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Partial Extended b-Metric Space and Some Fixed Point Theorems

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Abstract

In this paper, we introduce partial extended b-metric spaces (PEBMS) as a generalization that encompasses both extended b-metric spaces and partial b-metric spaces as special cases. This new structure incorporates a point-dependent control function from extended b-metrics together with the possibility of non-zero self-distance from partial b-metrics, providing a more flexible framework for studying generalized metric spaces. We establish fundamental properties and prove fixed point theorems for three distinct contraction types.

Keywords: contraction mapping; fixed point theorem; partial b-metric space; partial extended b-metric space.

1. Introduction

Fixed-point theory is a part of mathematics that has a fundamental role in mathematical analysis. Its classical foundation was created by the Banach Contraction Principle [1], which guarantees the existence and uniqueness of a fixed point for a contraction mapping on a complete metric space. As this field has evolved, various generalizations of metric spaces have been introduced to expand the scope of the theory. Bakhtin [2] and Czerwik [3] introduced b-metric spaces by modifying the triangle inequality using a constant $s \geq 1$, which subsequently produced many new fixed-point results [4–6]. On the other side, Matthews [7] introduced partial metric spaces, which allow the distance of a point from itself to be non-zero, with the primary motivation coming from the semantics of programming languages [8–10]. The combination of these two concepts resulted in partial metric-b spaces introduced by Shukla [11] and further developed by Dung and Hang [12] as well as Karahan and Isik [13]. Kamran et al. [14] then generalized the b-metric space to an extended b-metric space by replacing the constant $s \geq 1$ by a function $\theta : X \times X \to [1, \infty)$. Various fixed-point results within this framework have been studied in [15, 16].

Although these generalizations have produced many fixed-point theorems, most research still addresses each space structure separately. Some studies have attempted to combine two concepts, such as partial b-metric spaces, which only merge two concepts but are limited to certain types of contractions. In addition, the results in extended b-metric spaces generally focus on specific classes of contractions and do not simultaneously accommodate various types of contractions within a unified space structure. These limitations reveal a research gap in developing a space capable of integrating the characteristics of partial and extended b-metric structures while supporting various types of contraction mappings.

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Based on this background, this study is inspired to develop the concept of Partial Extended b-metric Space (PEBMS) as a generalization that combines the flexible distance structures of partial b-metric and extended b-metric spaces. Through this framework, it is expected that new fixed-point theorems for several types of contraction mappings can be obtained simultaneously, thus offering a more comprehensive contribution to the advancement of fixed-point theory.

2. Preliminaries

In this section, we recall several basics definitions and preliminary results related to extended b-metric spaces and partial b-metric spaces. In particular, we present some fundamental notions together with auxiliary lemmas and known fixed point results that will be used in the sequel. These concepts form the theoretical foundation for the main results on fixed points in partial extended b-metric space.

Definition 1. [14] Let X be a non empty set and $\theta : X \times X \to [1, \infty)$. A function $d_\theta : X \times X \to [0, \infty)$ is called extended b-metric (EBM) if for all $x, y, z \in X$ it satisfies:

1. $d_\theta(x, y) = 0 \iff x = y$
2. $d_\theta(x, y) = d_\theta(y, x)$
3. $d_\theta(x, z) \leq \theta(x, z)[d_\theta(x, y) + d_\theta(y, z)]$

The pair (X, d_θ) is called extended b-metric space (EBMS)

Remark 1. if $\theta(x, y) = s$ for $s \geq 1$ we obtain the definition of b-metric space [2, 3]

Example 1. Let $X = \{2, 3, 4\}$, $\theta : X \times X \to [1, \infty)$, and $d_\theta : X \times X \to [0, \infty)$ as follows:

$$\theta(x, y) = 1 + x + y$$

$$d_\theta(x, y) = \begin{cases} 0 & if \quad x = y \\ 20 & if \quad x \neq y \end{cases}$$

show that The pair (X, d_θ) is EBMS

Proof. from definition 1 (1) and (2) are done by replacing $x = y$. For (3) we have

$$\begin{aligned} d_\theta(2, 3) &= 20 \\ &\leq \theta(2, 3)[d_\theta(2, 4) + d_\theta(4, 3)] \\ &= 240 \end{aligned}$$

$$\begin{aligned} d_\theta(2, 4) &= 20 \\ &\leq \theta(2, 4)[d_\theta(2, 3) + d_\theta(3, 4)] \\ &= 280 \end{aligned}$$

in a similar way to $d_\theta(3, 4)$, Hence for all $x, y, z \in X$

$$d_\theta(x, y) \leq \theta(x, y)[d_\theta(x, z) + d_\theta(z, y)]$$

Then we have the pair of (X, d_θ) is a EBMS. $\square$

Definition 2. [1] Let (X, d) be a metric space. The mapping $T : X \to X$ is said to be Lipschitzian if there exist a constant $k > 0$ (called Lipschitz constant) such that

$$d(Tx, Ty) \leq kd(x, y), \quad (x, y) \in X \times X.$$

A Lipschitzian mapping with a Lipschitz constant $k < 1$ is called contraction.

Definition 3. [14] Let (X, d_θ) be an extended b-metric space. A sequence $\{x_n\}$ in X is said to be:

1. Cauchy if and only if $d(x_n, x_m) \to 0$ as $n, m \to \infty$;
2. Convergent if and only if there exist $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$ and we write $\lim_{n\to\infty} x_n = x$;
3. The b-metric space (X, d) is complete if for every Cauchy sequence is convergent.

Lemma 1. *[14] Let (X, d_θ) be an extended b-metric space. If d_θ is continuous, then every convergent sequence has a unique limit*

Definition 4. [11] Let X be a non-empty set and coeficient $s \geq 1$. A function $p : X \times X \to [0, \infty)$ is called partial b-metric (PBM) on X if for all $x, y, z \in X$ it satisfies:

1. $p(x, x) = p(x, y) = p(y, y)$ if and only if $x = y$
2. $p(x, x) \leq p(x, y)$
3. $p(x, y) = p(y, x)$
4. $p(x, z) \leq s[p(x, y) + p(y, z)] - p(y, y)$

The triplet of (X, p, s) is called partial b-metric space (PBMS).

Remark 2. ([11], Remarks 1 and 2)

1. In a partial b-metric space (X, b), condition $b(x, y) = 0$ implies that $x = y$; however, the converse implication does not necessarily hold.
2. A partial metric space can be viewed as a special case of a partial b-metric space with coefficient $s = 1$. Similarly, every b-metric space is a partial b-metric space with the same coefficient and zero self-distance. Nevertheless, the converse statements are not necessarily true.

Definition 5. [11] Let (X, p) be a partial b-metric space with coefficient s. Let $\{x_n\}$ be any sequence in X and $x \in X$. Then:

1. The sequence $\{x_n\}$ is said to be convergent to x, if $\lim_{n\to\infty} p(x_n, x) = p(x, x)$.
2. Sequence $\{x_n\}$ is said to be Cauchy sequence in (X, p) if $\lim_{n,m\to\infty} p(x_n, x_m)$ exist and is finite.
3. (X, p) is said to be a complete partial b-metric space if for every Cauchy sequence

$\{x_n\}$ in X there exist $x \in X$ such that

$$\lim_{n,m\to\infty} p(x_n, x_m) = \lim_{n\to\infty} p(x_n, x) = p(x, x).$$

3. Main Result

Recently, the concept of partial b-metric space was introduced by Shukla in [11] and further developed by Van Dung and Hang in [12]. Motivated by these developments, the partial extended b-metric is introduced as a natural of the existing structure.

Definition 6. Let X be a non-empty set and $\theta : X \times X \to [1, \infty)$. A function $p_\theta : X \times X \to [0, \infty)$ is called partial extended b-metric (PEBM) on X if for all $x, y, z \in X$ it satisfies:

1. $p_\theta(x, x) = p_\theta(x, y) = p_\theta(y, y)$ if and only if $x = y$
2. $p_\theta(x, x) \leq p_\theta(x, y)$
3. $p_\theta(x, y) = p_\theta(y, x)$
4. $p_\theta(x, z) \leq \theta(x, z)[p_\theta(x, y) + p_\theta(y, z)] - p_\theta(y, y)$

The triplet of (X, p_θ, θ) is called partial extended b-metric space (PEBMS).

While extended b-metric spaces [14] provide flexible triangle inequalities through point-dependent control functions, and partial b-metric spaces [11] allow non-zero self-distances, these structures have typically been studied separately with results tailored to specific contraction types. PEBMS combines both features, enabling a unified treatment of diverse contraction mappings (Banach, Kannan, Chatterjea types) in one framework. Notably, this combination is not merely additive: Example 2 shows that PEBMS admits structures impossible to capture by either EBMS or PBMS, demonstrating that the interplay between point-dependent control and non-zero self-distance provides genuinely new flexibility for fixed-point analysis.

Example 2. Let $X = [0, \infty)$. Define $\theta : X \times X \to [1, \infty)$ by $\theta(x, y) = e^{|x-y|} + x + y + 2$ and $p_\theta : X \times X \to [0, \infty)$ by $p_\theta(x, y) = e^{|x-y|} - 1 + x + y$. Then (X, p_θ, θ) is a partial extended b-metric space, and moreover no constant $s \geq 1$ can replace θ in the generalized triangle inequality; that is, (X, p_θ) is not a partial b-metric space for any fixed coefficient s.

Proof. for Axioms 1–3, Since $p_\theta(x, x) = 2x$ is injective in x, the condition $p_\theta(x, x) = p_\theta(y, y)$ already forces $x = y$, so Axiom 1 holds. For Axiom 2, using $e^t - 1 \geq t$ for all $t \in \mathbb{R}$ with $t = x - y$, we get $x - y \leq e^{|x-y|} - 1$, i.e.

$$p_\theta(x, x) = 2x \leq e^{|x-y|} - 1 + x + y = p_\theta(x, y).$$

Axiom 3 (symmetry) is immediate from the definition.
Next for no constant coefficient suffices, Fix $x = 0$, $z = t$, and take $y = t/2$. by asymptotic comparison

$$\frac{p_\theta(0, t)}{p_\theta(0, t/2) + p_\theta(t/2, t)} = \frac{e^t - 1 + t}{2\left(e^{t/2} - 1 + t/2\right)} \sim \frac{1}{2} e^{t/2} \longrightarrow \infty \quad \text{as } t \to \infty.$$

Hence for every constant $s \geq 1$ there exists t large enough for which this ratio exceeds s, so no partial b-metric coefficient can work uniformly a point-dependent control function is structurally necessary.

For axiom 4, by symmetry we may assume $x \leq z$; write $a = |x-y|$, $b = |y-z|$, $c = z-x$. *Case 1:* $y \in [x, z]$, so $a+b = c$. By convexity of $e^{(\cdot)}$, $e^a + e^b \geq 2e^{c/2}$, and by $e^t \geq 1+t$ (with $t = c/2$), $2e^{c/2} - 2 \geq c$. Hence

$$e^a + e^b - 2 \geq c.$$

Also, since $c(e^c+2) - (e^c - 1)$ vanishes at $c = 0$ and has derivative $2 + ce^c > 0$ for $c \geq 0$, we obtain $c(e^c + 2) \geq e^c - 1$. Combining,

$$\theta(x,z)(e^a + e^b - 2) \geq (e^c + 2) \cdot c \geq e^c - 1,$$

and adding the nonnegative remainder $\theta(x,z)(x+z) + 2y(\theta(x,z) - 1) \geq 0$ from both sides gives

$$p_\theta(x,z) \leq \theta(x,z)\left[p_\theta(x,y) + p_\theta(y,z)\right] - p_\theta(y,y).$$

Case 2: $y \notin [x,z]$, say $y > z$, so $a = b + c$ with $b = y - z \geq 0$. The bracket $p_\theta(x,y) + p_\theta(y,z)$ contains the term $e^b(e^c+1) - 2 + x + z + 2y$, which equals $p_\theta(x,z)$ exactly at $b = 0$ and is strictly increasing in b (as $e^c + 1 \geq 2 > 0$), while $\theta(x,z) \geq 1$ is fixed. Thus

$$\theta(x,z)\left[p_\theta(x,y) + p_\theta(y,z)\right] \geq \theta(x,z) \cdot p_\theta(x,z) \geq 2\, p_\theta(x,z) > p_\theta(x,z)$$

at $b = 0$, and the inequality persists for all $b \geq 0$. The case $y < x$ follows by the symmetric argument.

Therefore Axiom 4 holds for all $x, y, z \in X$, and (X, p_θ, θ) is a partial extended b-metric space. □

From Example 2, it can be seen that the example does not fall under EBMS, even though all elements such as the distance function, θ, and the set X have been provided. This means that PEBMS covers aspects not covered by EBMS. On the other hand, PEBMS further extends the space defined in the partial b-metric space in [11]. The following remarks clarify how PEBMS relates to, and differs from, the space structures discussed in Section 1, including those most structurally close to it.

Remark 3.

1. The notion of a partial extended b-metric extends the concept of an extended b-metric by allowing non–zero self-distance. In particular, while an extended b-metric satisfies $d_\theta(x,x) = 0$ for all $x \in X$, a partial extended b-metric admits $p_\theta(x,x) \geq 0$ and modifies the triangle inequality by the term $-p_\theta(y,y)$. Hence, every extended b-metric space can be regarded as a special case of a partial extended b-metric space when the self-distance is zero.
2. In a partial extended b-metric space (X, p_θ, θ), if $x, y \in X$ and $p_\theta(x,y) = 0$, it implies $x = y$. The reason is because of property of PEBMS number 2 in Definition 6. Clearly $p_\theta(x,x) \leq p_\theta(x,y)$ and $p_\theta(y,y) \leq p_\theta(x,y)$. Since it is given that $p_\theta(x,y) = 0$, we then get $p_\theta(x,x) = p_\theta(x,y) = p_\theta(y,y) = 0$. By property of PEBMS number 1 in Definition 6, $x = y$.
3. The concept of a partial extended b-metric space (PEBMS) used in this paper differs from the notion of an extended partial b-metric space (or partial b-metric space). The function of partial extended b-metric space $\theta : X \times X \to [1, \infty)$ acts as a control function in the generalized triangle inequality, whereas in the extended partial b-metric space (or partial b-metric space), the structure is governed by a function $\Omega : [0, \infty) \to [0, \infty)$ applied to the metric expression. Hence, these two notions are

not equivalent and lead to different analytical frameworks.

This definition is inspired by the notion of convergent sequences in partial b-metric spaces [11], and is formulated by adapting and extending the underlying ideas to suit the present context.

Definition 7. Let (X,p_θ,θ) be a partial extended b-metric space

1. A sequence $\{x_n\}$ is called convergent to x in X, written $\lim_{n\to\infty} x_n = x$ if

$$\lim_{n\to\infty} p_\theta(x_n, x) = p_\theta(x, x)$$

2. The sequence $\{x_n\}$ is said to be Cauchy sequence in (X, p_θ, θ) if $\lim_{n,m\to\infty} p_\theta(x_n, x_m)$ exist and is finite.
3. (X, p_θ, θ) is said to be a complete partial b-metric space if for every Cauchy sequence $\{x_n\}$ in X there exist $x \in X$ such that

$$\lim_{n,m\to\infty} p_\theta(x_n, x_m) = \lim_{n\to\infty} p_\theta(x_n, x) = p_\theta(x, x)$$

Example 3. Let $X = [0, 1]$. Define $\theta : X \times X \to [1, \infty)$ by $\theta(x, y) = 1 + x + y$. Define $p_\theta : X \times X \to [0, \infty)$ by $p_\theta(x, y) = \max\{x, y\}$. Consider the sequence $\{x_n\} \subset X$ with $x_n = \frac{1}{n}$. Then $x_n \to 0$ in (X, p_θ, θ), the sequence $\{x_n\}$ is Cauchy and (X, p_θ, θ) is a complete partial extended b-metric space.

Proposition 1. *Let (X, p_θ, θ) be a partial extended b-metric space. Define a function $d_p : X \times X \to [0, \infty)$ by*

$$d_p(x, y) = \begin{cases} 0, & x = y, \\ p_\theta(x, y), & x \neq y. \end{cases}$$

Then d_p is an extended b-metric on X with the same control function θ.

Proof. First, it is clear that $d_p : X \times X \to [0, \infty)$. (1) If $d_p(x, y) = 0$, then either $x = y$ or $p_\theta(x, y) = 0$. From the properties of p_θ, this implies $x = y$. Conversely, if $x = y$, then $d_p(x, y) = 0$. (2) Symmetry follows directly from the symmetry of p_θ, that is,

$$d_p(x, y) = d_p(y, x).$$

(3) For all $x, y, z \in X$, we consider two cases. If any two of x, y, z coincide, the inequality holds trivially. If $x \neq y \neq z$, then

$$d_p(x, z) = p_\theta(x, z) \leq \theta(x, z)\big[p_\theta(x, y) + p_\theta(y, z)\big] - p_\theta(y, y).$$

Since $p_\theta(y, y) \geq 0$, we obtain

$$d_p(x, z) \leq \theta(x, z)\big[d_p(x, y) + d_p(y, z)\big].$$

Thus, d_p satisfies the extended b-metric inequality. Therefore, d_p is an extended b-metric on X. ☐

The following lemma is a useful lemma that will be used in some of the next theorems.

Lemma 2. *Let $(a_n)_{n\geq 0}$ be a sequence of real numbers that converges to 0 and $t \in [0,1)$ be a real number. Then, $\lim\limits_{n\to\infty} \sum_{j=0}^{n-1} a_{n-j}t^j = 0$.*
(note: let the convention $t^0 = 1$ be permitted when $t = 0$)

Proof. When $t = 0$, it is trivial, so we only focus when $t \neq 0$ next. Let us consider $\epsilon > 0$ arbitrarily. Then there are $p_1, p_2 \in \mathbb{N}$ so that $t^j < \epsilon$ for all $j \geq p_1$ and $|a_j| < \epsilon$ for all $j \geq p_2$. Since $(a_n)_{n\geq 0}$ is convergent, so there is $M > 0$ for which $|a_n| \leq M$ for all n. Hence there exists $p_1 + p_2 \in \mathbb{N}$ so that for all $n \geq p_1 + p_2$, we have

$$\Big|\sum_{j=0}^{p_1-1} a_{n-j}t^j\Big| \leq \sum_{j=0}^{p_1-1} |a_{n-j}|t^j \leq \epsilon \cdot (1 + t + t^2 + \cdots + t^{p_1-1}) \leq \frac{\epsilon}{1-t}$$

and

$$\Big|\sum_{j=p_1}^{n-1} a_{n-j}t^j\Big| \leq \sum_{j=p_1}^{n-1} |a_{n-j}|t^j \leq M \cdot t^{p_1} \cdot (1 + t + t^2 + \cdots + t^{n-p_1-1}) \leq \frac{M\epsilon}{1-t},$$

and these imply

$$\Big|\sum_{j=0}^{n-1} a_{n-j}t^j\Big| \leq \Big|\sum_{j=0}^{p_1-1} a_{n-j}t^j\Big| + \Big|\sum_{j=p_1}^{n-1} a_{n-j}t^j\Big| \leq \epsilon \times \frac{M+1}{1-t}.$$

Therefore $\sum_{j=0}^{n-1} a_{n-j}t^j$ converges to 0 as n tends to infinity. □

The following theorem (Theorem 1) represents the fixed point results involving the combination of Banach and Kannan contractions in PEBMS.

Theorem 1. *Let (X, p_θ, θ) be a complete partial extended b-metric space. Let $T : X \to X$ be such that for all $x, y \in X$,*

$$p_\theta(Tx, Ty) \leq k_1 p_\theta(x, y) + k_2 p_\theta(x, Tx) + k_3 p_\theta(y, Ty). \tag{1}$$

(i) If $\theta(Tx, x) < \frac{1}{k_1+k_2+k_3}$, where $k_1, k_2, k_3 \in \mathbb{R}^+ \cup \{0\}$ and $0 < k_1 + k_2 + k_3 < 1$. If there is $x_0 \in X$ such that $\{p_\theta(T^n x_0, x_0) : n \in \mathbb{N}_0\}$ is bounded, then T has a unique fixed point $u \in X$. Moreover $p_\theta(u, u) = 0$ and $\lim\limits_{n\to\infty} T^n x_0 = u$.

(ii) Suppose that $k_1 = 0$ and $k_2 = k_3 = k$ (Kannan Type). If $\theta(Tx, x) < \frac{1}{k}$ for all $x \in X$, where $k \in (0, \frac{1}{2})$, then T has a unique fixed point $u \in X$. Moreover $p_\theta(u, u) = 0$ and $\lim\limits_{n\to\infty} T^n x_0 = u$.

Proof. *(i)* Let the sequence $\{x_n\}_{n\geq 0}$ with the initial term $x_0 \in X$ and recursive of x_0 generated by $x_{n+1} = Tx_n = T^n x_0$ for all $n \in \mathbb{N}$. Applying inequality (1) with $x = x_n$ and $y = x_{n-1}$, we have

$$p_\theta(x_{n+1}, x_n) \leq k_1 p_\theta(x_n, x_{n-1}) + k_2 p_\theta(x_n, x_{n+1}) + k_3 p_\theta(x_{n-1}, x_n) \qquad ,(\forall n \in \mathbb{N}). \tag{2}$$

It can be written as

$$\begin{aligned}
(1-k_2)\cdot p_\theta(x_{n+1},x_n) &\le (k_1+k_3)\cdot p_\theta(x_n,x_{n-1}) \qquad ,(\forall n\in\mathbb{N})\\
p_\theta(x_{n+1},x_n) &\le \left(\frac{k_1+k_3}{1-k_2}\right)\cdot p_\theta(x_n,x_{n-1}) \qquad ,(\forall n\in\mathbb{N})\\
p_\theta(x_{n+1},x_n) &\le \left(\frac{k_1+k_3}{1-k_2}\right)^2\cdot p_\theta(x_{n-1},x_{n-2}) \qquad ,(\forall n\in\mathbb{N}).
\end{aligned}$$

Continuing this process, we obtain

$$p_\theta(x_{n+1},x_n) \le \left(\frac{k_1+k_3}{1-k_2}\right)^n\cdot p_\theta(x_1,x_0) \qquad ,(\forall n\in\mathbb{N}).$$

It follows that

$$\lim_{n\to\infty} p_\theta(x_{n+1},x_n)=0.$$

Applying inequality (1) with $x=x_{n-1}$ and $y=x_{m-1}$, we have

$$p_\theta(x_n,x_m)-k_1p_\theta(x_{n-1},x_{m-1}) \le k_2p_\theta(x_n,x_{n-1})+k_3p_\theta(x_m,x_{m-1}) \qquad ,(\forall m,n\in\mathbb{N}). \tag{3}$$

Applying the telescopic sums involving (3), for all $m,n\in\mathbb{N}$ with $n\ge m$ satisfying:

$$\begin{aligned}
&\sum_{t=1}^{m} k_1^{t-1}\left(p_\theta(x_{n-t+1},x_{m-t+1})-k_1p_\theta(x_{n-t},x_{m-t})\right)\\
&\le \sum_{t=1}^{m} k_1^{t-1}\left(k_2p_\theta(x_{n-t+1},x_{n-t})+k_3p_\theta(x_{m-t+1},x_{m-t})\right)
\end{aligned}$$

or equivalently,

$$p_\theta(x_n,x_m)\le k_1^m p_\theta(x_{n-m},x_0)+k_2\sum_{t=0}^{m-1}k_1^t p_\theta(x_{n-t},x_{n-t-1})+k_3\sum_{t=0}^{m-1}k_1^t p_\theta(x_{m-t},x_{m-t-1}) \tag{4}$$

by permitting that $k_1^0=1$ when $k_1=0$. Since $0\le k_1<1$, $\lim\limits_{n\to\infty} p_\theta(x_{n+1},x_n)=0$, and $\{p_\theta(x_n,x_0): n\in\mathbb{N}_0\}$ is bounded; limiting $m,n\to\infty$ to (4) and also using Lemma 2 will yield that $\lim\limits_{m,n\to\infty} p_\theta(x_n,x_m)=0$. Hence $(x_n)_{n\ge 0}$ is Cauchy. Hence, there is $u\in X$ so that $p_\theta(u,u)=\lim_{n\to\infty}p_\theta(x_n,u)=\lim_{n\to\infty}p_\theta(x_n,x_m)=0$. We shall show that u is a fixed point of T. Note that for all $n\in\mathbb{N}$,

$$p_\theta(Tu,u)\le\theta(Tu,u)[p_\theta(Tu,x_{n+1})+p_\theta(x_{n+1},u)]. \tag{5}$$

By setting $(x,y)=(u,x_n)$ for all $n\in\mathbb{N}$ to (1), we have

$$p_\theta(Tu,x_{n+1})\le k_1p_\theta(u,x_n)+k_2p_\theta(Tu,u)+k_3p_\theta(x_n,x_{n+1}). \tag{6}$$

Combining both of (5) and (6) implies

$$(1-k_2\theta(Tu,u))p_\theta(Tu,u)\le\theta(Tu,u)[k_1p_\theta(u,x_n)+k_3p_\theta(x_{n+1},x_n)+p_\theta(x_{n+1},x_n)]. \tag{7}$$

Since $\theta(Tu,u)<\frac{1}{k_1+k_2+k_3}$, then $1-k_2\theta(Tu,u)\ne 0$. By limiting $n\to\infty$ to (7), it yields $p_\theta(Tu,u)=0$ and then $Tu=u$, i.e., u is a fixed point of T.
For uniqueness of fixed point, let w be another fixed point of T. Setting $(x,y):=(u,w)$ to (1) yields

$$p_\theta(u,w)\le k_1p_\theta(u,w)+k_2p_\theta(u,u)+k_3p_\theta(w,w)\le(k_1+k_2+k_3)p_\theta(u,w).$$

The fact $k_1+k_2+k_3<1$ implies $p_\theta(u,w)=0$, therefore $u=w$. In conclusion, fixed point must be unique.

(ii) Let us consider $x_0 \in X$ arbitrarily. Define sequence $(x_n)_{n\geq 0}$ in X with the initial x_0 and $x_n = Tx_{n-1}$, $n = 0, 1, 2, \ldots$. From inequality $p_\theta(Tx, Ty) \leq k\,[p_\theta(x, Tx) + p_\theta(y, Ty)]$ setting $x = x_{m-1}$ and $y = x_{n-1}$ we have

$$p_\theta(x_m, x_n) \leq k\,[p_\theta(x_m, x_{m-1}) + p_\theta(x_n, x_{n-1})] \quad ,\forall m, n \in \mathbb{N} \tag{8}$$

in a similar way setting $x = x_n$ and $y = x_{n-1}$ we have

$$p_\theta(x_{n+1}, x_n) \leq k\,[p_\theta(x_{n+1}, x_n) + p_\theta(x_n, x_{n-1})] \quad ,\forall n \in \mathbb{N}$$

$$(1-k)\,p_\theta(x_{n+1}, x_n) \leq k\,p_\theta(x_n, x_{n-1}) \quad ,\forall n \in \mathbb{N}$$

$$p_\theta(x_{n+1}, x_n) \leq \frac{k}{1-k}\,p_\theta(x_n, x_{n-1}) \quad ,\forall n \in \mathbb{N}$$

$$p_\theta(x_{n+1}, x_n) \leq \frac{k}{1-k}\,p_\theta(x_n, x_{n-1}) \leq \left(\frac{k}{1-k}\right)^2 p_\theta(x_{n-1}, x_{n-2}) \quad ,\forall n \in \mathbb{N}$$

By repeating the same recursive step, we obtain.

$$p_\theta(x_{n+1}, x_n) \leq \left(\frac{k}{1-k}\right)^n p_\theta(x_1, x_0) \quad ,\forall n \in \mathbb{N} \tag{9}$$

Since $k \in (0, \frac{1}{2})$, then $\left(\frac{k}{1-k}\right) \in (0, 1)$ and as $n \to \infty$ we have

$$\lim_{n\to\infty} p_\theta(x_{n+1}, x_n) = 0$$

Therefore by taking the limit $m, n \to \infty$ in inequality (8) we have

$$\lim_{m,n\to\infty} p_\theta(x_m, x_n) = 0$$

which means the sequence (x_n) is Cauchy. Since (X, p_θ, θ) is a complete, there is $u \in X$ such that $p_\theta(u, u) = \lim_{n\to\infty} p_\theta(x_n, u) = \lim_{m,n\to\infty} p_\theta(x_m, x_n) = 0$. Based on the triangle inequality of the partial extended b-metric, we obtain.

$$p_\theta(Tu, u) \leq \theta(Tu, u)[p_\theta(Tu, x_{n+1}) + p_\theta(x_{n+1}, u)] - p_\theta(x_{n+1}, x_{n+1}) \tag{10}$$

From inequality $p_\theta(Tx, Ty) \leq k\,[p_\theta(x, Tx) + p_\theta(y, Ty)]$ setting $x = u$ and $y = x_n$ we have

$$p_\theta(Tu, Tx_n) \leq k\,[p_\theta(Tu, u) + p_\theta(x_n, x_{n+1})] \quad ,\forall m, n \in \mathbb{N} \tag{11}$$

Combine the inequality (10) and (11) then we have.

$$\begin{aligned} p_\theta(Tu, u) \leq \theta(Tu, u) \cdot k \cdot p_\theta(Tu, u) + \theta(Tu, u) \cdot k \cdot p_\theta(x_{n+1}, x_n) \\ +\theta(Tu, u) p_\theta(x_{n+1}, u) - p_\theta(x_{n+1}, x_{n+1}) \end{aligned}$$

So that for each $n \in \mathbb{N}$.

$$\begin{aligned} [1 - \theta(Tu, u)k] \cdot p_\theta(Tu, u) \leq \theta(Tu, u) \cdot k \cdot p_\theta(x_{n+1}, x_n) + \theta(Tu, u) \cdot \\ p_\theta(x_{n+1}, u) - p_\theta(x_{n+1}, x_{n+1}) \end{aligned}$$

Therefore by limiting $n \to \infty$, we have $p_\theta(Tu, u) = 0$ since $[1 - \theta(Tu, u)k] \neq 0$. Consequently, T has a fixed point. For the uniqueness let $w \in X$ is a fixed point of T. setting $x = u$ and $y = w$ in inequality $p_\theta(Tx, Ty) \leq k[p_\theta(x, Tx) + p_\theta(y, Ty)]$ yields $p_\theta(u, w) \leq k[p_\theta(u, u) + p_\theta(w, w)] = kp_\theta(w, w)$. From the properties of a partial extended b-metric space, we have $p_\theta(w, w) \leq p_\theta(u, w)$. Consequently we obtain $p_\theta(u, w) \leq kp_\theta(w, w) \leq kp_\theta(u, w)$ then $p_\theta(u, w) = p_\theta(w, w) = p_\theta(u, u) = 0$ since $k \in (0, \frac{1}{2})$, it shows that the fixed point T is unique. □

Before presenting the following theorem, we note that its proof is based on a Kannan-type contraction condition [17], adapted from the previous result.

Theorem 2. *Let (X, p_θ, θ) be a complete PEBMS. Let $T : X \to X$ be a self-mapping such that*

$$p_\theta(Tx, Ty) \leq k[p_\theta(x, Ty) + p_\theta(y, Ty)]$$

for all $x, y \in X$ where k is a constant in $(0, \frac{1}{2})$. If there exist $x_0 \in X$ such that $\lim_{n\to\infty} p_\theta(T^n x_0, T^n x_0) = 0$ for $n \in \mathbb{N}$, Then T has a unique fixed point $u \in X$. Moreover there exists an element $x_0 \in X$ such that $u = \lim_{n\to\infty} T^n x_0$ and $p_\theta(u, u) = 0$.

Proof. Construct the sequence $(x_n)_{n\geq 0}$ with the initial term $x_0 \in X$ and recursion $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. Setting $x = x_{m-1}$ and $y = x_{n-1}$ to the inequality $p_\theta(Tx, Ty) \leq k[p_\theta(x, Ty) + p_\theta(y, Ty)]$ over $m, n \in \mathbb{N}$ implies

$$(\forall m, n \in \mathbb{N}) \quad p_\theta(x_m, x_n) \leq k\, p_\theta(x_{m-1}, x_n) + k\, p_\theta(x_n, x_{n-1}) \tag{12}$$

Fix $n \in \mathbb{N}$ and for all $\ell > n$, define $c_\ell := p_\theta(x_\ell, x_n)$. Inequality (12) state that $\ell \in \mathbb{N}$

$$c_\ell \leq k \cdot c_{\ell-1} + k \cdot p_\theta(x_n, x_{n-1}) \tag{13}$$

Applying inequality (13) repeatedly, starting from $\ell = n + m$ and descending to $\ell = n$, we obtain for every $m \in \mathbb{N}$:

$$\begin{aligned} c_{n+m} &\leq k \cdot c_{n+m-1} + k \cdot p_\theta(x_n, x_{n-1}) \\ &\leq k^2 \cdot c_{n+m-2} + (k^2 + k) \cdot p_\theta(x_n, x_{n-1}) \\ &\cdots \quad \cdots \\ &\cdots \quad \cdots \\ &\leq k^m \cdot c_n + (k^m + k^{m-1} + \cdots + k) \cdot p_\theta(x_n, x_{n-1}) \end{aligned} \tag{14}$$

Since $c_n = p_\theta(x_n, x_n)$ this gives for all $m, n \in \mathbb{N}$:

$$\begin{aligned} p_\theta(x_{n+m}, x_n) - k^m p_\theta(x_n, x_n) &= \sum_{t=1}^{m} \Big[k^{t-1} p_\theta(x_{n+m-t+1}, x_n) - k^t\, p_\theta(x_{n+m-t}, x_n)\Big] \\ &\leq \left(\sum_{t=1}^{m} k^t\right) p_\theta(x_n, x_{n-1}) \\ &\leq \frac{1}{1-k} \cdot p_\theta(x_n, x_{n-1}). \end{aligned}$$

Therefore, for all $m, n \in \mathbb{N}$ with $m > n$, and inequality property on PEBMS we have

$$\begin{aligned} p_\theta(x_m, x_n) &\leq k^{m-n} p_\theta(x_n, x_n) + \frac{1}{1-k} \cdot p_\theta(x_n, x_{n-1}) \\ &\leq k^{m-n} p_\theta(x_{n+1}, x_n) + \frac{1}{1-k} \cdot p_\theta(x_n, x_{n-1}) \end{aligned} \tag{15}$$

Besides that, the following inequality holds for all $n \in \mathbb{N}$, which is obtained by using induction and applying (12) when $(m, n) := (n + 1, n)$.

$$p_\theta(x_{n+1}, x_n) \leq k^n p_\theta(x_1, x_0) + k \sum_{t=1}^{n} k^{n-t} p_\theta(x_t, x_t). \tag{16}$$

Since $0 < k < 1$ and $\lim_{n\to\infty} p_\theta(x_n, x_n) = 0$, limiting $n \to \infty$ to (16) and also applying Lemma 2 will yield $\lim_{n\to\infty} p_\theta(x_{n+1}, x_n) = 0$. Then, limiting $m, n \to \infty$ to (15) we have

$$\lim_{m,n\to\infty} p_\theta(x_m, x_n) = 0.$$

Hence $(x_n)_{n\geq 0}$ is a Cauchy sequence. Since (X, p_θ, θ) is complete, then there exist $u \in X$ such that $p_\theta(u, u) = \lim_{n\to\infty} p_\theta(x_n, u) = \lim_{n\to\infty} p_\theta(x_m, x_n) = 0$. We will show that u is a fixed point of T. According to triangle inequality of partial extended b-metric space, we have $(\forall n \in \mathbb{N})$

$$p_\theta(Tu, u) \leq \theta(Tu, u)[p_\theta(Tu, x_{n+1}) + p_\theta(x_{n+1}, u)] - p_\theta(x_{n+1}, x_{n+1}). \tag{17}$$

Setting $x = u$ and $y = x_n$ to $p_\theta(Tx, Ty) \leq k[p_\theta(x, Ty) + p_\theta(y, Ty)]$ yields

$$(\forall n \in \mathbb{N}) \quad p_\theta(Tu, x_{n+1}) \leq k[p_\theta(u, x_{n+1}) + p_\theta(x_n, x_{n+1})] \tag{18}$$

Combining (17) and (18), we obtain $\forall n \in \mathbb{N}$:

$$p_\theta(Tu, u) \leq \theta(Tu, u)(k+1) \cdot p_\theta(x_{n+1}, u) + \theta(Tu, u)kp_\theta(x_{n+1}, x_n) - p_\theta(x_{n+1}, x_{n+1}). \tag{19}$$

Limiting $n \to \infty$ to (19) yields $p_\theta(Tu, u) = 0$. Hence $Tu = u$, i.e. u is the fixed point of T. For the uniqueness of fixed point, let w be a fixed point of T. Setting $(x, y) = (u, w)$ and $(x, y) = (w, u)$ to the inequality $p_\theta(Tx, Ty) \leq k[p_\theta(x, Ty) + p_\theta(y, Ty)]$, we will get $p_\theta(u, w) \leq \frac{k}{1-k} \cdot p_\theta(w, w) \leq p_\theta(w, w)$ and $p_\theta(u, w) \leq \frac{k}{1-k} \cdot p_\theta(u, u) \leq p_\theta(u, u)$. Since $p_\theta(x, x) \leq p_\theta(x, y)$ for all $x, y \in X$, then $p_\theta(u, w) = p_\theta(u, u) = p_\theta(w, w)$. Therefore $u = w$. In conclusion, T has a unique fixed point u where $p_\theta(u, u) = 0$ and $u = \lim_{n\to\infty} T^n x_0$. □

The following one is a Chatterjea-type fixed point theorem version for our PEBMS.

Theorem 3. *Let (X, p_θ, θ) be a complete PEBMS. Let $k \in (0, \frac{1}{2})$ be a constant and $T : X \to X$ be a self-mapping such that for all $x, y \in X$,*

$$p_\theta(Tx, Ty) \leq k[p_\theta(x, Ty) + p_\theta(y, Tx)].$$

If there exists an element $x_0 \in X$ such that $\theta(T^a x_0, T^b x_0) \leq \theta(x_0, x_0) < \lambda$, where $\lambda = \frac{1}{2}\left(\sqrt{1 + \frac{4}{k}} - 1\right)$ for all $a, b \in \mathbb{N}_0$, then T has a unique fixed point. Moreover $u = \lim_{n\to\infty} T^n x_0$ and $p_\theta(u, u) = 0$.

Proof. Let $(x_n)_{n\geq 0}$ be a sequence with initial term x_0 and recursion $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. Then we have $\theta(x_a, x_b) \leq \lambda$ for all $a, b \in \mathbb{N}_0$. Hence, for all $a, b \in \mathbb{N}_0$,

$$\frac{k\theta(x_a, x_b)}{1 - k\theta(x_a, x_b)} \leq \frac{k \cdot \frac{1}{2} \cdot \left(\sqrt{1 + \frac{4}{k}} - 1\right)}{1 - k \cdot \frac{1}{2} \cdot \left(\sqrt{1 + \frac{4}{k}} - 1\right)} = \left(\frac{\sqrt{1 + \frac{4}{k}} - 1}{2}\right)^{-1} = \lambda^{-1}.$$

Since $0 < k < \frac{1}{2}$, then $\lambda > 1$.
Next, setting $x := x_{n-1}$ and $y := x_n$ to inequality $p_\theta(Tx, Ty) \leq k[p_\theta(x, Ty) + p_\theta(y, Tx)]$ over $n \in \mathbb{N}$, we have

$$p_\theta(x_n, x_{n+1}) \leq k[p_\theta(x_{n-1}, x_{n+1}) + p_\theta(x_n, x_n)] \leq k \cdot \theta(x_{n-1}, x_{n+1}) \cdot [p_\theta(x_{n-1}, x_n) + p_\theta(x_n, x_{n+1})].$$

Hence for all $n \in \mathbb{N}$,

$$p_\theta(x_n, x_{n+1}) \leq \frac{k\theta(x_{n-1}, x_{n+1})}{1 - k\theta(x_{n-1}, x_{n+1})} \cdot p_\theta(x_{n-1}, x_n).$$

It implies that for all $n \in \mathbb{N}_0$,

$$p_\theta(x_n, x_{n+1}) \leq p_\theta(x_0, x_1) \prod_{t=1}^{n} \frac{k\theta(x_{t-1}, x_{t+1})}{1 - k\theta(x_{t-1}, x_{t+1})} \leq \lambda^{-n} p_\theta(x_0, x_1).$$

Next, we observe the inequality

$$\begin{aligned} p_\theta(x_m, x_n) &\leq \theta(x_m, x_n)(p_\theta(x_m, x_{n+1}) + p_\theta(x_{n+1}, x_n)) - p_\theta(x_{n+1}, x_{n+1}) \\ &\leq \theta(x_m, x_n) p_\theta(x_m, x_{n+1}) + \theta(x_m, x_n) \lambda^{-n} p_\theta(x_0, x_1) \end{aligned}$$

or then

$$p_\theta(x_m, x_n) - \theta(x_m, x_n) p_\theta(x_m, x_{n+1}) \leq \lambda^{-n} \theta(x_m, x_n) p_\theta(x_0, x_1)$$

which holds over $m, n \in \mathbb{N}_0$. Using the above inequality, the following inequality holds for all integers $m, n \in \mathbb{N}_0$ with $m \geq n + 1$,

$$\begin{aligned} &p_\theta(x_m, x_{n+1}) - \lambda^{-m} p_\theta(x_0, x_1) \theta(x_0, x_0)^{m-n} \\ &\leq p_\theta(x_m, x_{n+1}) - p_\theta(x_m, x_{m+1}) \prod_{t=1}^{m-n} \theta(x_m, x_{n+t}) \\ &\leq \frac{1}{\theta(x_m, x_n)} \sum_{t=1}^{m-n} \left([p_\theta(x_m, x_{n+t}) - \theta(x_m, x_{n+t}) p_\theta(x_m, x_{n+t+1})] \cdot \prod_{j=0}^{t-1} \theta(x_m, x_{n+j}) \right) \\ &\leq \frac{1}{\theta(x_m, x_n)} \sum_{t=1}^{m-n} \left(\lambda^{-n-t} \theta(x_m, x_{n+t}) p_\theta(x_0, x_1) \cdot \prod_{j=0}^{t-1} \theta(x_m, x_{n+j}) \right) \\ &= \lambda^{-n} p_\theta(x_0, x_1) \sum_{t=1}^{m-n} \lambda^{-t} \theta(x_m, x_{n+1}) \theta(x_m, x_{n+2}) \cdots \theta(x_m, x_{n+t}) \\ &\leq \lambda^{-n} p_\theta(x_0, x_1) \sum_{t=1}^{m-n} (\lambda^{-1} p_\theta(x_0, x_0))^t \end{aligned}$$

Consequently, we obtain

$$p_\theta(x_m, x_{n+1}) \leq \lambda^{-n} p_\theta(x_0, x_1) \sum_{t=0}^{m-n} (\lambda^{-1} p_\theta(x_0, x_0))^t \leq \frac{\lambda^{-n} p_\theta(x_0, x_1)}{1 - \lambda^{-1} \theta(x_0, x_0)}. \tag{20}$$

Limiting $m, n \to \infty$ to the inequality (20) yields $\lim_{m,n\to\infty} p_\theta(x_m, x_n) = 0$ and $(x_n)_{n\geq 0}$ is Cauchy. Since (X, p_θ, θ) is complete, there is $u \in X$ so that $p_\theta(u, u) = \lim_{n\to\infty} p_\theta(u, x_n) = \lim_{m,n\to\infty} p_\theta(x_m, x_n) = 0$. We shall show that this u is a fixed point of T.
Note that from the generalized triangle inequality of PEBMS, for all $n \in \mathbb{N}$,

$$p_\theta(Tu, u) \leq \theta(Tu, u)[p_\theta(Tu, x_n) + p_\theta(u, x_n)]. \tag{21}$$

Setting $x := u$ and $y := x_{n-1}$ over $n \in \mathbb{N}$ to inequality $p_\theta(Tx, Ty) \leq k[p_\theta(x, Ty) + p_\theta(y, Tx)]$, we get

$$p_\theta(Tu, x_n) - k p_\theta(Tu, x_{n-1}) \leq k p_\theta(u, x_n). \tag{22}$$

Applying telescopic sums which involve (22), that is,

$$p_\theta(Tu, x_n) - k^n p_\theta(Tu, x_0) = \sum_{t=1}^{n}(k^{t-1}p_\theta(Tu, x_{n-t+1}) - k^t p_\theta(Tu, x_{n-t})) \le \sum_{t=1}^{n} k^t p_\theta(u, x_{n-t+1})$$

$$\implies \quad p_\theta(Tu, x_n) \le k^n p_\theta(Tu, x_0) + \sum_{t=1}^{n} k^t p_\theta(u, x_{n-t+1}). \tag{23}$$

By noticing $0 < k < 1$, $\lim_{n\to\infty} p_\theta(u, x_n) = 0$, and using Lemma 2, the right hand side of (23) tends to 0 as n goes to infinity. Hence, $\lim_{n\to\infty} p_\theta(Tu, x_n) = 0$. By limiting $n \to \infty$ to (21), we conclude $p_\theta(Tu, u) = 0$ and $Tu = u$, i.e., u is a fixed point of T.
For uniqueness of fixed point, let w be another fixed point of T. Setting $x := u$ and $y := w$ to inequality $p_\theta(Tx, Ty) \le k[p_\theta(x, Ty) + p_\theta(y, Tx)]$, will yield that $p_\theta(u, w) = 0$ and then $u = w$, so the fixed point must be unique. □

4. Conclusion

This paper introduces partial extended b-metric spaces (PEBMS) as a structure combining a point-dependent control function θ with possibly non-zero self-distance. We showed that PEBMS strictly generalizes partial b-metric spaces in the sense that no constant coefficient s can replace θ in Example 3.2, and that every extended b-metric space arises as a special case of PEBMS when self-distance vanishes. Within this framework, we established existence and uniqueness of fixed points for three specific contraction types: a combined Banach–Kannan contraction (Theorem 3.8), a Kannan-type contraction (Theorem 3.9), and a Chatterjea-type contraction (Theorem 3.10), each under explicit conditions on θ and the contraction constants. These results are limited to the contraction classes and hypotheses stated above; extending them to other contraction types, and clarifying the relationship between PEBMS and other existing generalized metric structures, remain open directions for future work.

CRediT Authorship Contribution Statement

Muhamad Abdillah Ahen: Conceptualization, Formal Analysis, Investigation, Validation, Writing Original Draft, Corresponding Author. **Ivan Hadinata:** Formal Analysis, validation, Collaborated with first author to prove Some Theorems. **Ariqah Rif'at Rafilah Sapsuha:** Identified areas requiring revision and correcting grammar.

Declaration of Generative AI and AI-assisted technologies

The authors improved the language and checked for typos using ChatGPT (OpenAI) and Claude (Sonnet 4.6). This research paper's authors created all of its content.

Declaration of Competing Interest

The authors declare no competing interest

Ethics Approval and Consent to Participate

Ethics approval was not required for this study

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Data and Code Availability

There are no computational codes or empirical data used in this theoretical study. The authors developed all result, proofs, and analyses analytically.

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