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Existence and Uniqueness Property On a Generalized Ledin-Brousseau Sum

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Abstract

In this paper, we present the existence and uniqueness property on Ledin-Brousseau sum involving a polynomial and a homogeneous linear recurrence sequence. This sum is of the form $\sum_{k=1}^{n} P(k)s_{hk+r}$ where n is a positive integer, $P(x)$ is a polynomial in $\mathbb{C}[x]$, h and r are some integers, and $(s_k)_{k\in\mathbb{Z}}$ is a homogeneous linear recurrence sequence of degree $m \geq 2$ with some constraints.

Keywords: Ledin-Brousseau Sum; Linear Recurrence Sequence; Polynomial.

1. Introduction and Preliminaries

One of the frequently discussed topics involving finite sum and linear recurrence sequence is the Ledin-Brousseau sum (or simply the Brousseau sum) and its generalizations. In the beginning, Ledin [1] and Brousseau [2] respectively studied the finite sums $\sum_{k=1}^{n} k^m F_k$ and $\sum_{k=1}^{n} k^m F_{k+r}$ where $(F_k)_{k\geq 0}$ is the classic Fibonacci sequence. Ledin [1] presented the identity

$$S(m,n) \stackrel{\text{def}}{=} \sum_{k=1}^{n} k^m F_k = F_{n+1}P_2(m,n) + F_n P_1(m,n) + C(m) \tag{1}$$

where $P_1(m,n)$ and $P_2(m,n)$ are polynomials in n of degree m and $C(m)$ is a constant depending on m. In the same year with Ledin, Brousseau [2] formulated the sum

$$S_r(m,n) \stackrel{\text{def}}{=} \sum_{k=0}^{n-1} k^m F_{k+r} = \sum_{t=0}^{m} (-1)^t \Delta^t(n^m) F_{n+r+2t+1} + C \tag{2}$$

where C is a constant independent of n and Δ is the forward difference operator with the properties $\Delta(f(n)) = f(n+1) - f(n)$, $\Delta^{t+1}(f(n)) = \Delta(\Delta^t(f(n)))$, and $\Delta^0(f(n)) = f(n)$.

Many various methods have been developed to evaluate the formula of $S(m,n)$ and $S_r(m,n)$. Ledin [1] used an integration method to find the recursion of $P_1(m,n)$ and $P_2(m,n)$. Brousseau [2] used finite difference method to obtain the expression Eq. (2). Shannon and Ollerton, in their works [3] and [4], used the matrix method and formulated the linear recurrence relation

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of $S(m,n)$. Besides that, partial differentiation method was applied by Adegoke [5] to obtain the recursive relation for $P_1(m,n)$, $P_2(m,n), C(m)$ and then the polynomial form of $S(m,n)$. Lastly, Dresden [6] evaluated $S(m,n)$ by applying a convolution representation of $F_n - n^m$ which involves binomial coefficients.

Ledin's and Brousseau's works ([1, 2]) have also motivated many researchers afterward to investigate the more general sums. Mostly, the Ledin-Brousseau sum is generalized by replacing the Fibonacci sequence in the sums Eqs. (1) and (2) with other linear recurrence sequences.

Zeitlin [7] extended the summations by using the sequence $(H_n)_{n\geq 0}$ which satisfies the recurrence relation $H_{n+2} = H_{n+1} + H_n$. He obtained that the following formula holds for all $n \in \mathbb{N}$ and all $r, m \in \mathbb{N}_0$:

$$\sum_{k=0}^{n-1} k^m H_{k+r} = H_{n+r} \sum_{s=0}^{m} \left(\binom{m}{s} \sum_{t=0}^{m} (-1)^t t! F_{2t} \left\{ {m-s \atop t} \right\} \right) n^s + H_{n+r+1} \sum_{s=0}^{m} \left(\binom{m}{s} \sum_{t=0}^{m} (-1)^t t! F_{2t+1} \left\{ {m-s \atop t} \right\} \right) n^s + C_2 \quad (3)$$

where

$$C_2 = -H_r \sum_{t=0}^{m} (-1)^t t! F_{2t} \left\{ {m \atop t} \right\} - H_{r+1} \sum_{t=0}^{m} (-1)^t t! F_{2t+1} \left\{ {m \atop t} \right\}$$

and $\left\{ {n \atop k} \right\}$ are Stirling numbers of second kind (see page 66 in [8] for explanation about Stirling numbers).

Nair [9] generalized Ledin-Brousseau sum by replacing with Horadam sequence $(W_n)_{n\geq 0}$ which satisfies the recurrence relation $W_{n+2} = pW_{n+1} - qW_n$ where p, q are non-zero integers and $\lambda = p - q - 1 \neq 0$. He obtained that for all $n \in \mathbb{N}$ and $m \in \mathbb{N}_0$,

$$\sum_{k=1}^{n} k^m W_k = \frac{\mathscr{C}_{(p,q)}^{(m)}(n)}{\lambda} W_{n+1} - \frac{q\mathscr{C}_{(p,q)}^{(m)}(n+1)}{\lambda} W_n + \frac{-\mathscr{C}_{(p,q)}^{(m)}(0)W_1 + q\mathscr{C}_{(p,q)}^{(m)}(1)W_0}{\lambda} \quad (4)$$

where $\mathscr{C}_{(p,q)}^{(m)}(x) \in \mathbb{C}[x]$ is a polynomial determined by m, p, q.

Nair and Karunakaran [10] explored the Brousseau sum which is generated by k-Fibonacci sequence $(F_{k,n})_{n\geq 0}, k \in \mathbb{N}$, which satisfies the recurrence relation $F_{k,n+2} = kF_{k,n+1} + F_{k,n}$ and initial values $F_{k,0} = 0, F_{k,1} = 1$. They obtained that for all $n \in \mathbb{N}$ and $m, p \in \mathbb{N}_0$,

$$\sum_{i=1}^{n} i^p F_{k,m+i} = \frac{C_k^{(p)}(n)}{k} F_{k,m+n+1} + \frac{C_k^{(p)}(n+1)}{k} F_{k,m+n} - \frac{C_k^{(p)}(0)F_{k,m+1} + C_k^{(p)}(1)F_{k,m}}{k} \quad (5)$$

where $C_k^{(p)}(x)$ is some polynomial in $\mathbb{C}[x]$ which depends on p and k.

Hadinata [11] considered the sum $\sum_{k=1}^{n} P(k)S_{k-1}$ where $P(x)$ is a polynomial of real coefficients and the sequence $(S_n)_{n\geq 0}$ is defined by recurrence relation $S_{n+2} = aS_{n+1} + bS_n$ and initial values $S_0 = c_0, S_1 = c_1$ where $a, b \in \mathbb{N}$ and $(c_0, c_1) \in \mathbb{Z}^2 \backslash \{(0,0)\}$. He obtained that there exists a triple $(F_1(x), G_1(x), H_1(x))$ of real coefficients polynomials such that for all $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} P(k)S_{k-1} = F_1(n)S_{n+1} + G_1(n)S_n + H_1(n). \quad (6)$$

Moreover, he showed that if both of $2c_1 - (a + \sqrt{a^2+4b})c_0$ and $2c_1 - (a - \sqrt{a^2+4b})c_0$ are non-zero, then the triple $(F_1(x), G_1(x), H_1(x)) \in \mathbb{R}[x]$ which satisfies Eq. (6) is unique. Otherwise, if one of $2c_1 - (a + \sqrt{a^2+4b})c_0$ or $2c_1 - (a - \sqrt{a^2+4b})c_0$ is zero, then there are infinitely many triples $(F_1(x), G_1(x), H_1(x)) \in \mathbb{R}[x]$ which satisfy Eq. (6).

The Brousseau sum of Tribonacci numbers was presented by Nair and Karunakaran [12]. They established the formula for the sum $\sum_{k=0}^{n} k^m T_k$ where $m \in \mathbb{N}_0$ and the Tribonacci sequence

$(T_k)_{k\geq -1}$ is defined by recurrence relation $T_{n+3} = T_{n+2} + T_{n+1} + T_n$ and initial terms $T_{-1} = T_0 = 0, T_1 = 1$. The following formula holds for all $m, n \in \mathbb{N}_0$:

$$\sum_{k=0}^{n} k^m T_k = \frac{\psi_A^{(m)}(n)}{2} T_{n+1} + \frac{\psi_B^{(m)}(n)}{2} T_n + \frac{\psi_C^{(m)}(n)}{2} T_{n-1} - \frac{\psi_A^{(m)}(0)}{2} \tag{7}$$

where $\psi_A^{(m)}(x)$, $\psi_B^{(m)}(x)$, and $\psi_C^{(m)}(x)$ are some polynomials in $\mathbb{C}[x]$ which depends on m.

Besides that, Nair [13] extended the Brousseau sum for generalized Tetranacci numbers. Let the generalized Tetranacci numbers $(T'_k)_{k\geq -2}$ be defined by recursion $T'_k = T'_{k-1} + T'_{k-2} + T'_{k-3} + T'_{k-4}$ for $k > 1$ and T_1, T_0, T_{-1}, T_{-2} are arbitrary initial terms. He found that for all $m \in \mathbb{N}_0$ and all $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} k^m T'_k = \frac{\varphi_0^{(m)}(n)}{3} T'_{n+1} + \frac{\varphi_1^{(m)}(n)}{3} T'_n + \frac{\varphi_2^{(m)}(n)}{3} T'_{n-1} + \frac{\varphi_3^{(m)}(n)}{3} T'_{n-2} + \frac{\varphi_4^{(m)}(n)}{3} \tag{8}$$

where $\varphi_0^{(m)}(x), \varphi_1^{(m)}(x), \varphi_2^{(m)}(x), \varphi_3^{(m)}(x), \varphi_4^{(m)}(x)$ are some polynomials in $\mathbb{C}[x]$ which depends on m and $\varphi_4^{(m)}(x)$ is a constant polynomial with

$$\varphi_4^{(m)}(x) = -\left(\varphi_0^{(m)}(0) T'_1 + \varphi_1^{(m)}(0) T'_0 + \varphi_2^{(m)}(0) T'_{-1} + \varphi_3^{(m)}(0) T'_{-2}\right).$$

The most recently, Nair [14] presented the Brousseau sums for Padovan sequence $(P_n)_{n\geq 0}$, Perrin sequence $(Q_n)_{n\geq 0}$, and Van der Laan sequence $(V_n)_{n\geq 0}$. These sequences satisfy the recursion $P_n = P_{n-2} + P_{n-3}$, $Q_n = Q_{n-2} + Q_{n-3}$, $V_n = V_{n-2} + V_{n-3}$ and initial terms $P_0 = 1$, $P_1 = P_2 = 0$, $Q_0 = 3$, $Q_1 = 0$, $Q_2 = 2$, $V_0 = 1$, $V_1 = 0$, $V_2 = 1$. He showed that if given $m, r \in \mathbb{N}_0$, then there exists a polynomial $\mathcal{C}^{(m)}(x) \in \mathbb{C}[x]$ of degree m such that the following three identities hold for all $n \in \mathbb{N}$:

$$\sum_{k=1}^{n} k^m P_{k+r} = \mathcal{C}^{(m)}(n) P_{n+r} + \mathcal{C}^{(m)}(n-2) P_{n+r+1} + \mathcal{C}^{(m)}(n-1) P_{n+r+2} - \mathcal{C}^{(m)}(0) P_r - \mathcal{C}^{(m)}(-2) P_{r+1} - \mathcal{C}^{(m)}(-1) P_{r+2}, \tag{9}$$

$$\sum_{k=1}^{n} k^m Q_k = \mathcal{C}^{(m)}(n) Q_n + \mathcal{C}^{(m)}(n-2) Q_{n+1} + \mathcal{C}^{(m)}(n-1) Q_{n+2} - 3\mathcal{C}^{(m)}(0) - 2\mathcal{C}^{(m)}(-1), \tag{10}$$

$$\sum_{k=1}^{n} k^m V_k = \mathcal{C}^{(m)}(n) V_n + \mathcal{C}^{(m)}(n-2) V_{n+1} + \mathcal{C}^{(m)}(n-1) V_{n+2} - \mathcal{C}^{(m)}(0) - \mathcal{C}^{(m)}(-1). \tag{11}$$

By observing the formulas Eqs. (1), (3), (4), (5), (6), (7), (8), (9), (10), and (11), it may be reasonable to conjecture that if $P(x) \in \mathbb{C}[x]$ is a polynomial and $(s_k)_{k\in\mathbb{Z}}$ is a m^{th} order homogeneous linear recurrence sequence, then there exist $m+1$ polynomials $p_1(x), p_2(x), \ldots, p_{m+1}(x)$ in $\mathbb{C}[x]$ such that for all $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} P(k) s_k = \left(\sum_{k=1}^{m} p_k(n) s_{n+k}\right) + p_{m+1}(n).$$

More generally, if $h, r \in \mathbb{Z}$, we may conjecture that

$$\sum_{k=1}^{n} P(k) s_{hk+r} = \left(\sum_{k=1}^{m} P_k(n) s_{(n+k)h+r}\right) + P_{m+1}(n), \quad \forall n \in \mathbb{N} \tag{12}$$

for some polynomials $P_1(x), P_2(x), \ldots, P_{m+1}(x)$ in $\mathbb{C}[x]$. In this paper, we investigate these hypothesizes specifically when $(s_k)_{k\in\mathbb{Z}}$ is a sequence introduced in Definition 1.

For convenience, in all next parts of this paper, let the notation $(s_k)_{k\in\mathbb{Z}}$ be referred to the sequence defined in Definition 1.

Definition 1. Let $m \geq 2$ be a positive integer. Let $(s_k)_{k\in\mathbb{Z}}$ be a homogeneous linear recurrence sequence of minimal degree[a] m which satisfies the following properties:

- $(s_k)_{k\in\mathbb{Z}}$ satisfies the recurrence relation

$$s_{k+m} = a_m s_{k+m-1} + a_{m-1}s_{k+m-2} + \cdots\cdots + a_2 s_{k+1} + a_1 s_k \quad (\forall k \in \mathbb{Z}) \tag{13}$$

  where $a_1, a_2, \ldots, a_m$ are complex constants with $a_1 \neq 0$.
- $s_1, s_2, \ldots, s_m$ are not simultaneously zero.
- The characteristic polynomial of recurrence relation Eq. (13), that is,

$$\mathcal{CP}(x) = x^m - a_m x^{m-1} - a_{m-1}x^{m-2} - \ldots - a_2 x - a_1 \tag{14}$$

  has m distinct roots. Moreover, all roots of $\mathcal{CP}(x)$ are non-zero because of $a_1 \neq 0$.

[a]The clause "m is the minimal degree of $(s_k)_{k\in\mathbb{Z}}$" means that m is the smallest degree of homogeneous linear recurrence relations of complex constant coefficients satisfied by $(s_k)_{k\in\mathbb{Z}}$.

The motivation in this paper is not just about generalizing the Ledin-Brousseau sum to more general homogeneous linear recurrence sequence. Most of previous results just formulated the Ledin-Brousseau sum as similar as the identity Eq. (12) and showed the existence of polynomials $P_1(x), P_2(x), \ldots, P_{m+1}(x)$ without showing their uniqueness. But, the first author in [11] initiated to explore the uniqueness of these polynomials in the Ledin-Brousseau of second degree homogeneous linear recurrence sequence. This uniqueness is our research gap and in this paper, we explore it in m^{th} degree homogeneous linear recurrence sequence $(s_k)_{k\in\mathbb{Z}}$ as introduced in Definition 1. We find the result that if m is the minimal degree of $(s_k)_{k\in\mathbb{Z}}$ and h, r are arbitrary integers for which the h^{th} power[1] of the roots of the monic characteristic polynomial of $(s_k)_{k\in\mathbb{Z}}$ with minimal degree (that is $\mathcal{CP}(x)$ in Eq. (14)) are distinct and not equal to 1, then the existence of polynomials $P_1(x), P_2(x), \ldots, P_{m+1}(x)$ in the right hand side of Eq. (12) is unique.

The given conditions "m is the minimal degree of $(s_k)_{k\in\mathbb{Z}}$" and "the h^{th} power of the roots of $\mathcal{CP}(x)$ are distinct and not equal to 1" play significant roles in the uniqueness property of $P_1(x), P_2(x), \ldots, P_{m+1}(x)$. It is because if one of them is not satisfied, this uniqueness may not hold probably. As an example, let us assume that m is not minimal degree of $(s_k)_{k\in\mathbb{Z}}$. Let $m = 3$ and $(s_k)_{k\in\mathbb{Z}}$ be defined by third degree recursion $s_{k+3} = s_k$ and initial terms $s_1 = s_2 = s_3 = 1$. It is clear that $(s_k)_{k\in\mathbb{Z}}$ is a constant sequence and hence the minimal degree of $(s_k)_{k\in\mathbb{Z}}$ is 1. If we set $P(x) = 2x$ and $(h, r) = (1, 0)$, the identity

$$\sum_{k=1}^{n} P(k)s_k = n^2 + n = P_1(n)s_{n+1} + P_2(n)s_{n+2} + P_3(n)s_{n+3} + P_4(n)$$

holds for all $n \in \mathbb{N}$ and all $(P_1(x), P_2(x), P_3(x), P_4(x)) \in \mathbb{C}[x]^4$ for which $P_1(x) + P_2(x) + P_3(x) + P_4(x) = x^2 + x$. It shows the non-uniqueness of $P_1(x), P_2(x), \ldots, P_{m+1}(x)$ with respect to the expected formula Eq. (12).

The next sections of our paper are structured as follows. Section 2 presents two main theorems of this paper: Theorem 1 and Theorem 2. Section 3 presents the explicit formula of $(s_k)_{k\in\mathbb{Z}}$. Section 4 presents some basic lemmas which are needed to show Theorem 1 and 2. Section 5 presents the complete proof of Theorem 1. Section 6 presents the complete proof of Theorem 2. Section 7 presents numerical computation of some examples of Ledin-Brousseau sums. Finally, Section 8 gives the conclusion which is followed by further discussion and some open problems in Section 9.

[1]If α is a non-zero complex numbers and β is an integer, then the phrase "the β^{th} power of α" means the number α^β. For example, the $(-4)^{th}$ of 2 equals $2^{-4} = \frac{1}{16}$. Hence, in this paper, we allow this phrase when β is a non-positive integer.

2. Main Results

Let $P(x) \in \mathbb{C}[x]$ be a polynomial and $h, r \in \mathbb{Z}$ be such that h^{th} power of the roots of the characteristic polynomial Eq. (14) are distinct and not equal to 1. The generalized Ledin-Brousseau sum $\sum_{k=1}^{n} P(k)s_{hk+r}$ are established and investigated. There are two main results in this paper: Theorem 1 and Theorem 2.

Theorem 1. *Let $P(x) \in \mathbb{C}[x]$ be a polynomial and h, r be integers such that h^{th} power of the roots of the characteristic polynomial Eq. (14) are distinct and not equal to 1. Then, there exist $m+1$ polynomials $P_1(x), P_2(x), \ldots, P_{m+1}(x)$ in $\mathbb{C}[x]$ such that the following identity holds:*

$$\sum_{k=1}^{n} P(k)s_{hk+r} = \left(\sum_{k=1}^{m} P_k(n)s_{(n+k)h+r}\right) + P_{m+1}(n), \ \forall n \in \mathbb{N}. \tag{15}$$

To show the existence of $(m+1)$-tuple of polynomials $(P_1(x), P_2(x), \ldots, P_{m+1}(x)) \in \mathbb{C}[x]^{m+1}$ which satisfies Eq. (15), it is enough to find only one possible tuple satisfying Eq. (15). But, the more intricate problem to solve is "is the tuple $(P_1(x), P_2(x), \ldots, P_{m+1}(x)) \in \mathbb{C}[x]^{m+1}$ which satisfy Eq. (15) unique or not ?". The following theorem answers this question.

Theorem 2. *Let $P(x) \in \mathbb{C}[x]$ be a polynomial and h, r be integers such that h^{th} power of the roots of the characteristic polynomial Eq. (14) are distinct and not equal to 1. Then, the number of possible tuples $(P_1(x), P_2(x), \ldots, P_{m+1}(x)) \in \mathbb{C}[x]^{m+1}$ which satisfy the identity Eq. (15) is one.*

To give the imagination of Theorem 1 and 2, we provide some examples below.

Example 1. Let $m = 2$, $s_{k+2} = s_{k+1} + s_k$, and $s_0 = 0$, $s_1 = 1$. Hence $s_k = F_k$ where $(F_k)_{k\in\mathbb{Z}}$ is Fibonacci sequence with integer indices. One can obtain that

$$\sum_{k=1}^{n} kF_{k-1} = -F_n + (n-1)F_{n+1} + 1, \ \forall n \in \mathbb{N}$$

which confirms Theorem 1 when $P(x) = x$ and $(h, r) = (1, -1)$. By Theorem 2, the only triple $(P_1(x), P_2(x), P_3(x)) \in \mathbb{C}[x]^3$ which satisfies the identity

$$\sum_{k=1}^{n} kF_{k-1} = P_1(n)F_n + P_2(n)F_{n+1} + P_3(n), \ \forall n \in \mathbb{N}$$

is $(P_1(x), P_2(x), P_3(x)) = (-1, x-1, 1)$.

Example 2. Let $m = 2$, $s_{k+2} = s_{k+1} + s_k$, and $s_0 = 2, s_1 = 1$. Hence $s_k = L_k$ where $(L_k)_{k\in\mathbb{Z}}$ is Lucas sequence with integer indices. The following identity is true:

$$\sum_{k=1}^{n} k^2 L_k = (-2n+3)L_{n+1} + (n^2 - 2n + 5)L_{n+2} - 18, \ \forall n \in \mathbb{N}.$$

This is a special case of Theorem 1 when $P(x) = x^2$ and $(h, r) = (1, 0)$. By Theorem 2, the only triple $(P_1(x), P_2(x), P_3(x)) \in \mathbb{C}[x]^3$ which satisfies the identity

$$\sum_{k=1}^{n} k^2 L_k = P_1(n)L_{n+1} + P_2(n)L_{n+2} + P_3(n), \ \forall n \in \mathbb{N}$$

is $(P_1(x), P_2(x), P_3(x)) = (-2x+3, x^2-2x+5, -18)$.

3. Overview of $(s_k)_{k\in\mathbb{Z}}$

Let m roots of $\mathcal{CP}(x) = x^m - a_m x^{m-1} - a_{m-1}x^{m-2} - \ldots - a_2 x - a_1$ be r_1, r_2, ..., $r_m \in \mathbb{C}$ where all of $r_1, r_2, \ldots, r_m$ are distinct and $|r_1| \leq |r_2| \leq \cdots \leq |r_m|$. Since $a_1 \neq 0$, then all of $r_1, r_2, \ldots, r_m$ are non-zero. By all properties in Definition 1, the explicit formula of each s_k is represented as

$$s_k = \Omega_1 r_1^k + \Omega_2 r_2^k + \cdots + \Omega_m r_m^k, \quad \forall k \in \mathbb{Z} \tag{16}$$

where Ω_1, Ω_2, ..., Ω_m are some non-zero constants in $\mathbb{C}$. The values of Ω_1, Ω_2, ..., Ω_m are obtained from system of m equations $s_k = \Omega_1 r_1^k + \Omega_2 r_2^k + \cdots + \Omega_m r_m^k$ $(k = 1, 2, \ldots, m)$. Hence, $(\Omega_1, \Omega_2, \ldots, \Omega_m)$ is the solution of matrix equation

$$\begin{bmatrix} r_1 & r_2 & r_3 & \ldots & r_m \\ r_1^2 & r_2^2 & r_3^2 & \ldots & r_m^2 \\ r_1^3 & r_2^3 & r_3^3 & \ldots & r_m^3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_1^m & r_2^m & r_3^m & \ldots & r_m^m \end{bmatrix} \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \\ \vdots \\ \Omega_m \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ \vdots \\ s_m \end{bmatrix}.$$

By using Cramer rule, for every $k \in \{1, 2, 3, \ldots, m\}$,

$$\Omega_k = \frac{\det(A_k)}{\det \begin{bmatrix} r_1 & r_2 & r_3 & \ldots & r_m \\ r_1^2 & r_2^2 & r_3^2 & \ldots & r_m^2 \\ r_1^3 & r_2^3 & r_3^3 & \ldots & r_m^3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_1^m & r_2^m & r_3^m & \ldots & r_m^m \end{bmatrix}} = \frac{\det(A_k)}{\prod_{i_1=1}^{m} r_{i_1} \times \prod_{1\leq i_1<i_2\leq m}(r_{i_2} - r_{i_1})},$$

where A_k is the $m \times m$ matrix obtained by replacing the k^{th} column of square matrix

$$\begin{bmatrix} r_1 & r_2 & r_3 & \ldots & r_m \\ r_1^2 & r_2^2 & r_3^2 & \ldots & r_m^2 \\ r_1^3 & r_2^3 & r_3^3 & \ldots & r_m^3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_1^m & r_2^m & r_3^m & \ldots & r_m^m \end{bmatrix} \text{ with the column vector } \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ \vdots \\ s_m \end{bmatrix}.$$

Remark 1. The formula Eq. (16) is obviously implied since $\mathcal{CP}(x)$ has m distinct roots. The constants $\Omega_1, \Omega_2, ..., \Omega_m$ must be non-zero because if one of them is equal to zero, then $(s_k)_{k\in\mathbb{Z}}$ will satisfy a homogeneous linear recurrence relations of complex constant coefficients with degree less than m, contradicting to the condition that m is the minimal degree of $(s_k)_{k\in\mathbb{Z}}$.

4. Some Basic Lemmas

In this section, we present some basic lemmas that will be used for showing Theorem 1 and 2.

Lemma 1. *Let A be an infinite subset of $\mathbb{N}$. If $P(x) \in \mathbb{C}[x]$ is a polynomial and $\lim_{\substack{n\to\infty \\ n\in A}} P(n) = 0$, then $P(x)$ is a zero polynomial.*

Proof. If $P(x)$ is not a constant polynomial, then $P(x) = \lambda_u x^u + \lambda_{u-1}x^{u-1} + \cdots + \lambda_1 x + \lambda_0$ for some $u \in \mathbb{N}$ and $\lambda_0, \lambda_1, \ldots, \lambda_{u-1}, \lambda_u \in \mathbb{C}$ with $\lambda_u \neq 0$. We have

$$0 = \lim_{\substack{n\to\infty \\ n\in A}} |P(n)| = \lim_{\substack{n\to\infty \\ n\in A}} |\lambda_u n^u + \lambda_{u-1}n^{u-1} + \cdots + \lambda_1 n + \lambda_0|,$$

but

$$\begin{aligned}\lim_{\substack{n\to\infty \\ n\in A}} |\lambda_u n^u + \lambda_{u-1}n^{u-1} + \cdots + \lambda_1 n + \lambda_0| &= \lim_{\substack{n\to\infty \\ n\in A}} |\lambda_u n^u| \cdot \left|1 + \frac{\lambda_{u-1}}{\lambda_u n} + \frac{\lambda_{u-2}}{\lambda_u n^2} + \cdots + \frac{\lambda_0}{\lambda_u n^u}\right| \\ &= \lim_{\substack{n\to\infty \\ n\in A}} |\lambda_u n^u| \cdot \lim_{\substack{n\to\infty \\ n\in A}} \left|1 + \frac{\lambda_{u-1}}{\lambda_u n} + \frac{\lambda_{u-2}}{\lambda_u n^2} + \cdots + \frac{\lambda_0}{\lambda_u n^u}\right| \\ &= \infty \cdot 1 = \infty,\end{aligned}$$

so it is a contradiction and $P(x)$ can not be non-constant.
If $P(x)$ is a constant, let $P(x) \equiv c$, then the condition $\lim\limits_{\substack{n\to\infty \\ n\in A}} P(n) = 0$ implies that $c = 0$ and $P(x) \equiv 0$.
Hence $P(x)$ is a zero polynomial. □

If $P(x)$ is a polynomial in $\mathbb{C}[x]$ and y is a complex number with $|y| > 1$, then the growth rate of $|y|^n$ is faster than $|P(n)|$ for all big enough positive integers n. It will cause that $P(n)/y^n \to 0$ as positive integer n goes to ∞. This intuition motivates the following lemma:

Lemma 2. *Let A be an infinite subset of $\mathbb{N}$, $P(x)$ be a polynomial in $\mathbb{C}[x]$, and $y_1, y_2 \in \mathbb{C}$ be non-zero with $|y_1| > 1 > |y_2|$. Then,*

1. $\lim\limits_{\substack{n\to\infty \\ n\in A}} \frac{P(n)}{y_1^n} = 0.$
2. $\lim\limits_{\substack{n\to\infty \\ n\in A}} P(n)y_2^n = 0.$

Proof. It suffices to show the first limit. Because if the first limit is true, by setting $y_1 := 1/y_2$ where $y_2 \in \mathbb{C}$ is non-zero with $|y_2| < 1$, it yields the second limit.
Here is proof of the first limit: Let $P(x) = \lambda_u x^u + \lambda_{u-1}x^{u-1} + \cdots + \lambda_1 x + \lambda_0$ for some $u \in \mathbb{N}_0$ and $\lambda_0, \lambda_1, \ldots, \lambda_u \in \mathbb{C}$. For all $n \in A$, we have

$$\begin{aligned}0 \leq \left|\frac{P(n)}{y_1^n}\right| = \frac{|\lambda_u n^u + \lambda_{u-1}n^{u-1} + \cdots + \lambda_1 n + \lambda_0|}{|y_1|^n} &\leq \frac{|\lambda_u| n^u + |\lambda_{u-1}| n^{u-1} + \cdots + |\lambda_1| n + |\lambda_0|}{|y_1|^n} \\ &\leq \frac{M \cdot n^u}{|y_1|^n}\end{aligned}$$

where $M = |\lambda_0| + |\lambda_1| + \cdots + |\lambda_u|$.
Besides that, for all $n \in A$ with $n > \frac{(u+2)!}{(\ln|y_1|)^{u+2}}$, we have

$$0 < \frac{n^{u+1}}{|y_1|^n} = \frac{n^{u+1}}{\sum_{a=0}^{\infty} \frac{(n\ln|y_1|)^a}{a!}} \leq \frac{n^{u+1}}{\frac{(n\ln|y_1|)^{u+2}}{(u+2)!}} = \frac{(u+2)!}{n(\ln|y_1|)^{u+2}} < 1 \implies 0 < \frac{n^u}{|y_1|^n} < \frac{1}{n}.$$

Since $\lim\limits_{\substack{n\to\infty \\ n\in A}} 0 = \lim\limits_{\substack{n\to\infty \\ n\in A}} \frac{1}{n} = 0$, by squeeze theorem, we get $\lim\limits_{\substack{n\to\infty \\ n\in A}} \frac{n^u}{|y_1|^n} = 0$. Then $\lim\limits_{\substack{n\to\infty \\ n\in A}} 0 = \lim\limits_{\substack{n\to\infty \\ n\in A}} \frac{M\cdot n^u}{|y_1|^n} = 0$. By squeeze theorem again, we obtain $\lim\limits_{\substack{n\to\infty \\ n\in A}} \left|\frac{P(n)}{y_1^n}\right| = 0$, then $\lim\limits_{\substack{n\to\infty \\ n\in A}} \frac{P(n)}{y_1^n} = 0$. □

The linear recurrence relation Eq. (13) shows us that every term in $(s_k)_{k\in\mathbb{Z}}$ is representable as a fixed linear combination of m previous terms. Then, the new question arises: "For a non-zero integer ℓ, can $s_{m\ell+q}$ be represented as a linear combination of $s_{(m-1)\ell+q}, s_{(m-2)\ell+q}, \ldots, s_{\ell+q}, s_q$ for every integers q? If yes, are the coefficients of those linear combination fixed over all $q \in \mathbb{Z}$?". The answer is positive, by the constraint that $r_1^\ell, r_2^\ell, \ldots, r_m^\ell$ are distinct.

Definition 2. For every non-zero integers ℓ, let $e_1(\ell), e_2(\ell), \ldots, e_m(\ell)$ be complex numbers defined by

$$e_{m+1-k}(\ell) = \sum_{1\le i_1<i_2\cdots<i_k\le m} (-1)^{k+1}(r_{i_1}r_{i_2}\cdots r_{i_k})^\ell, \quad \forall k \in \{1,2,3,\ldots,m\}.$$

Lemma 3. *Let ℓ be a non-zero integer so that $r_1^\ell, r_2^\ell, \ldots, r_m^\ell$ are distinct, then for all $q \in \mathbb{Z}$:*

$$s_{m\ell+q} = e_m(\ell)s_{(m-1)\ell+q} + e_{m-1}(\ell)s_{(m-2)\ell+q} + \cdots + e_2(\ell)s_{\ell+q} + e_1(\ell)s_q. \tag{17}$$

Proof. Let us consider an arbitrary $k' \in \{0,1,2,\ldots,\ell-1\}$. Let $T_t(k') = s_{k'+t\ell}$ for all $t \in \mathbb{Z}$. Then, for all $t \in \mathbb{Z}$,

$$\begin{aligned} T_t(k') = s_{k'+t\ell} &= L_1 r_1^{k'+t\ell} + L_2 r_2^{k'+t\ell} + \cdots + L_m r m^{k'+t\ell} \\ &= L_1 r_1^{k'}(r_1^\ell)^t + L_2 r_2^{k'}(r_2^\ell)^t + \cdots + L_m r_m^{k'}(r_m^\ell)^t. \end{aligned}$$

Since $r_1^\ell, r_2^\ell, \cdots, r_m^\ell$ are distinct and $L_1 r_1^{k'}, L_2 r_2^{k'}, \cdots, L_m r_m^{k'}$ are non-zero constant, then the sequence $(T_t(k'))_{t\in\mathbb{Z}}$ satisfies a homogeneous linear recurrence relation of degree m with constant coefficients. That is, there are m complex constants $f_1, f_2, \cdots, f_m$ so that $f_1 \neq 0$ and

$$T_{t+m}(k') = f_m T_{t+m-1}(k') + f_{m-1}T_{t+m-2}(k') + \cdots + f_1 T_t(k')$$

for all $t \in \mathbb{Z}$. This recurrence relation has the characteristic polynomial $x^m - f_m x^{m-1} - f_{m-1}x^{m-2} - \cdots - f_2 x - f_1$ and the m roots of this polynomial are $r_1^\ell, r_2^\ell, \cdots, r_m^\ell$. By Vieta's theorem, for all $k \in \{1,2,\cdots,m\}$,

$$f_{m+1-k} = \sum_{1\le i_1<i_2<\cdots<i_k\le m} (-1)^{k+1}(r_{i_1}r_{i_2}\cdots r_{i_k})^\ell = e_{m+1-k}(\ell).$$

Therefore, for all $k' \in \{0,1,2,\ldots,\ell-1\}$ and all $t \in \mathbb{Z}$,

$$s_{k'+(t+m)\ell} = e_m(\ell)s_{k'+(t+m-1)\ell} + e_{m-1}(\ell)s_{k'+(t+m-2)\ell} + \cdots + e_2(\ell)s_{k'+(t+1)\ell} + e_1(\ell)s_{k'+t\ell}.$$

and it is equivalent to

$$s_{m\ell+q} = e_m(\ell)s_{(m-1)\ell+q} + e_{m-1}(\ell)s_{(m-2)\ell+q} + \cdots + e_2(\ell)s_{\ell+q} + e_1(\ell)s_q \quad (\forall q \in \mathbb{Z})$$

as the lemma said. □

The next two lemmas present the properties involving the limit to infinity of a linear combination of exponential functions exp(...).

Lemma 4. *Let q be a positive integer, $c_1, c_2, \ldots, c_q$ be non-zero complex numbers, and $b_1, b_2, \ldots, b_q \in (0, 2\pi)$ be distinct real numbers. Then, the limit*

$$\lim_{\substack{n\to\infty \\ n\in\mathbb{N}}} (c_1\exp(inb_1) + c_2\exp(inb_2) + \cdots + c_q\exp(inb_q))$$

does not exist.

Proof. Assume the contrary that the limit $\lim\limits_{\substack{n\to\infty \\ n\in\mathbb{N}}}(c_1\exp(inb_1)+c_2\exp(inb_2)+\cdots+c_q\exp(inb_q))$ exists, then there is a complex number T such that $\lim\limits_{\substack{n\to\infty \\ n\in\mathbb{N}}}(c_1\exp(inb_1)+c_2\exp(inb_2)+\cdots+c_q\exp(inb_q))=T$. It implies that for all $\alpha\in\mathbb{Z}$, we have

$$\lim_{\substack{n\to\infty \\ n\in\mathbb{N}}}(c_1\exp(i(n+\alpha)b_1)+c_2\exp(i(n+\alpha)b_2)+\cdots+c_q\exp(i(n+\alpha)b_q))=T. \tag{18}$$

By setting $\alpha=0,1,2,\ldots,q-1$ to Eq. (18), we get the limit equation of matrix

$$\lim_{\substack{n\to\infty \\ n\in\mathbb{N}}}\mathcal{M}\begin{bmatrix}c_1e^{inb_1}\\ c_2e^{inb_2}\\ \vdots\\ c_qe^{inb_q}\end{bmatrix}=\begin{bmatrix}T\\ T\\ \vdots\\ T\end{bmatrix} \tag{19}$$

where $\mathcal{M}$ is $q\times q$ square matrix defined by

$$\mathcal{M}=\begin{bmatrix}1 & 1 & \ldots & 1\\ e^{ib_1} & e^{ib_2} & \ldots & e^{ib_q}\\ (e^{ib_1})^2 & (e^{ib_2})^2 & \ldots & (e^{ib_q})^2\\ \vdots & \vdots & \ddots & \vdots\\ (e^{ib_1})^{q-1} & (e^{ib_2})^{q-1} & \ldots & (e^{ib_q})^{q-1}\end{bmatrix}.$$

We observe that $\mathcal{M}$ is invertible since $\det(\mathcal{M})=\prod_{1\le i_1<i_2\le q}(e^{ib_{i_2}}-e^{ib_{i_1}})\neq 0$. Therefore, by equation Eq. (19), we obtain

$$\lim_{\substack{n\to\infty \\ n\in\mathbb{N}}}\begin{bmatrix}c_1e^{inb_1}\\ c_2e^{inb_2}\\ \vdots\\ c_qe^{inb_q}\end{bmatrix}=\mathcal{M}^{-1}\begin{bmatrix}T\\ T\\ \vdots\\ T\end{bmatrix}=\begin{bmatrix}\mathscr{D}_1\\ \mathscr{D}_2\\ \vdots\\ \mathscr{D}_q\end{bmatrix}$$

where $\mathscr{D}_1,\mathscr{D}_2,\ldots,\mathscr{D}_q$ are some complex numbers. Then, for all $k\in\{1,2,...,q\}$, $\lim\limits_{\substack{n\to\infty \\ n\in\mathbb{N}}}e^{inb_k}=\mathscr{D}_k/c_k$. It is an impossibility, because the condition $b_1,b_2,\ldots,b_q\in(0,2\pi)$ should imply that all the limits $\lim\limits_{\substack{n\to\infty \\ n\in\mathbb{N}}}e^{inb_1}$, $\lim\limits_{\substack{n\to\infty \\ n\in\mathbb{N}}}e^{inb_2}$, $\cdots$, $\lim\limits_{\substack{n\to\infty \\ n\in\mathbb{N}}}e^{inb_q}$ do not converge. □

Lemma 5. *Let q be a positive integer, $c_1,c_2,...,c_{q+1}$ be complex numbers, and $b_1,b_2,...,b_q\in(0,2\pi)$ be distinct real numbers. Then,*

$$\lim_{\substack{n\to\infty \\ n\in\mathbb{N}}}(c_1\exp(inb_1)+c_2\exp(inb_2)+\cdots+c_q\exp(inb_q))=c_{q+1}$$

if and only if $c_1=c_2=\cdots=c_{q+1}=0$.

Proof. For the "if" part, it is trivial: If $c_1=c_2=\cdots=c_{q+1}=0$, then the limit equation $\lim\limits_{\substack{n\to\infty \\ n\in\mathbb{N}}}(c_1\exp(inb_1)+c_2\exp(inb_2)+\cdots+c_q\exp(inb_q))=c_{q+1}$ holds.

For the "only if" part: Let be given the limit equation $\lim\limits_{\substack{n\to\infty \\ n\in\mathbb{N}}}(c_1\exp(inb_1)+c_2\exp(inb_2)+\cdots+c_q\exp(inb_q))=c_{q+1}$, then we must show $c_1=c_2=\cdots=c_{q+1}=0$. If one of $c_1,c_2,...,c_q$ is non-zero, then $\{k\in\{1,2,...,q\}:c_k\neq 0\}=\{\tau_1,\tau_2,...,\tau_j\}$ for some $j,\tau_1,\tau_2,...,\tau_j\in\{1,2,...,q\}$ with $\tau_1<\tau_2<\cdots<\tau_j$. It implies

$$\lim_{\substack{n\to\infty \\ n\in\mathbb{N}}}(c_{\tau_1}\exp(inb_{\tau_1})+c_{\tau_2}\exp(inb_{\tau_2})+\cdots+c_{\tau_j}\exp(inb_{\tau_j}))=c_{q+1}. \tag{20}$$

According to Lemma 4, the limit $\lim\limits_{\substack{n\to\infty \\ n\in\mathbb{N}}}(c_{\tau_1}\exp(inb_{\tau_1})+c_{\tau_2}\exp(inb_{\tau_2})+\cdots+c_{\tau_j}\exp(inb_{\tau_j}))$ does not exist and this condition contradicts with Eq. (20). Consequently, all of $c_1,c_2,...,c_q$ must be zero and it implies $c_{q+1}=0$. □

5. Proof of Theorem 1

First, let us define some notations used in this chapter as the following (definition 3 and 4).

Definition 3. For every $(d,\ell)\in\mathbb{N}_0\times(\mathbb{Z}\setminus\{0\})$ so that $r_1^\ell,r_2^\ell,\ldots,r_m^\ell$ are distinct and not equal to 1, let $M(d,\ell)=(x_{i_1,i_2}(\ell))_{i_1,i_2=1}^{d+1}$ be a $(d+1)\times(d+1)$ matrix where $x_{i_1,i_2}(\ell)$ is the i_1^{th}-row and i_2^{th}-column entry of $M(d,\ell)$ defined by

$$x_{i_1,i_2}(\ell)=\begin{cases}\binom{i_2-1}{i_1-1}\sum\limits_{i_3=1}^{m}e_i(\ell)(m+1-i_3)^{i_2-i_1}, & \text{if } i_1<i_2\\ -1+\sum\limits_{i_3=1}^{m}e_{i_3}(\ell), & \text{if } i_1=i_2\\ 0, & \text{if } i_1>i_2.\end{cases}$$

It can be checked that $M(d,\ell)$ is invertible with $det(M(d,\ell))=(-1+\sum_{i_3=1}^{m}e_i(\ell))^{d+1}=(-\prod_{i_3=1}^{m}(1-r_{i_3}^\ell))^{d+1}\neq 0$. Then, the equation

$$M(d,\ell)\mathbf{v}=\begin{pmatrix}\binom{d}{0}m^d & \binom{d}{1}m^{d-1} & \binom{d}{2}m^{d-2} & \cdots & \binom{d}{d}m^0\end{pmatrix}^T$$

has a unique solution in $\mathbf{v}\in\mathbb{C}^{d+1}$. Let this unique solution be denoted by $(\delta_0(\ell)\ \ \delta_1(\ell)\ \ \cdots\ \ \delta_d(\ell))^T$ and hence

$$\begin{pmatrix}\delta_0(\ell)\\ \delta_1(\ell)\\ \delta_2(\ell)\\ \vdots\\ \delta_d(\ell)\end{pmatrix}=(M(d,\ell))^{-1}\begin{pmatrix}\binom{d}{0}m^d\\ \binom{d}{1}m^{d-1}\\ \binom{d}{2}m^{d-2}\\ \vdots\\ \binom{d}{d}m^0\end{pmatrix}. \tag{21}$$

Definition 4. For every $(d,\ell,r)\in\mathbb{N}_0\times(\mathbb{Z}\setminus\{0\})\times\mathbb{Z}$ so that $r_1^\ell,r_2^\ell,\ldots,r_m^\ell$ are distinct and not equal to 1, let the polynomials $P_{1,d,\ell,r}(x),P_{2,d,\ell,r}(x),\ldots,P_{m+1,d,\ell,r}(x)$ be defined as

follows.

$$P_{k,d,\ell,r}(x) = \begin{cases} P_{m,d,\ell,r}(x-m+k) - \sum\limits_{i_3=k+1}^{m} e_{i_3}(\ell) P_{m,d,\ell,r}(x+k+1-i_3), & \text{if } 1 \le k \le m-1 \\ \delta_d(\ell)x^d + \delta_{d-1}(\ell)x^{d-1} + \cdots + \delta_1(\ell)x + \delta_0(\ell), & \text{if } k = m \\ s_{\ell+r} - \sum\limits_{i_3=1}^{m} P_{i_3,d,\ell,r}(1) s_{(i_3+1)\ell+r}, & \text{if } k = m+1. \end{cases}$$

We shall show Lemma 6 as a special case of Theorem 1 when $P(x) = x^d$ with $d \in \mathbb{N}_0$. Hence, Theorem 1 will be automatically true for general polynomial $P(x)$. Before we direct to Lemma 6, let us notice the following remark. This remark explains why we "suddenly" construct the polynomials in Definition 4 and how they are influenced by $\delta_0(\ell), \delta_1(\ell), ..., \delta_d(\ell)$ as defined in Definition 3.

Remark 2. We note that the polynomials introduced in Definition 4 play importantly to show Theorem 1. How can these polynomials be obtained? It is motivated by claiming that the formula Eq. (12) holds true for all $n \in \mathbb{N}$. Then, we have two equations as follows:

$$\sum_{k=1}^{n} P(k) s_{hk+r} = P_1(n) s_{h(n+1)+r} + \cdots + P_m(n) s_{h(n+m)+r} + P_{m+1}(n), \tag{22}$$

$$\sum_{k=1}^{n+1} P(k) s_{hk+r} = P_1(n+1) s_{h(n+2)+r} + \cdots + P_m(n+1) s_{h(n+m+1)+r} + P_{m+1}(n+1). \tag{23}$$

Subtracting the Eq. (22) from (23) yields

$$\begin{aligned}(P_m(n+1) - P_m(n)) + P_m(n+1) s_{h(n+m+1)r} &= (P_1(n) + P(n+1)) s_{h(n+1)+r} \\ &\quad + \sum_{k=2}^{m} (P_k(n) - P_{k-1}(n+1)) s_{h(n+k)+r}. \end{aligned} \tag{24}$$

If we compare the Eq. (24) with the linear recurrence relation Eq. (17) when $(\ell, q) := (h, hn+h+r)$, we expect that $P_{m+1}(x)$ is a constant polynomial and the following system of equations must hold:

$$\begin{cases} P_1(x) + P(x+1) = e_1(h) P_m(x+1) \\ P_k(x) - P_{k-1}(x+1) = e_k(h) P_m(x+1), \quad if \ \ 2 \le k \le m. \end{cases} \tag{25}$$

By Eq. (25), it should imply

$$P_m(x) + P(x+m) = e_1(h) P_m(x+m) + e_2(h) P_m(x+m-1) + \cdots + e_m(h) P_m(x+1). \tag{26}$$

By the given condition that the h^{th} power of every roots of $\mathcal{CP}(x)$ are not equal to 1 (i.e., $r_k^h \neq 1$ for $1 \le k \le m$), we have $-1 + \sum_{i_3=1}^{m} e_i(h) = -\prod_{i_3=1}^{m}(1 - r_{i_3}^h) \neq 0$ and $e_1(h) + e_2(h) + \cdots + e_m(h) \neq 1$. By Eq. (26), the degrees of $P_m(x)$ and $P(x)$ are equal. For the case when $P(x) = x^d$ with $d \in \mathbb{N}_0$, by comparing the equality of coefficients between the left and right hand sides of Eq. (26), it turns out that $\delta_k(h)$ is the coefficient of x^k in $P_m(x)$. Therefore, $\{P_k(x) := P_{k,d,h,r}(x)\}_{k=1}^{m+1}$ is a solution of Eq. (22) for the case that $P(x) = x^d$. However, this step is not allowed as a formal proofing, because at first we assume directly that the formula Eq. (12) is true for all $n \in \mathbb{N}$. Instead, the mathematical induction is the suitable way to apply as we will see later in the proof of Lemma 6.

Lemma 6. *Given $(d,h,r) \in \mathbb{N}_0 \times (\mathbb{Z}\setminus\{0\}) \times \mathbb{Z}$ such that $r_1^h, r_2^h, \ldots, r_m^h$ are distinct and not equal to 1, then*

$$\sum_{k=1}^{n} k^d s_{hk+r} = \Big[\sum_{k=1}^{m} P_{k,d,h,r}(n) s_{(n+k)h+r}\Big] + P_{m+1,d,h,r}(n), \quad \forall n \in \mathbb{N}.$$

Proof. We shall show the above identity by using induction. For all $n \in \mathbb{N}$, let $S(n)$ be the statement that $\sum_{k=1}^{n} k^d s_{hk+r} = \big[\sum_{k=1}^{m} P_{k,d,h,r}(n) s_{(n+k)h+r}\big] + P_{m+1,d,h,r}(n)$. The definition of $P_{m+1,d,h,r}(x)$ implies that $S(1)$ is true. By considering an arbitrary $n_1 \in \mathbb{N}$ which makes $S(n_1)$ true, we then have

$$\sum_{k=1}^{n_1} k^d s_{hk+r} = \Big[\sum_{k=1}^{m} P_{k,d,h,r}(n_1) s_{(n_1+k)h+r}\Big] + P_{m+1,d,h,r}(n_1) \tag{27}$$

By definition 4 and noting that $\sum_{i_3=m+1}^{m} e_{i_3}(h) P_{m,d,h,r}(x+m+1-i_3) = 0$, each $k \in \{2,3,\ldots,m\}$ satisfies

$$P_{k,d,h,r}(x) = P_{m,d,h,r}(x-m+k) - \sum_{i_3=k+1}^{m} e_{i_3}(h) P_{m,d,h,r}(x+k+1-i_3)$$

and

$$P_{k-1,d,h,r}(x+1) = P_{m,d,h,r}(x-m+k) - \sum_{i_3=k}^{m} e_{i_3}(h) P_{m,d,h,r}(x+k+1-i_3),$$

then

$$P_{k,d,h,r}(x) - P_{k-1,d,h,r}(x+1) = e_k(h) P_{m,d,h,r}(x+1) \tag{28}$$

Replacing x in Eq. (28) with $x+m-k$ implies

$$P_{k,d,h,r}(x+m-k) - P_{k-1,d,h,r}(x+m-k+1) = e_k(h) P_{m,d,h,r}(x+m-k+1).$$

Therefore, we obtain

$$\sum_{k=2}^{m} (P_{k,d,h,r}(x+m-k) - P_{k-1,d,h,r}(x+m-k+1)) = \sum_{k=2}^{m} e_k(h) P_{m,d,h,r}(x+m-k+1)$$

$$\implies \quad P_{m,d,h,r}(x) - P_{1,d,h,r}(x+m-1) = \sum_{k=2}^{m} e_k(h) P_{m,d,h,r}(x+m-k+1). \tag{29}$$

Next, we observe that the given matrix equation

$$M(d,h) \begin{bmatrix} \delta_0(h) \\ \delta_1(h) \\ \vdots \\ \delta_d(h) \end{bmatrix} = \begin{bmatrix} \binom{d}{0} m^d \\ \binom{d}{1} m^{d-1} \\ \vdots \\ \binom{d}{d} m^0 \end{bmatrix}$$

is equivalent to the system of $d+1$ equations

$$\sum_{i_1=1}^{m} \sum_{i_2=k}^{d} e_{i_1}(h) \delta_{i_2}(h) \binom{i_2}{k} (m+1-i_1)^{i_2-k} = \binom{d}{k} m^{d-k} + \delta_k(h), \quad k = 0,1,2,\ldots,d. \tag{30}$$

Besides that, for all $k \in \{0, 1, 2, \ldots, d\}$, the coefficients of x^k in the polynomials $(x+m)^d + P_{m,d,h,r}(x)$ and $\sum_{i_1=1}^{m} e_{i_1}(h) P_{m,d,h,r}(x+m+1-i_1)$ are respectively

$$\binom{d}{k} m^{d-k} + \delta_k(h) \quad \text{and} \quad \sum_{i_1=1}^{m} \sum_{i_2=k}^{d} e_{i_1}(h)\delta_{i_2}(h) \binom{i_2}{k} (m+1-i_1)^{i_2-k}.$$

Since the degrees of $(x+m)^d + P_{m,d,h,r}(x)$ and $\sum_{i_3=1}^{m} e_{i_3}(h) P_{m,d,h,r}(x+m+1-i_3)$ are at most d, comparing to Eq. (30) yields

$$(x+m)^d + P_{m,d,h,r}(x) = \sum_{i_3=1}^{m} e_{i_3}(h) P_{m,d,h,r}(x+m+1-i_3). \tag{31}$$

We then subtract the Eq. (29) from (31) to obtain

$$(x+m)^d + P_{1,d,h,r}(x+m-1) = e_1(h) P_{m,d,h,r}(x+m). \tag{32}$$

Replacing x in Eq. (32) with $x-m+1$ yields the equation

$$(x+1)^d + P_{1,d,h,r}(x) = e_1(h) P_{m,d,h,r}(x+1). \tag{33}$$

Combining Eqs. (28) and (33) implies that the ratio of polynomials $P_{m,d,h,r}(x+1) : (P_{m,d,h,r}(x) - P_{m-1,d,h,r}(x+1)) : (P_{m-1,d,h,r}(x) - P_{m-2,d,h,r}(x+1)) : \cdots\cdots : (P_{2,d,h,r}(x) - P_{1,d,h,r}(x+1)) : ((x+1)^d + P_{1,d,h,r}(x))$ is equal to $1 : e_m(h) : e_{m-1}(h) : \cdots : e_2(h) : e_1(h)$. By combining with the recurrence relation Eq. (2) when $\ell := h$ and $q := (n_1+1)h + r$, we will get the equation

$$\begin{aligned} P_{m,d,h,r}(x+1) s_{(n_1+m+1)h+r} = {} & ((x+1)^d + P_{1,d,h,r}(x)) s_{(n_1+1)h+r} \\ & + \sum_{k=2}^{m} (P_{k,d,h,r}(x) - P_{k-1,d,h,r}(x+1)) s_{(n_1+k)h+r} \end{aligned}$$

which is equivalent to

$$\begin{aligned} (x+1)^d s_{(n_1+1)h+r} = {} & -P_{1,d,h,r}(x) s_{(n_1+1)h+r} + P_{m,d,h,r}(x+1) s_{n_1+m+1} \\ & + \sum_{k=2}^{m} (P_{k-1,d,h,r}(x+1) - P_{k,d,h,r}(x)) s_{(n_1+k)h+r}. \end{aligned} \tag{34}$$

Setting $x := n_1$ to Eq. (34) yields the equation

$$\begin{aligned} (n_1+1)^d s_{(n_1+1)h+r} = {} & -P_{1,d,h,r}(n_1) s_{(n_1+1)h+r} + P_{m,d,h,r}(n_1+1) s_{n_1+m+1} \\ & + \sum_{k=2}^{m} (P_{k-1,d,h,r}(n_1+1) - P_{k,d,h,r}(n_1)) s_{(n_1+k)h+r}. \end{aligned} \tag{35}$$

By summing up the equations Eqs. (27) and (35), we obtain

$$\sum_{k=1}^{n_1+1} k^d s_{hk+r} = \Big[\sum_{k=1}^{m} P_{k,d,h,r}(n_1+1) s_{(n_1+k+1)h+r} \Big] + P_{m+1,d,h,r}(n_1+1).$$

Then, $S(n_1+1)$ is true. By induction, $S(n)$ is true for all $n \in \mathbb{N}$, then the proof (of Lemma 6) completes. □

By considering a polynomial $P(x) \in \mathbb{C}[x]$, we have $P(x) = \sum_{d\in\mathbb{N}_0} Coef(d, P(x))x^d$ where $Coef(d, P(x))$ is the coefficient of x^d in $P(x)$. Then, a suitable example of polynomials $P_1(x), P_2(x), \ldots, P_{m+1}(x)$ in $\mathbb{C}[x]$ which satisfy the identity Eq. (15) is

$$P_k(x) := \mathscr{P}_k(x) \stackrel{\text{def}}{=} \sum_{d\in\mathbb{N}_0} Coef(d, P(x))P_{k,d,h,r}(x), \quad \forall k \in \{1, 2, \ldots, m+1\}.$$

Hence, the proof (of Theorem 1) completes.

6. Proof of Theorem 2

Let us consider a polynomial $P(x)$ in $\mathbb{C}[x]$. Next, we suppose that $\mathscr{S}$ is the set of $(m+1)$-tuple of polynomials $(\gamma_1(x), \gamma_2(x), \ldots, \gamma_{m+1}(x)) \in \mathbb{C}[x]^{m+1}$ which satisfy

$$\gamma_1(n)s_{(n+1)h+r} + \gamma_2(n)s_{(n+2)h+r} + \cdots + \gamma_m(n)s_{(n+m)h+r} + \gamma_{m+1}(n) = 0 \quad (\forall n \in \mathbb{N}) \tag{36}$$

and $\mathscr{S}(P(x))$ is the set of all possible $(P_1(x), P_2(x), \ldots, P_{m+1}(x))$ in $\mathbb{C}[x]^{m+1}$ which satisfy the identity Eq. (15).

Then, we have an important lemma as follows.

Lemma 7. $|\mathscr{S}(P(x))| = |\mathscr{S}|$.

Proof. As we have said in the last paragraph of Section 5, $(\mathscr{P}_1(x), \mathscr{P}_2(x), \ldots, \mathscr{P}_{m+1}(x)) \in \mathscr{S}(P(x))$. Let us observe that for every $(P_1(x), P_2(x), \ldots, P_{m+1}(x))$ in $\mathscr{S}(P(x))$, then $(P_1(x) - \mathscr{P}_1(x), P_2(x) - \mathscr{P}_2(x), \ldots, P_{m+1}(x) - \mathscr{P}_{m+1}(x)) \in \mathscr{S}$. So, we can construct a mapping $T : \mathscr{S}(P(x)) \to \mathscr{S}$ defined by

$$T(P_1(x), P_2(x), \ldots, P_{m+1}(x)) = (P_1(x) - \mathscr{P}_1(x), P_2(x) - \mathscr{P}_2(x), \ldots, P_{m+1}(x) - \mathscr{P}_{m+1}(x))$$

for all $(P_1(x), P_2(x), \ldots, P_{m+1}(x)) \in \mathscr{S}(P(x))$.
T is injective because if

$$(P_1^{(1)}(x), P_2^{(1)}(x), \cdots, P_{m+1}^{(1)}(x)) \in \mathscr{S}(P(x)) \text{ and } (P_1^{(2)}(x), P_2^{(2)}(x), \cdots, P_{m+1}^{(1)}(x)) \in \mathscr{S}(P(x))$$

satisfy the condition

$$T(P_1^{(1)}(x), P_2^{(1)}(x), \cdots, P_{m+1}^{(1)}(x)) = T(P_1^{(2)}(x), P_2^{(2)}(x), \cdots, P_{m+1}^{(1)}(x)),$$

then for all $k \in \{1, 2, ..., m+1\}$,

$$P_k^{(1)}(x) - \mathscr{P}_k(x) = P_k^{(2)}(x) - \mathscr{P}_k(x) \quad \Longrightarrow \quad P_k^{(1)}(x) = P_k^{(2)}(x),$$

hence $(P_1^{(1)}(x), P_2^{(1)}(x), \cdots, P_{m+1}^{(1)}(x)) = (P_1^{(2)}(x), P_2^{(2)}(x), \cdots, P_{m+1}^{(2)}(x))$.
Besides that, T is surjective because for every $(P_1^{(3)}(x), P_2^{(3)}(x), \cdots, P_{m+1}^{(3)}(x)) \in \mathscr{S}$, there exists $(P_1^{(3)}(x) + \mathscr{P}_1(x), P_2^{(3)}(x) + \mathscr{P}_2(x), \cdots, P_{m+1}^{(3)}(x) + \mathscr{P}_{m+1}(x)) \in \mathscr{S}(P(x))$ such that

$$T(P_1^{(3)}(x)+\mathscr{P}_1(x), P_2^{(3)}(x)+\mathscr{P}_2(x), \cdots, P_{m+1}^{(3)}(x)+\mathscr{P}_{m+1}(x)) = (P_1^{(3)}(x), P_2^{(3)}(x), \cdots, P_{m+1}^{(3)}(x)).$$

Thus, T is a bijection and the result follows. □

By replacing each of $s_{(n+1)h+r}, s_{(n+2)h+r}, \ldots, s_{(n+m)h+r}$ in Eq. (36) with the formula Eq. (16), the identity Eq. (36) is equivalent to

$$\gamma_{m+1}(n) + \sum_{i_1=1}^{m}\sum_{i_2=1}^{m} \gamma_{i_1}(n)\Omega_{i_2} r_{i_2}^{(n+i_1)h+r} = 0 \quad (\forall n \in \mathbb{N}). \tag{37}$$

By defining the polynomials $Q_1(x), Q_2(x), \ldots, Q_m(x)$ with

$$Q_{i_2}(x) = \Omega_{i_2} r_{i_2}^{h+r} \sum_{i_1=1}^{m} \gamma_{i_1}(x) r_{i_2}^{(i_1-1)h}, \quad \forall i_2 = 1, 2, ..., m, \tag{38}$$

the identity Eq. (37) is equivalent to

$$Q_1(n) r_1^{hn} + Q_2(n) r_2^{hn} + \cdots + Q_m(n) r_m^{hn} + \gamma_{m+1}(n) = 0, \ \forall n \in \mathbb{N}. \tag{39}$$

We shall show that all of $Q_1(x), Q_2(x), \ldots, Q_m(x)$ are zero polynomials. Assume the contrary that there exists a polynomial among $Q_1(x), Q_2(x), \ldots, Q_m(x)$ which is not a zero polynomial. Let $\{t_1, t_2, \cdots, t_p\}$, with $p \in \{1, 2, ..., m\}$ and $t_1 < t_2 < \cdots < t_p$, be the set of all indices i_2 in $\{1, 2, ..., m\}$ for which $Q_{i_2}(x)$ is not zero polynomial. Then, the identity Eq. (39) becomes

$$\gamma_{m+1}(n) + \sum_{k=1}^{p} Q_{t_k}(n) r_{t_k}^{hn} = 0, \ \forall n \in \mathbb{N}. \tag{40}$$

Suppose that $\{|r_{t_k}^h| : k \in \{1, 2, ..., p\}\} = \{\rho_1, \rho_2, ..., \rho_w\}$ for some $w \in \mathbb{N}$ with $w \leq p$ and $0 < \rho_1 < \rho_2 < \cdots < \rho_w$. Therefore, we can construct a function $\kappa : \{0, 1, 2, ..., w\} \to \{0, 1, 2, ..., p\}$ in such a way that $0 = \kappa(0) < \kappa(1) < \cdots < \kappa(w) = p$ and for every $i_1 \in \{1, 2, ..., w\}$,

$$\rho_{i_1} = |r_{t_{\kappa(i_1-1)+i_2}}^h|, \quad \forall i_2 \in \{1, 2, ..., \kappa(i_1) - \kappa(i_1 - 1)\}.$$

Without loss of generality, for every $i_1 \in \{1, 2, ..., w\}$ and $i_2 \in \{1, 2, ..., \kappa(i_1) - \kappa(i_1 - 1)\}$, let $r_{t_{\kappa(i_1-1)+i_2}}^h = \rho_{i_1} \exp(i\theta_{i_1,i_2})$ where $0 \leq \theta_{i_1,1} < \theta_{i_1,2} < \cdots < \theta_{i_1,\kappa(i_1)-\kappa(i_1-1)} < 2\pi$,and let the polynomials $Q_{i_1,i_2}(x) = Q_{t_{\kappa(i_1-1)+i_2}}(x)$. Then, the identity Eq. (40) is equivalent to

$$\gamma_{m+1}(n) + \sum_{i_1=1}^{w} \rho_{i_1}^n V_n(i_1) = 0, \ \ \forall n \in \mathbb{N}, \tag{41}$$

where $V_n(i_1) = \sum_{i_2=1}^{\kappa(i_1)-\kappa(i_1-1)} Q_{i_1,i_2}(n) \exp(in\theta_{i_1,i_2})$ for every $(i_1, n) \in \{1, 2, ..., w\} \times \mathbb{N}$. Let us consider the identity Eq. (41) into 3 cases: $\rho_w < 1$, $\rho_w > 1$, and $\rho_w = 1$.

CASE $\rho_w < 1$:

We note that $\rho_1, \rho_2, ..., \rho_w < 1$. By Lemma 2, for each $i_1 \in \{1, 2, ..., w\}$ and $i_2 \in \{1, 2, ..., \kappa(i_1) - \kappa(i_1 - 1)\}$, we have

$$\lim_{\substack{n\to\infty \\ n\in\mathbb{N}}} |\rho_{i_1}^n Q_{i_1,i_2}(n) \exp(in\theta_{i_1,i_2})| = \lim_{\substack{n\to\infty \\ n\in\mathbb{N}}} |\rho_{i_1}^n Q_{i_1,i_2}(n)| = 0 \implies \lim_{\substack{n\to\infty \\ n\in\mathbb{N}}} \rho_{i_1}^n Q_{i_1,i_2}(n) \exp(in\theta_{i_1,i_2}) = 0.$$

Limiting $n \to \infty$ to Eq. (41) yields $\lim\limits_{\substack{n\to\infty \\ n\in\mathbb{N}}} \gamma_{m+1}(n) = 0$. By Lemma 1, $\gamma_{m+1}(x)$ is a zero polynomial. Then, the identity Eq. (41) is equivalent to

$$\sum_{i_1=1}^{w} \frac{\rho_{i_1}^n}{\rho_w^n} V_n(i_1) = 0, \quad \forall n \in \mathbb{N}. \tag{42}$$

If $w = 1$, then the identity Eq. (42) yields $V_n(w) = 0$ for all $n \in \mathbb{N}$, so we can write $\lim\limits_{\substack{n\to\infty \\ n\in\mathbb{N}}} V_n(w) = 0$.

If $w > 1$, then for all $i_1 \in \{1, 2, ..., w - 1\}$, $\frac{\rho_{i_1}}{\rho_w} < 1$. By triangle inequality and Lemma 2, each i_1 in $\{1, 2, ..., w - 1\}$ satisfies

$$\lim_{\substack{n\to\infty \\ n\in\mathbb{N}}} \left| \frac{\rho_{i_1}^n}{\rho_w^n} V_n(i_1) \right| \leq \lim_{\substack{n\to\infty \\ n\in\mathbb{N}}} \sum_{i_2=1}^{\kappa(i_1)-\kappa(i_1-1)} \left| \frac{\rho_{i_1}^n}{\rho_w^n} Q_{i_1,i_2}(n) \right| = 0,$$

hence $\lim\limits_{\substack{n\to\infty\\ n\in\mathbb{N}}} \frac{\rho_{i_1}^n}{\rho_w^n} V_n(i_1) = 0$. Limiting $n \to \infty$ to Eq. (42) implies $\lim\limits_{\substack{n\to\infty\\ n\in\mathbb{N}}} V_n(w) = 0$.
Now, we focus on the limit $\lim\limits_{\substack{n\to\infty\\ n\in\mathbb{N}}} V_n(w) = 0$. It is clear that there is a $w' \in \{1, 2, ..., \kappa(w) - \kappa(w-1)\}$ in such a way that $Q_{w,w'}(x)$ has the highest degree among the polynomials $Q_{w,1}(x)$, $Q_{w,2}(x)$,...,$Q_{w,\kappa(w)-\kappa(w-1)}(x)$. Then, $\lim\limits_{\substack{n\to\infty\\ n\in\mathbb{N}}} \frac{1}{Q_{w,w'}(n)}$ equals zero or a complex constant. It implies the following limit.

$$\lim_{\substack{n\to\infty\\ n\in\mathbb{N}}} \sum_{i_2=1}^{\kappa(w)-\kappa(w-1)} \frac{Q_{w,i_2}(n)}{Q_{w,w'}(n)} \exp(in\theta_{w,i_2}) = \lim_{\substack{n\to\infty\\ n\in\mathbb{N}}} \frac{V_n(w)}{Q_{w,w'}(n)} = 0 \tag{43}$$

For all $i_2 \in \{1, 2, ..., \kappa(w) - \kappa(w-1)\}$, let us define $\mathcal{J}_{i_2} \overset{\text{def}}{=} \lim\limits_{\substack{n\to\infty\\ n\in\mathbb{N}}} \frac{Q_{w,i_2}(n)}{Q_{w,w'}(n)}$. Then, each i_2 in $\{1, 2, ..., \kappa(w) - \kappa(w-1)\}$ satisfies the limit

$$\lim_{\substack{n\to\infty\\ n\in\mathbb{N}}} \left(\mathcal{J}_{i_2} - \frac{Q_{w,i_2}(n)}{Q_{w,w'}(n)}\right) \exp(in\theta_{w,i_2}) = 0,$$

which derives from

$$\lim_{\substack{n\to\infty\\ n\in\mathbb{N}}} \left| \left(\mathcal{J}_{i_2} - \frac{Q_{w,i_2}(n)}{Q_{w,w'}(n)}\right) \exp(in\theta_{w,i_2}) \right| = \lim_{\substack{n\to\infty\\ n\in\mathbb{N}}} \left| \mathcal{J}_{i_2} - \frac{Q_{w,i_2}(n)}{Q_{w,w'}(n)} \right| = |\mathcal{J}_{i_2} - \mathcal{J}_{i_2}| = 0.$$

Consequently,

$$\lim_{\substack{n\to\infty\\ n\in\mathbb{N}}} \sum_{i_2=1}^{\kappa(w)-\kappa(w-1)} \left(\mathcal{J}_{i_2} - \frac{Q_{w,i_2}(n)}{Q_{w,w'}(n)}\right) \exp(in\theta_{w,i_2}) = 0. \tag{44}$$

Summing up the limit equation Eqs. (43) and (44) yields

$$\lim_{\substack{n\to\infty\\ n\in\mathbb{N}}} \left(\mathcal{J}_1 \exp(in\theta_{w,1}) + \cdots + \mathcal{J}_{\kappa(w)-\kappa(w-1)} \exp(in\theta_{w,\kappa(w)-\kappa(w-1)})\right) = 0 \tag{45}$$

If $\kappa(w) - \kappa(w-1) = 1$, we have $w' = 1$ and

$$\lim_{\substack{n\to\infty\\ n\in\mathbb{N}}} |\mathcal{J}_1| = \lim_{\substack{n\to\infty\\ n\in\mathbb{N}}} |\mathcal{J}_1 \exp(in\theta_{w,1})| = 0 \implies \mathcal{J}_1 = 0.$$

While by definition, $\mathcal{J}_1 = \lim\limits_{\substack{n\to\infty\\ n\in\mathbb{N}}} \frac{Q_{w,1}(n)}{Q_{w,1}(n)} = 1$, a contradiction.
If $\kappa(w) - \kappa(w-1) > 1$, by Eq. (45) and using Lemma 5, we must have $\mathcal{J}_1 = \mathcal{J}_2 = \cdots = \mathcal{J}_{\kappa(w)-\kappa(w-1)} = 0$. It also contradicts to the fact $\mathcal{J}_{w'} = \lim\limits_{\substack{n\to\infty\\ n\in\mathbb{N}}} \frac{Q_{w,w'}(n)}{Q_{w,w'}(n)} = 1$.
CASE $\rho_w > 1$:
The identity Eq. (41) is equivalent to

$$\frac{\gamma_{m+1}(n)}{\rho_w^n} + \sum_{i_1=1}^{w} \frac{\rho_{i_1}^n}{\rho_w^n} V_n(i_1) = 0, \quad \forall n \in \mathbb{N}. \tag{46}$$

By Lemma 2, $\lim\limits_{\substack{n\to\infty\\ n\in\mathbb{N}}} \frac{\gamma_{m+1}(n)}{\rho_w^n} = 0$.
If $w = 1$, by limiting $n \to \infty$ to Eq. (46), we get $\lim\limits_{\substack{n\to\infty\\ n\in\mathbb{N}}} V_n(w) = 0$.
If $w > 1$, then $\frac{\rho_{i_2}}{\rho_w} < 1$ for all $i_2 \in \{1, 2, ..., w-1\}$. By triangle inequality and Lemma 2, each i_1 in $\{1, 2, ..., w-1\}$ satisfies

$$\lim_{\substack{n\to\infty\\ n\in\mathbb{N}}} \left| \frac{\rho_{i_1}^n}{\rho_w^n} V_n(i_1) \right| \leq \lim_{\substack{n\to\infty\\ n\in\mathbb{N}}} \sum_{i_2=1}^{\kappa(i_1)-\kappa(i_1-1)} \left| \frac{\rho_{i_1}^n}{\rho_w^n} Q_{i_1,i_2}(n) \right| = 0,$$

then $\lim\limits_{\substack{n\to\infty\\ n\in\mathbb{N}}} \frac{\rho_{i_1}^n}{\rho_w^n} V_n(i_1) = 0$. By limiting $n \to \infty$ to Eq. (46), we obtain $\lim\limits_{\substack{n\to\infty\\ n\in\mathbb{N}}} V_n(w) = 0$.
The implication of the limit $\lim\limits_{\substack{n\to\infty\\ n\in\mathbb{N}}} V_n(w) = 0$ is as similar as our step in the case when $\rho_w < 1$. There will be a contradiction.
CASE $\rho_w = 1$:
If $w = 1$, the identity Eq. (41) is equivalent to $V_n(w) + \gamma_{m+1}(n) = 0$, $\forall n \in \mathbb{N}$, hence we get the limit $\lim\limits_{\substack{n\to\infty\\ n\in\mathbb{N}}} (V_n(w) + \gamma_{m+1}(n)) = 0$.
If $w > 1$, we have $\rho_{i_1} < 1$ for all $i_1 \in \{1, 2, ..., w-1\}$. By triangle inequality and Lemma 2, for all $i_1 \in \{1, 2, ..., w-1\}$,

$$\lim_{\substack{n\to\infty\\ n\in\mathbb{N}}} |\rho_{i_1}^n V_n(i_1)| \leq \lim_{\substack{n\to\infty\\ n\in\mathbb{N}}} \sum_{i_2=1}^{\kappa(i_1)-\kappa(i_1-1)} |\rho_{i_1}^n Q_{i_1,i_2}(n)| = 0,$$

hence $\lim\limits_{\substack{n\to\infty\\ n\in\mathbb{N}}} \rho_{i_1}^n V_n(i_1) = 0$. Therefore, by limiting $n \to \infty$ to Eq. (41), we get $\lim\limits_{\substack{n\to\infty\\ n\in\mathbb{N}}} (V_n(w) + \gamma_{m+1}(n)) = 0$.
Suppose $\mathcal{Q}(x)$ is the polynomial among $Q_{w,1}(x), Q_{w,2}(x), \ldots, Q_{w,\kappa(w)-\kappa(w-1)}(x), \gamma_{m+1}(x)$ which has the highest degree, so $\lim\limits_{\substack{n\to\infty\\ n\in\mathbb{N}}} \frac{1}{\mathcal{Q}(n)}$ equals zero or a complex constant. We obtain

$$\lim_{\substack{n\to\infty\\ n\in\mathbb{N}}} \left(\frac{\gamma_{m+1}(n)}{\mathcal{Q}(n)} + \sum_{i_2=1}^{\kappa(w)-\kappa(w-1)} \frac{Q_{w,i_2}(n)}{\mathcal{Q}(n)} \exp(in\theta_{w,i_2}) \right) = \lim_{\substack{n\to\infty\\ n\in\mathbb{N}}} \frac{V_n(w) + \gamma_{m+1}(n)}{\mathcal{Q}(n)} = 0. \tag{47}$$

Next, suppose $\mathscr{J}_0 \overset{\text{def}}{=} \lim\limits_{\substack{n\to\infty\\ n\in\mathbb{N}}} \frac{\gamma_{m+1}(n)}{\mathcal{Q}(n)}$ and $\mathscr{J}_{i_2} \overset{\text{def}}{=} \lim\limits_{\substack{n\to\infty\\ n\in\mathbb{N}}} \frac{Q_{w,i_2}(n)}{\mathcal{Q}(n)}$ for every $i_2 \in \{1, 2, ..., \kappa(w) - \kappa(w-1)\}$. Let us observe that, for all $i_2 \in \{1, 2, ..., \kappa(w) - \kappa(w-1)\}$,

$$\lim_{\substack{n\to\infty\\ n\in\mathbb{N}}} \left| \left(\mathscr{J}_{i_2} - \frac{Q_{w,i_2}(n)}{\mathcal{Q}(n)} \right) \exp(in\theta_{w,i_2}) \right| = \lim_{\substack{n\to\infty\\ n\in\mathbb{N}}} \left| \mathscr{J}_{i_2} - \frac{Q_{w,i_2}(n)}{\mathcal{Q}(n)} \right| = |\mathscr{J}_{i_2} - \mathscr{J}_{i_2}| = 0$$

and therefore,

$$\lim_{\substack{n\to\infty\\ n\in\mathbb{N}}} \left(\mathscr{J}_{i_2} - \frac{Q_{w,i_2}(n)}{\mathcal{Q}(n)} \right) \exp(in\theta_{w,i_2}) = 0.$$

So, we get the following limit of summation:

$$\lim_{\substack{n\to\infty\\ n\in\mathbb{N}}} \sum_{i_2=1}^{\kappa(w)-\kappa(w-1)} \left(\mathscr{J}_{i_2} - \frac{Q_{w,i_2}(n)}{\mathcal{Q}(n)} \right) \exp(in\theta_{w,i_2}) = 0. \tag{48}$$

Summing up the limit Eqs. (47) and (48) implies the new limit equation

$$\lim_{\substack{n\to\infty\\ n\in\mathbb{N}}} \left(\mathscr{J}_0 + \sum_{i_2=1}^{\kappa(w)-\kappa(w-1)} \mathscr{J}_{i_2} \exp(in\theta_{w,i_2}) \right) = 0. \tag{49}$$

Since $r_1^h, r_2^h, ..., r_m^h$ are not equal to 1 and $\rho_w = 1$, then we must have $0 < \theta_{w,1} < \theta_{w,2} < \cdots < \theta_{w,\kappa(w)-\kappa(w-1)} < 2\pi$. By applying Lemma 5 to Eq. (49), we obtain $\mathscr{J}_0 = \mathscr{J}_1 = \cdots = \mathscr{J}_{\kappa(w)-\kappa(w-1)} = 0$. It is clearly a contradiction because one of $\mathscr{J}_0, \mathscr{J}_1, ..., \mathscr{J}_{\kappa(w)-\kappa(w-1)}$ should be equal to 1.

Finally, all of $Q_1(x), Q_2(x), ..., Q_m(x)$ are zero polynomials. By Eq. (39) and Lemma 1, $\gamma_{m+1}(x)$ is a zero polynomial. By Eq. (38), we obtain the following matrix equation:

$$\begin{bmatrix} 1 & r_1^h & r_1^{2h} & \dots & r_1^{(m-1)h} \\ 1 & r_2^h & r_2^{2h} & \dots & r_2^{(m-1)h} \\ 1 & r_3^h & r_3^{2h} & \dots & r_3^{(m-1)h} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & r_m^h & r_m^{2h} & \dots & r_m^{(m-1)h} \end{bmatrix} \begin{bmatrix} \gamma_1(x) \\ \gamma_2(x) \\ \gamma_3(x) \\ \vdots \\ \gamma_m(x) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{50}$$

The matrix

$$\begin{bmatrix} 1 & r_1^h & r_1^{2h} & \dots & r_1^{(m-1)h} \\ 1 & r_2^h & r_2^{2h} & \dots & r_2^{(m-1)h} \\ 1 & r_3^h & r_3^{2h} & \dots & r_3^{(m-1)h} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & r_m^h & r_m^{2h} & \dots & r_m^{(m-1)h} \end{bmatrix}$$

is invertible, because its determinant is $\prod_{1\le i_1<i_2\le m}(r_{i_2}^h - r_{i_1}^h) \neq 0$. Therefore, Eq. (50) implies that all of $\gamma_1(x)$, $\gamma_2(x)$, ..., $\gamma_m(x)$ are zero polynomials. In conclusion, $\mathscr{S}$ has only one element. By Lemma 7, $\mathscr{S}(P(x))$ has one element too. It means that there is uniquely $(m+1)$-tuple $(P_1(x), P_2(x), \dots, P_{m+1}(x)) \in \mathbb{C}[x]^{m+1}$ which satisfies the identity

$$\sum_{k=1}^{n} P(k)s_{hk+r} = \left(\sum_{k=1}^{m} P_k(n)s_{(n+k)h+r}\right) + P_{m+1}(n),\ \forall n \in \mathbb{N},$$

and the proof completes.

7. Some Examples

We provide examples of numerical computation of Ledin-Brousseau sum in two perspectives: computing directly and applying Lemma 6. Here we consider Example 1 and 2 as the samples.

Example 3. Let $m = 2$, $s_{k+2} = s_{k+1} + s_k$, and $s_0 = 0$, $s_1 = 1$. Hence $s_k = F_k$ where $(F_k)_{k\in\mathbb{Z}}$ is Fibonacci sequence with integer indices. Let $n \in \mathbb{N}$ be arbitrary. A well-known identity[a] involving Fibonacci numbers states that $F_1 + F_2 + \cdots + F_n = F_{n+2} - 1$. If we compute the sum $\sum_{k=1}^{n} kF_{k-1}$ directly, we can obtain as follows.

$$\begin{aligned} \sum_{k=1}^{n} kF_{k-1} = (n+1)\sum_{k=0}^{n-1} F_k - \sum_{k=0}^{n-1}\sum_{j=0}^{k} F_j &= (n+1)(F_{n+1}-1) - \sum_{k=0}^{n-1}(F_{k+2}-1) \\ &= (n+1)(F_{n+1}-1) - (F_{n+3} - F_1 - 1 - n) \\ &= (n+1)F_{n+1} - (n+1) - (2F_{n+1} + F_n - n - 2) \\ &= -F_n + (n-1)F_{n+1} + 1. \end{aligned}$$

If we apply Lemma 6, we have

$$\sum_{k=1}^{n} kF_{k-1} = \mathfrak{P}_1(n)F_n + \mathfrak{P}_2(n)F_{n+1} + \mathfrak{P}_3(n), \tag{51}$$

where $\mathfrak{P}_1(x), \mathfrak{P}_2(x), \mathfrak{P}_3(x)$ are polynomials defined by

$$\begin{cases} \mathfrak{P}_1(x) = \mathfrak{P}_2(x-1) - \mathfrak{P}_2(x), \\ \mathfrak{P}_2(x) = \mathfrak{h}_1 x + \mathfrak{h}_0, \\ \mathfrak{P}_3(x) = F_0 - (\mathfrak{P}_1(1)F_1 + \mathfrak{P}_2(1)F_2), \end{cases} \tag{52}$$

and $\mathfrak{h}_0$ and $\mathfrak{h}_1$ are determined by

$$\begin{pmatrix}\mathfrak{h}_0\\ \mathfrak{h}_1\end{pmatrix}=\begin{pmatrix}1 & 3\\ 0 & 1\end{pmatrix}^{-1}\begin{pmatrix}2\\ 1\end{pmatrix}=\begin{pmatrix}-1\\ 1\end{pmatrix}.$$

Setting $(\mathfrak{h}_0,\mathfrak{h}_1)=(-1,1)$ to Eq. (52) implies $\mathfrak{P}_2(x)=x-1$, $\mathfrak{P}_1(x)=(x-2)-(x-1)=-1$, and $\mathfrak{P}_3(x)=F_0-(-F_1+0)=1$. Hence, by Eq. (51), we get the explicit formula

$$\sum_{k=1}^{n}kF_{k-1}=-F_n+(n-1)F_{n+1}+1.$$

[a] see [15], table 1.1, page 19, identity (15).

Example 4. Let $m=2$, $s_{k+2}=s_{k+1}+s_k$, and $s_0=2, s_1=1$. Hence $s_k=L_k$ where $(L_k)_{k\in\mathbb{Z}}$ is Lucas sequence with integer indices. Let $n\in\mathbb{N}$ be arbitrary. It has been known[a] that $L_1+L_2+....+L_n=L_{n+2}-3$, hence it implies

$$\begin{aligned}\sum_{k=1}^{n}kL_k=(n+1)\sum_{k=1}^{n}L_k-\sum_{k=1}^{n}\sum_{j=1}^{k}L_j&=(n+1)(L_{n+2}-3)-\sum_{k=1}^{n}(L_{k+2}-3)\\&=(n+1)(L_{n+2}-3)-(L_{n+4}-3-L_1-L_2-3n)\\&=(n+1)L_{n+2}-L_{n+4}+4\\&=(n+1)L_{n+2}-(L_{n+1}+2L_{n+2})+4\\&=-L_{n+1}+(n-1)L_{n+2}+4\end{aligned}$$

and then

$$\begin{aligned}\sum_{k=1}^{n}k^2L_k&=(n+1)\sum_{k=1}^{n}kL_k-\sum_{k=1}^{n}(n+1-k)kL_k\\&=(n+1)\sum_{k=1}^{n}kL_k-\sum_{k=1}^{n}\sum_{j=1}^{k}jL_j\\&=(n+1)\sum_{k=1}^{n}kL_k-\sum_{k=1}^{n}(-L_{k+1}+(k-1)L_{k+2}+4)\\&=(n+1)\sum_{k=1}^{n}kL_k+\sum_{k=2}^{n+1}L_k-\sum_{k=3}^{n+2}kL_k+3\sum_{k=3}^{n+2}L_k-4n\\&=(n+1)(-L_{n+1}+(n-1)L_{n+2}+4)+(L_{n+3}-3-L_1)\\&\qquad-(-L_{n+3}+(n+1)L_{n+4}+4-L_1-2L_2)+3(L_{n+4}-3-L_1-L_2)-4n\\&=-(n+1)L_{n+1}+(n^2-1)L_{n+2}+2L_{n+3}-(n-2)L_{n+4}-18\\&=-(n+1)L_{n+1}+(n^2-1)L_{n+2}+2(L_{n+1}+L_{n+2})-(n-2)(L_{n+1}+2L_{n+2})-18\\&=(-2n+3)L_{n+1}+(n^2-2n+5)L_{n+2}-18.\end{aligned}$$

Next, let us apply Lemma 6 to compute $\sum_{k=1}^{n}k^2L_k$. We have the identity of the form

$$\sum_{k=1}^{n}k^2L_k=\mathfrak{Q}_1(n)L_{n+1}+\mathfrak{Q}_2(n)L_{n+2}+\mathfrak{Q}_3(n),\tag{53}$$

where $\mathfrak{Q}_1(x)$, $\mathfrak{Q}_2(x)$, $\mathfrak{Q}_3(x)$ are polynomials defined by

$$\begin{cases} \mathfrak{Q}_1(x) = \mathfrak{Q}_2(x-1) - \mathfrak{Q}_2(x), \\ \mathfrak{Q}_2(x) = \mathfrak{g}_2 x^2 + \mathfrak{g}_1 x + \mathfrak{g}_0, \\ \mathfrak{Q}_3(x) = L_1 - (\mathfrak{Q}_1(1)L_2 + \mathfrak{Q}_2(1)L_3) \end{cases} \tag{54}$$

and $\mathfrak{g}_0, \mathfrak{g}_1, \mathfrak{g}_2$ are determined by

$$\begin{pmatrix} \mathfrak{g}_0 \\ \mathfrak{g}_1 \\ \mathfrak{g}_2 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 5 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}.$$

Setting $(\mathfrak{g}_0, \mathfrak{g}_1, \mathfrak{g}_2) = (5, -2, 1)$ to Eq. (54) implies $\mathfrak{Q}_2(x) = x^2 - 2x + 5$, $\mathfrak{Q}_1(x) = ((x-1)^2 - 2(x-1) + 5) - (x^2 - 2x + 5) = -2x + 3$, and $\mathfrak{Q}_3(x) = L_1 - (L_2 + 4L_3) = -18$. Plugging these polynomials to Eq. (53) yields

$$\sum_{k=1}^{n} k^2 L_k = (-2n+3)L_{n+1} + (n^2 - 2n + 5)L_{n+2} - 18.$$

[a] see [15], table 1.1, page 19, identity (16).

8. Conclusion

This paper extends several previous results on Ledin-Brousseau sum. We generalized this sum for a more general linear recurrence sequence and found the similarity of the formula with previous results. Besides that, we have established a new property on the representation of Ledin-Brousseau sum, that is the uniqueness of polynomials which generate its formula. It is hoped that this paper inspires the future research especially in the realm of number theory. We have several ideas for the future development of Ledin-Brousseau sum and will discuss these in the next section.

9. Discussion and Open Problems

We propose some open problems by modifying some given constraints, either the roots $r_1, r_2, ..., r_m$ of characteristic polynomial $\mathcal{CP}(x)$ or the choice of h.

Since all roots of $\mathcal{CP}(x)$ must be distinct in our results, it is natural to ask what will happen if one of $r_1, r_2, ..., r_m$ may be same. Does the existence and uniqueness property still hold as similar as Theorem 1 and 2?

Problem 1. Let $P(x) \in \mathbb{C}[x]$ be a polynomial and $r \in \mathbb{Z}$. Let $(s_k)_{k\in\mathbb{Z}}$ be a sequence defined in Definition 1, where the only modified constraint is that two of the roots of $\mathcal{CP}(x)$ can be equal.

(i) Does there exist an ordered pair of polynomials $(P_1(x), P_2(x), ..., P_{m+1}(x))$ in $\mathbb{C}[x]^{m+1}$ such that $\sum_{k=1}^{n} P(k)s_{k+r} = P_1(n)s_{n+1+r} + P_2(n)s_{n+2+r} + \cdots + P_m(n)s_{n+m+r} + P_{m+1}(n)$ for all $n \in \mathbb{N}$? If yes, is this pair unique?

(ii) What will happen if we extend to the sum $\sum_{k=1}^{n} P(k)s_{hk+r}$ for general $h \in \mathbb{Z}$?

Next, suppose $(s_k)_{k\in\mathbb{Z}}$ is a sequence defined in Definition 1 without any modification and we only focus on the choice of h. It is written in Theorem 1 and 2 that all of $r_1^h, r_2^h, ..., r_m^h$ are not equal to 1. It turns out that the value of h satisfying this condition can affect the construction of matrix $M(d, \ell)$ when $\ell := h$. It means that, in our results, the coefficients

of polynomials of $P_1(x), P_2(x), ..., P_{m+1}(x)$ are generated by $M(d,h)$ to make the identity $\sum_{k=1}^{n} P(k)s_{hk+r} = P_1(n)s_{h(n+1)+r} + P_2(n)s_{h(n+2)+r} + \cdots + P_m(n)s_{h(n+m)+r} + P_{m+1}(n)$ holds for all $n \in \mathbb{N}$. Unfortunately, the weakness of this construction is when the matrix $M(d,h)$ is singular, hence the constructions in Definition 4 and Eq. (21) are no longer reliable. We note that $M(d,h)$ is singular if and only if one of $r_1^h, r_2^h, ..., r_m^h$ is equal to 1. Therefore, it is natural to ask whether Theorem 1 and 2 still hold or not when one of $r_1, r_2, ..., r_m$ is a h^{th} root of unity.

Problem 2. Let $P(x) \in \mathbb{C}[x]$ be a polynomial and $(s_k)_{k\in\mathbb{Z}}$ be a sequence as defined in Definition 1 without any modification. Let h, r be integers such that one of the roots of characteristic polynomial $\mathcal{CP}(x)$ is a h^{th} root of unity. Does there exist an ordered pair of polynomials $(P_1(x), P_2(x), ..., P_{m+1}(x))$ in $\mathbb{C}[x]^{m+1}$ such that the sum $\sum_{k=1}^{n} P(k)s_{hk+r}$ can be represented by $P_1(n)s_{h(n+1)+r} + P_2(n)s_{h(n+2)+r} + \cdots + P_m(n)s_{h(n+m)+r} + P_{m+1}(n)$ for all $n \in \mathbb{N}$? If yes, is this pair unique?

Another modification for the value of h is that two of $r_1^h, r_2^h, ..., r_m^h$ may be equal. Suppose $(s_k)_{k\in\mathbb{Z}}$ is still a sequence in Definition 1 without any modification. Suppose that all roots of $\mathcal{CP}(x)$ are not the h^{th} roots of unity. For $r \in \mathbb{Z}$, in our results, we use the mutually distinct condition of $r_1^h, r_2^h, ..., r_m^h$ to imply that $(s_{hk+r})_{k\in\mathbb{Z}}$ satisfies a homogeneous linear recurrence relation with complex constant coefficients, see Lemma 3 when $(\ell, q) := (h, r)$. If we modify that two of $r_1^h, r_2^h, ..., r_m^h$ can be equal, does $(s_{hk+r})_{k\in\mathbb{Z}}$ still satisfy a linear recurrence relation as similar as Lemma 3 ? Are there the other methods to show the existence and uniqueness property in this case? Hence, we propose the following problem.

Problem 3. Let $P(x) \in \mathbb{C}[x]$ be a polynomial and $(s_k)_{k\in\mathbb{Z}}$ be a sequence as defined in Definition 1 without any modification. Let h, r be integers such that none of $r_1, r_2, ..., r_m$ is a h^{th} root of unity and two of $r_1^h, r_2^h, ..., r_m^h$ can be equal. Does there exist an ordered pair of polynomials $(P_1(x), P_2(x), ..., P_{m+1}(x))$ in $\mathbb{C}[x]^{m+1}$ such that the sum $\sum_{k=1}^{n} P(k)s_{hk+r}$ can be represented by $P_1(n)s_{h(n+1)+r} + P_2(n)s_{h(n+2)+r} + \cdots + P_m(n)s_{h(n+m)+r} + P_{m+1}(n)$ for all $n \in \mathbb{N}$? If yes, is this pair unique?

CRediT Authorship Contribution Statement

Ivan Hadinata: Conceptualization, Methodology, Formal Analysis, Investigation, Validation, Writing Original Draft, Visualization, Corresponding Author. **Muhamad Abdillah Ahen:** Validation, Writing Review and Editing. Author 2 reviewed the mathematical arguments and proofs, provided critical feedback, identified areas requiring revision, and collaborated with Author 1 in improving the manuscript.

Declaration of Generative AI and AI-assisted technologies

The authors used ChatGPT (OpenAI), Gemini (3.5 Flash), and Claude (Sonnet 4.6) to improve the language and check for typographical errors. All content in this research paper was produced entirely by the authors.

Declaration of Competing Interest

The authors declare no competing interests.

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Data and Code Availability

There are no computational codes or empirical data used in this theoretical study. The authors developed all results, proofs, and analyses analytically. As a result, this manuscript was prepared without the use of any datasets or code.

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