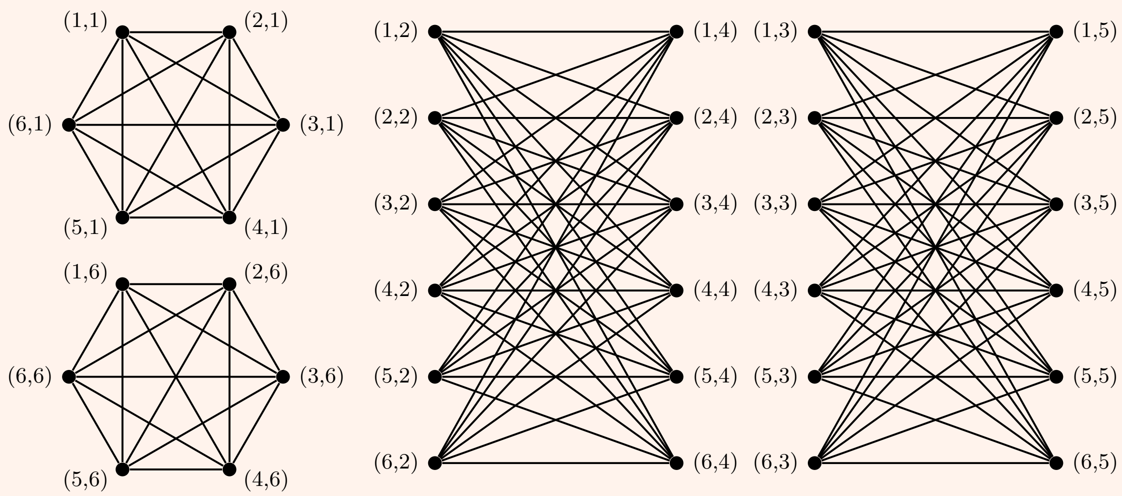

Fig. 2: $\mathrm{Cr}_2(\mathbb{Z}_7)$

Then, we can state the following result:

(i) The order of graph $\mathrm{Cr}_2(\mathbb{Z}_7)$ is

$$|V(\mathrm{Cr}_2(\mathbb{Z}_7))| = |U_{\mathrm{reg}}(\mathbb{Z}_7) \setminus \{0\}| \times |U(\mathbb{Z}_7)| = 6 \times 6 = 36$$

and the total number of edges in $\mathrm{Cr}_2(\mathbb{Z}_7)$ is

$$2 \times 15 + 2 \times 36 = 30 + 72 = 102.$$

(ii) The degree of a vertex (x,u) in $\mathrm{Cr}_2(\mathbb{Z}_7)$ depends on whether u is self-inverse or not. For $u \in U'(\mathbb{Z}_7)$, the degree is $\deg((x,u)) = 6 - 1 = 5$. On the other hand, for $u \in U''(\mathbb{Z}_7)$, the degree is $\deg((x,u)) = 6$. Therefore, the degree of vertices in $\mathrm{Cr}_2(\mathbb{Z}_7)$ can be summarized as follows:

$$\deg((x,u)) = \begin{cases} 5, & \text{if } u \in U'(\mathbb{Z}_7), \\ 6, & \text{if } u \in U''(\mathbb{Z}_7). \end{cases}$$

(iii) In $\mathbb{Z}_7$, we have $U'(\mathbb{Z}_7) = \{1,6\}$ and $U''(\mathbb{Z}_7) = \{2,3,4,5\}$ with inverse pairs $\{2,4\}$ and $\{3,5\}$. By Theorem 3, for vertices sharing the same second coordinate in $U'(\mathbb{Z}_7)$, the distance is 1, i.e., $d((x,1),(y,1)) = 1$, while vertices with distinct second coordinates in $U'(\mathbb{Z}_7)$ are disconnected, giving $d((x,1),(y,6)) = \infty$. For vertices whose second coordinates form an inverse pair in $U''(\mathbb{Z}_7)$, the distance is also 1, namely $d((x,2),(y,4)) = d((x,3),(y,5)) = 1$. On the other hand, vertices sharing the same second coordinate in $U''(\mathbb{Z}_7)$ are at distance 2, that is, $d((x,2),(y,2)) = d((x,3),(y,3)) = d((x,4),(y,4)) = d((x,5),(y,5)) = 2$. Finally, any two vertices whose second coordinates belong to different sets $U'(\mathbb{Z}_7)$ and $U''(\mathbb{Z}_7)$ are disconnected, so their distance is ∞. This implies that $\text{diam}(\text{Cr}_2(\mathbb{Z}_7)) = \infty$.

(iv) There are 4 components in $\text{Cr}_2(\mathbb{Z}_7)$ and $2\binom{6}{3} = 40$ triangles.

Theorem 5. *Let* $\text{Cr}_2(\mathbb{Z}_p)$ *be the clear graph with* $p > 3$ *and* p *prime. Then the clique number of* $\text{Cr}_2(\mathbb{Z}_p)$ *is* $p - 1$*.*

Proof. By Proposition 1, the graph $\text{Cr}_2(\mathbb{Z}_p)$ is isomorphic to the union of several disconnected components consisting of a complete graph K_{p-1} and a complete bipartite graph $K_{p-1,p-1}$. A complete bipartite graph has a clique number of 2, whereas a complete graph has a clique number equal to the order of the graph itself. In the graph $\text{Cr}_2(\mathbb{Z}_p)$, there is component K_{p-1}, so the clique number of the graph K_{p-1} is $p-1$. It follows that the complete subgraph K_{p-1} is the largest clique in the graph $\text{Cr}_2(\mathbb{Z}_p)$. Therefore, the clique number of the graph $\text{Cr}_2(\mathbb{Z}_p)$ is $p - 1$. □

Theorem 6. *The chromatic number of* $\text{Cr}_2(\mathbb{Z}_p)$ *is* $p - 1$*.*

Proof. It is well known that the complete graph K_{p-1} has chromatic number $p - 1$, and the complete bipartite graph $K_{p-1,p-1}$ has chromatic number 2. Thus, using Proposition 1, the chromatic number of $\text{Cr}_2(\mathbb{Z}_p)$ is $\chi(\text{Cr}_2(\mathbb{Z}_p)) = \max(p - 1, 2) = p - 1$, which follows from the fact that $p > 3$ implies $p - 1 > 2$. □

Theorem 7. *The domination number of* $\text{Cr}_2(\mathbb{Z}_p)$ *is* $p - 1$*.*

Proof. By Proposition 1, every complete subgraph K_{p-1} in a graph $\text{Cr}_2(\mathbb{Z}_p)$ can clearly be dominated by a vertex in that subgraph. Meanwhile, for every complete bipartite subgraph $K_{p-1,p-1}$, it is dominated by two vertices. Since there are $2K_{p-1}$ subgraphs and $\frac{p-3}{2}K_{p-1,p-1}$ subgraphs, the domination number of graph $\text{Cr}_2(\mathbb{Z}_p)$ is $2 + 2(\frac{p-3}{2}) = p - 1$. □

Theorem 8. *The independence number of* $\text{Cr}_2(\mathbb{Z}_p)$ *is*

$$\alpha(\text{Cr}_2(\mathbb{Z}_p)) = \frac{p^2 - 4p + 7}{2}.$$

Proof. We know that every complete subgraph of K_{p-1}, there is only one vertex that is independent. Meanwhile, in every complete bipartite subgraph $K_{p-1,p-1}$, there are $p - 1$ vertices that are mutually independent. By Proposition 1, the number of independent vertices in graph $\text{Cr}_2(\mathbb{Z}_p)$ is $2(1) + \frac{(p-3)}{2}(p - 1) = \frac{p^2-4p+7}{2}$. □

Example 3. Let $\text{Cr}_2(\mathbb{Z}_7)$ be the clear graph as in Example 2. Recall that $U'(\mathbb{Z}_7) = \{1, 6\}$ and $U''(\mathbb{Z}_7) = \{2, 3, 4, 5\}$ with inverse pairs $\{2, 4\}$ and $\{3, 5\}$. Since $\text{Cr}_2(\mathbb{Z}_7)$ contains K_6 as a component, the largest clique in $\text{Cr}_2(\mathbb{Z}_7)$ is K_6 itself. Therefore, the clique number is $\omega(\text{Cr}_2(\mathbb{Z}_7)) = 6$. In addition, as K_6 is a component of $\text{Cr}_2(\mathbb{Z}_7)$, at least 6 colors are required to properly color $\text{Cr}_2(\mathbb{Z}_7)$. On the other hand, the components $K_{6,6}$ are 2-colorable. Thus, 6 colors suffice to color the entire graph, giving $\chi(\mathrm{Cr}_2(\mathbb{Z}_7)) = 6$. A 6-coloring of $\mathrm{Cr}_2(\mathbb{Z}_7)$ is shown in Fig. 3. Since $\mathrm{Cr}_2(\mathbb{Z}_7)$ is a disconnected graph, its domination number is the sum of the domination numbers of each component. Each component K_6 has domination number 1, while each component $K_{6,6}$ has domination number 2. Therefore, the domination number of $\mathrm{Cr}_2(\mathbb{Z}_7)$ is $\gamma(\mathrm{Cr}_2(\mathbb{Z}_7)) = 1 + 1 + 2 + 2 = 6$. An independent set in $\mathrm{Cr}_2(\mathbb{Z}_7)$ consists of vertices that are mutually non-adjacent. In each component K_6, the maximum independent set has size 1, while in each component $K_{6,6}$, the maximum independent set has size 6, that is one full partition. Since $\mathrm{Cr}_2(\mathbb{Z}_7)$ has two of K_6 and two of $K_{6,6}$, the independence number is $\alpha(\mathrm{Cr}_2(\mathbb{Z}_7)) = 1 + 1 + 6 + 6 = 14$.

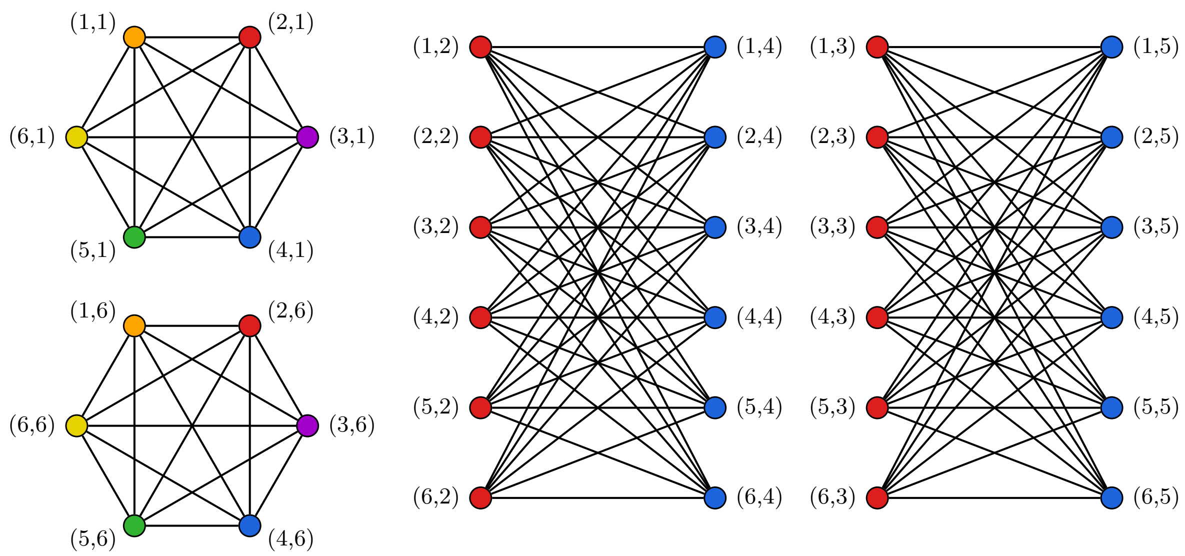

Fig. 3: $\mathrm{Cr}_2(\mathbb{Z}_7)$ with 6-colorable

4. Conclusion

In this paper, we have investigated the structural and combinatorial properties of the clear graph $\mathrm{Cr}_2(\mathbb{Z}_p)$ for $p > 3$ with p prime, as a direct extension of the work initiated by Mir et al. [11]. By analyzing its algebraic construction, we established a detailed description of its structure and derived several fundamental graph invariants. We note that, although several results follow from the known decomposition of $\mathrm{Cr}_2(\mathbb{Z}_p)$, the explicit and unified determination of these invariants constitutes the primary contribution of this work.

Specifically, we determine the order and size of the graph and characterize its vertex degrees and distance properties. Furthermore, we compute important invariants including the clique number, chromatic number, domination number, and independence number. These results provide a comprehensive and self-contained reference for the combinatorial properties of $\mathrm{Cr}_2(\mathbb{Z}_p)$, making explicit what was previously only implicit in the decomposition.

The findings of this study not only extend the theory of clear graphs but also contribute to the broader interaction between ring theory and graph theory. Future research may consider investigating clear graphs in more general ring classes, such as matrix rings, polynomial rings, or quotient rings over $\mathbb{Z}_p$, in order to further explore the relationship between algebraic properties and graph-theoretic characteristics.

CRediT Authorship Contribution Statement

Mohammad Ilham Maulana: Conceptualization, Methodology, Writing–Original Draft. **Vira Hari Krisnawati:** Supervision, validation, Writing–Review & Editing. **Ratno Bagus Edy**

Wibowo: Supervision, validation, Writing–Review & Editing.

Declaration of Generative AI and AI-assisted technologies

The authors declare that no generative AI or AI-assisted technologies were used to generate or modify the results of this research. The author utilized Gemini AI and ChatGPT as a tool for grammatical and language editing only.

Declaration of Competing Interest

The authors declare no competing interests.

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