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A Multiplicative Extension of Weighted Hadamard Fractional Integrals

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Abstract

This paper presents an innovative category of weighted Hadamard fractional integral operators situated within the paradigm of multiplicative calculus. The operator introduced herein broadens the traditional weighted Hadamard fractional integral by integrating the multiplicative framework of non-Newtonian calculus, thereby establishing a cohesive linkage between weighted fractional analysis and multiplicative fractional calculus. We delineate the formulation of both the left- and right-sided multiplicative weighted Hadamard fractional integral operators and explore their core analytical characteristics. Specifically, a multiplicative linearity property is demonstrated, the continuity of the logarithmic representation of the operator is affirmed, and boundedness results are extracted in weighted Lebesgue-type spaces through the application of Hölder's inequality. Furthermore, we establish that the proposed operators comply with a semigroup property, which ensures alignment with the classical framework of fractional integration. These findings illustrate that the introduced operators maintain crucial structural attributes of weighted fractional integrals while seamlessly extending them to the multiplicative context. The theoretical framework developed herein lays a foundational basis for subsequent explorations in multiplicative fractional calculus, fractional integral inequalities, and associated applications within the realm of mathematical analysis.

Keywords: Hadamard fractional integral; Weighted fractional integral; Fractional operators

1. Introduction

Generalizing the order of differentiation to non-integer values has been a fundamental question since the early development of calculus itself. Its origins can be traced back to a question posed by Gottfried Wilhelm Leibniz in the 17th century, with further developments by mathematicians such as Joseph Liouville and Bernhard Riemann, who introduced the foundational formulations. Fractional calculus, generalizes the concepts of differentiation and integration to non-integer (fractional) orders. Fractional calculus provides a more flexible mathematical framework to describe complex systems and processes. This generalization has proven to be particularly effective in capturing memory effects, hereditary properties, and anomalous behaviors in various fields of science and engineering.

In 1892, Jacques Hadamard, introduced an alternative form of fractional integration characterized by logarithmic kernels, different from classical Riemann–Liouville fractional integral. This

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formulation arises when dealing with functions defined on intervals of the form $(0, \infty)$, where scaling and multiplicative structures play a central role. Unlike other fractional operators, the Hadamard approach incorporates logarithmic expressions such as $\ln(x/t)$, making it particularly suitable for problems exhibiting scale invariance and multiplicative behavior.

Through the years, Hadamard-type operators have been further generalized and studied within the framework of fractional calculus, with applications in areas such as differential equations, integral inequalities, and mathematical physics, see ([1–4]). In particular, the introduction of weight functions into Hadamard fractional operators has attracted considerable attention, as weighted formulations provide greater flexibility in analyzing functions with nonuniform behavior. These weighted Hadamard-type operators extend the classical framework by incorporating an additional weight term, allowing the development of more general analytical results and broader applications in fractional integral inequalities and functional analysis. The weighted Hadamard fractional integral operator is defined as follows.

Definition 1. [5] Let $f \in X_w^p(a,b)$ and $w \neq 0$ be a function on interval (a,b), for $\alpha > 0$, the left and right-sided weighted Hadamard fractional integral operator are described as,

$$({}^H\mathcal{I}_{a+}^{\alpha;w} f)(x) = \frac{w^{-1}(x)}{\Gamma(\alpha)} \int_a^x w(t) \left(\ln \frac{x}{t}\right)^{\alpha-1} \frac{f(t)dt}{t}, \quad (a < x),$$

and

$$({}^H\mathcal{I}_{b-}^{\alpha;w} f)(x) = \frac{w^{-1}(x)}{\Gamma(\alpha)} \int_x^b w(t) \left(\ln \frac{t}{x}\right)^{\alpha-1} \frac{f(t)dt}{t}, \quad (x < b).$$

On the other hand, the fundamental structure of classical calculus is additive in nature. This has motivated researchers to develop alternative frameworks capable of describing exponential growth and multiplicative processes more naturally, leading to the development of multiplicative calculus. In 1967, Grossman and Katz developed as an alternative framework to Newtonian calculus (classic calculus), a non-Newtonian calculus as an alternative analytical framework. In this framework linear operations in classical calculus with multiplicative operations, addition is replaced by multiplication and subtraction is replaced by division. This framework is called as multiplicative calculus and excels for modeling nonlinear phenomena with exponential and geometric growth. Its multiplicative derivative is defined as:

Definition 2. [6] Let $f(x) > 0$, multiplicative derivative of f,

$$f^*(x) = \frac{d^*}{dx^*} f(x) = \lim_{h \to 0} \left(\frac{f(x+h)}{f(x)}\right)^{\frac{1}{h}}.$$

The concept of fractional calculus began to be integrated with multiplicative calculus. Between 2014 and 2016, Thabet Abdeljawad and several other researchers developed fractional derivative and integral operators within the framework of multiplicative calculus, particularly of the Riemann–Liouville fractional integral and Caputo types [7]. These studies opened new directions in the investigation of semigroup properties, integral inequalities, and the development of other multiplicative fractional operators.

Subsequently, in 2025, Tingsong et al. [8] formulated a more general form of the Hadamard-type multiplicative fractional integral operator. This generalized form was obtained through an extension of the logarithmic kernel structure in the multiplicative integral. In addition to introducing the generalized definition, Tingsong et al. also investigated several fundamental properties of the operator, including the semigroup property and integral operator inequalities.

Definition 3. [8] Given $\alpha > 0$, let $f(x)$ a positive function on interval $[a, b]$, the left and right multiplicative Hadamard fractional integral of function f on $[a, b]$ are described as,

$$ {}_{a+}^{H}\mathcal{I}_{*}^{\alpha} f(x) = \exp\{ {}^{H}\mathcal{I}_{a+}^{\alpha} \ln(f(x))\} \exp\left(\frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln \frac{x}{t}\right)^{\alpha-1} \frac{\ln(f(t))}{t} dt\right), \quad a < x, $$

and

$$ {}_{*}^{H}I_{b-}^{\alpha} f(x) = \exp\{ {}^{H}\mathcal{I}_{b-}^{\alpha} \ln(f(x))\} = \exp\left(\frac{1}{\Gamma(\alpha)} \int_x^b \left(\ln \frac{t}{x}\right)^{\alpha-1} \frac{\ln(f(t))}{t} dt\right), \quad x < b, $$

where $\Gamma(\cdot)$ is the gamma function.

Motivated by the works mentioned previously, we defined weighted Hadamard fractional integrals on multiplicative framework. The main contribution of this work is the extension of weighted Hadamard fractional integration from the classical additive framework to the multiplicative framework. We define a weighted Hadamard fractional integral acting on positive-valued function via a logarithmic transformation of the classical operator and develop its associated operator theory in weighted Lebesgue spaces. We establish boundedness and semigroup properties of the operator, thereby providing a multiplicative counterpart to several fundamental results from the classical theory.

2. Preliminaries

In this section, we give some theorem that will be used throughout this paper and introduce the definitions of multiplicative integral of a function. First, we show

Theorem 1. *(Hölder's inequality,[9]) Let p and q be real numbers with $1 < p < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$ \sum_{k=1}^{n} |x_k y_k| \leq \left(\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} |y_k|^q\right)^{\frac{1}{q}} $$

For further theoretical background on Hölder's inequality and Lebesgue spaces, readers are referred to [9]. Next, we will give the definition of the multiplicative integral.

Definition 4. [7] Let $f(x) > 0$, multiplicative integrals of f,

$$ \int_a^b f(x)^{dx} = \exp\left(\int_a^b \ln(f(x)) dx\right), \quad x \in \mathbb{R}. $$

Then, we will show its properties.

Theorem 2. *[10] Let f and g be a multiplicative integrable function on interval $I[a, b]$, some properties of multiplicative integral are as follows*

1. $\displaystyle \int_a^b (f(x)^p)^{dx} = \left(\int_a^b f(x)^{dx}\right)^p, \quad p \in \mathbb{R}$
2. $\displaystyle \int_a^b (f(x)g(x))^{dx} = \int_a^b f(x)^{dx} \cdot \int_a^b g(x)^{dx},$

3. $\int_a^b \left(\frac{f(x)}{g(x)}\right)^{dx} = \frac{\int_a^b f(x)^{dx}}{\int_a^b g(x)^{dx}}$,
4. $\int_a^b f(x)^{dx} = \int_a^c f(x)^{dx} \cdot \int_c^b f(x)^{dx}, \quad a \le c \le b.$

For the details and further discussion about multiplicative calculus, can be found in [10].

3. Results and Discussion

In this section, we introduced the integral operator and investigated several fundamental properties of the operator, including linearity and the semigroup property, in order to examine the behavior of the operator in the setting of multiplicative calculus. Based on the definition of weighted Hadamard fractional integral operator and the multiplicative fractional integral operator, we define of the multiplicative weighted Hadamard fractional integral operator as follows

Definition 5. Given $\alpha > 0$, let $f(x)$ be a positive function on interval $[a,b] \subset (0,\infty)$, and $w(x)$ be a positive function on interval $[a,b]$. The left and right multiplicative weighted Hadamard functional fractional integral of function f on $[a,b]$ are

$$\begin{aligned} {}_{a+}^{H}\mathcal{I}_{*}^{\alpha;w} f(x) &= \exp\left\{ {}^{H}I_{a+}^{\alpha;w}(\ln f(x)) \right\} \\ &= \exp\left\{ \frac{w^{-1}(x)}{\Gamma(\alpha)} \int_a^x \frac{w(t)}{t} \left(\ln \frac{x}{t}\right)^{\alpha-1} \ln(f(t))dt \right\}, (a < x), \end{aligned}$$

and

$$\begin{aligned} {}_{*}^{H}\mathcal{I}_{b-}^{\alpha;w} f(x) &= \exp\left\{ {}^{H}I_{b-}^{\alpha;w}(\ln f(x)) \right\} \\ &= \exp\left\{ \frac{w^{-1}(x)}{\Gamma(\alpha)} \int_x^b \frac{w(t)}{t} \left(\ln \frac{t}{x}\right)^{\alpha-1} \ln(f(t))dt \right\}, (x < b), \end{aligned}$$

where $\Gamma(\cdot)$ is the gamma function and

$$w^{-1}(x) = \frac{1}{w(x)}.$$

The above definition requires f to be positive so that $\ln(f)$ is well-defined and throughout the remainder of this paper, we further assume that $\ln(f) \in X_w^p(a,b)$, ensuring that the corresponding classical integral operator of $ln(f)$ exists. We formally define the weighted Lebesgue space $X_w^p(a,b)$ later.

Remark 1. Setting $w = 1$ in Definition 5, then we obtain the multiplicative Hadamard functional fractional integral,

$$ {}_{a+}^{H}\mathcal{I}_{*}^{\alpha} f(x) = \exp\left\{ \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln \frac{x}{t}\right)^{\alpha-1} \ln(f(t)) \frac{dt}{t} \right\}.$$

Next, we state the following theorem, concerning the linearity property of the operator introduced before within multiplicative calculus.

Theorem 3. *Let $f, g : [a, b] \to \mathbb{R}^+$ be both integrable functions and $w(x) > 0$ for any $x \in [a, b]$, then for $m, n \in \mathbb{R}$ and any $\alpha > 0$, we have*

$$ {}_{a+}^{H}\mathcal{I}_{*}^{\alpha;w}(f^m g^n)(x) = \left[{}_{a+}^{H}\mathcal{I}_{*}^{\alpha;w} f(x)\right]^m \left[{}_{a+}^{H}\mathcal{I}_{*}^{\alpha;w} g(x)\right]^n . $$

Proof. Using Definition 5, it follows that

$$ \begin{aligned} {}_{a+}^{H}\mathcal{I}_{*}^{\alpha;w}(f^m g^n)(x) &= \exp\left\{\frac{w^{-1}(x)}{\Gamma(\alpha)}\int_a^x \frac{w(t)}{t}\left(\ln\frac{x}{t}\right)^{\alpha-1} \ln(f^m g^n(t))dt\right\} \\ &= \exp\left\{\frac{w^{-1}(x)}{\Gamma(\alpha)}\int_a^x \frac{w(t)}{t}\left(\ln\frac{x}{t}\right)^{\alpha-1} [m\ln(f(t)) + n\ln(g(t))]dt\right\} \\ &= \exp\left\{\frac{mw^{-1}(x)}{\Gamma(\alpha)}\int_a^x \frac{w(t)}{t}\left(\ln\frac{x}{t}\right)^{\alpha-1} [\ln(f(t))]dt\right\} \\ &\quad \exp\left\{\frac{nw^{-1}(x)}{\Gamma(\alpha)}\int_a^x \frac{w(t)}{t}\left(\ln\frac{x}{t}\right)^{\alpha-1} [\ln(g(t))]dt\right\} \\ &= \left[{}_{a+}^{H}\mathcal{I}_{*}^{\alpha;w} f(x)\right]^m \left[{}_{a+}^{H}\mathcal{I}_{*}^{\alpha;w} g(x)\right]^n . \end{aligned} $$

□

The continuity result for the fractional integral operator is presented in the following theorem.

Theorem 4. *Let $f : [a, b] \to \mathbb{R}^+$ and w be a strictly positive function on $[a, b]$, which is continuous on $[a, b]$, then for any $\alpha > 1$, $\ln({}_{a+}^{H}\mathcal{I}_{*}^{\alpha;w} f(x))$ is continuous on interval $[a, b]$.*

Proof. Taking a sufficiently small constant h such that $h > 0$, and for any $x \in [a, b]$ there is $x + h < b$. By using the continuity of f, it can be obtained that

$$ \begin{aligned} &|\ln({}_{a+}^{H}\mathcal{I}_{*}^{\alpha;w} f(x+h)) - \ln({}_{a+}^{H}\mathcal{I}_{*}^{\alpha;w} f(x))| \\ &= \left|\frac{w^{-1}(x+h)}{\Gamma(\alpha)}\int_a^{x+h} \frac{w(t)}{t}\left(\ln\frac{x+h}{t}\right)^{\alpha-1} \ln(f(t))dt\right. \\ &\left. - \frac{w^{-1}(x)}{\Gamma(\alpha)}\int_a^{x} \frac{w(t)}{t}\left(\ln\frac{x}{t}\right)^{\alpha-1} \ln(f(t))dt\right| \\ &= \frac{1}{\Gamma(\alpha)}\left|w^{-1}(x+h)\int_a^{x} \frac{w(t)}{t}\left(\ln\frac{x+h}{t}\right)^{\alpha-1} \ln(f(t))dt\right. \\ &\left. - w^{-1}(x)\int_a^{x} \frac{w(t)}{t}\left(\ln\frac{x}{t}\right)^{\alpha-1} \ln(f(t))dt + w^{-1}(x+h)\int_x^{x+h} \frac{w(t)}{t}\left(\ln\frac{x+h}{t}\right)^{\alpha-1} \ln(f(t))dt\right| \\ &\le \frac{1}{\Gamma(\alpha)}\left|\int_a^{x} \left(w^{-1}(x+h)\left(\ln\frac{x+h}{t}\right)^{\alpha-1} - w^{-1}(x)\left(\ln\frac{x}{t}\right)^{\alpha-1}\right)\frac{w(t)}{t}\ln(f(t))dt\right| \\ &+ \frac{1}{\Gamma(\alpha)}\left|w^{-1}(x+h)\int_x^{x+h} \frac{w(t)}{t}\left(\ln\frac{x+h}{t}\right)^{\alpha-1} \ln(f(t))dt\right| \\ &\le \frac{1}{\Gamma(\alpha)}\left|\int_a^{x} \left(w^{-1}(x+h)\left(\ln\frac{x+h}{t}\right)^{\alpha-1} - w^{-1}(x)\left(\ln\frac{x}{t}\right)^{\alpha-1}\right)\frac{w(t)}{t}\ln(f(t))dt\right| \\ &+ \frac{1}{\Gamma(\alpha)}w^{-1}(x+h)\left(\ln\frac{x+h}{x}\right)^{\alpha-1}\int_x^{x+h} \frac{w(t)}{t}|\ln(f(t))|dt. \end{aligned} \tag{1} $$

For the first term of (1), define

$$G_h(x) = \left(w^{-1}(x+h)\left(\ln\frac{x+h}{t}\right)^{\alpha-1} - w^{-1}(x)\left(\ln\frac{x}{t}\right)^{\alpha-1}\right)\frac{w(t)}{t}\ln(f(t)).$$

Since w^{-1} are continuous, then $|w^{-1}(x+h)| \leq M$, we can show that

$$|G_h(x)| \leq \left(\left(\ln\frac{x+h}{t}\right)^{\alpha-1} + \left(\ln\frac{x}{t}\right)^{\alpha-1}\right)\frac{M|w(t)||\ln(f(t))|}{t}.$$

Next, suppose $|h| < \delta, x+h \leq x+\delta$, we get

$$|G_h(x)| \leq \left(\ln\frac{x+\delta}{t}\right)^{\alpha-1}\frac{C|w(t)||\ln(f(t))|}{t}.$$

Then we take the limit as $h \to 0$ of the first term of (1), using Lebesgue's dominated convergence theorem, we obtain

$$\lim_{h\to 0}\frac{1}{\Gamma(\alpha)}\left|\int_a^x G_h(x)dt\right| = \frac{1}{\Gamma(\alpha)}\left|\int_a^x \lim_{h\to 0} G_h(x)dt\right| = 0.$$

For the second term of (1), taking the limit as $h \to 0$ similarly yields

$$\lim_{h\to 0} w^{-1}(x+h)\left(\ln\frac{x+h}{x}\right)^{\alpha-1} = 0.$$

Furthermore, since both f and w are continuous, we obtain

$$\int_x^{x+h}\frac{w(t)}{t}|\ln(f(t))|dt < \infty.$$

Therefore,

$$\lim_{h\to 0}|\ln({}_{a+}^{H}\mathcal{I}_*^{\alpha;w}f(x+h)) - \ln({}_{a+}^{H}\mathcal{I}_*^{\alpha;w}f(x))| = 0.$$

It follows that $\ln({}_{a+}^{H}\mathcal{I}_*^{\alpha;w}f(x))$ is continuous for all $x \in [a,b]$. □

Before presenting the main boundedness results for the weighted multiplicative fractional integral operators, we first need to establish the definition of the function space in which the operators act. The introduction of this space is essential, since it establishes the norm, where its bounded.

Definition 6. [11] Let $w \neq 0$ be a function on interval $[a,b]$, g be a differentiable strictly increasing function on interval $[a,b]$. The space $X_w^p(a,b), 1 \leq p \leq \infty$ is a space of all Lebesgue measurable function f on interval $[a,b]$, for which $\|f\|_{X_w^p} < \infty$, where

$$\|f\|_{X_w^p(a,b)} = \left(\int_a^b |w(x)f(x)|^p\frac{dx}{x}\right)^{\frac{1}{p}}, 1 \leq p < \infty$$

and

$$\|f\|_{X_w^\infty(a,b)} = ess\sup_{a\leq x\leq b}|w(x)f(x)|, \quad p = \infty.$$

We first, prove that the fractional integral operator is finite for the functions on the weighed space, before proceeding to the boundedness.

Theorem 5. *If $f : [a,b] \to \mathbb{R}^+$ is an integrable function and w is a strictly positive measurable function on $[a,b]$, then for $a > 0$,*

$$|{}_{a+}^{H}\mathcal{I}_{*}^{\alpha;w} f(x)| \leq \exp\left\{ \frac{w^{-1}(x)\|\ln(f(x))\|_{X_w^\infty(a,b)}}{\Gamma(\alpha+1)} \left(\ln\frac{x}{a}\right)^{\alpha} \right\}.$$

Proof. Using Definition 5, it follows that

$$\begin{aligned}
|{}_{a+}^{H}\mathcal{I}_{*}^{\alpha;w} f(x)| &= \left| \exp\left\{ \frac{w^{-1}(x)}{\Gamma(\alpha)} \int_a^x \frac{w(t)}{t}\left(\ln\frac{x}{t}\right)^{\alpha-1} \ln(f(t))dt \right\} \right| \\
&\leq \exp\left\{ \left| \frac{w^{-1}(x)}{\Gamma(\alpha)} \int_a^x \frac{w(t)}{t}\left(\ln\frac{x}{t}\right)^{\alpha-1} \ln(f(t))dt \right| \right\} \\
&\leq \exp\left\{ \frac{w^{-1}(x)}{\Gamma(\alpha)} \int_a^x \left(\ln\frac{x}{t}\right)^{\alpha-1} |w(t)\ln(f(t))| \frac{dt}{t} \right\} \\
&\leq \exp\left\{ \frac{w^{-1}(x)\|\ln(f(x))\|_{X_w^\infty(a,b)}}{\Gamma(\alpha)} \int_a^x \left(\ln\frac{x}{t}\right)^{\alpha-1} \frac{dt}{t} \right\} \\
&= \exp\left\{ \frac{w^{-1}(x)\|\ln(f(x))\|_{X_w^\infty(a,b)}}{\Gamma(\alpha+1)} \left(\ln\frac{x}{a}\right)^{\alpha} \right\}.
\end{aligned}$$

□

After verifying that the fractional integral operator is well-defined on the weighted space, the following theorem presents its boundedness of the function $\ln({}_{a+}^{H}\mathcal{I}_{*}^{\alpha;w} f(x))$ on weighted space $\|\cdot\|_{X_w^p(a,b)}$.

Theorem 6. *If $f : [a,b] \to \mathbb{R}^+$ is an integrable function and w is a strictly positive measurable function on $[a,b]$, then for $p > 1$ and $p\alpha > 1$,*

$$\|\ln({}_{a+}^{H}\mathcal{I}_{*}^{\alpha;w} f(x))\|_{X_w^p(a,b)} \leq \frac{\|\ln(f(x))\|_{X_w^p(a,b)}(p-1)^{\frac{p-1}{p}} \ln\left(\frac{b}{a}\right)^{\alpha}}{\Gamma(\alpha)(p\alpha)^{1/p}(p\alpha-1)^{\frac{p-1}{p}}}.$$

Proof. For $1 < p < \infty$, using Definition 5 and Hölder's inequality, we obtain

$$\begin{aligned}
&\|\ln({}_{a+}^{H}\mathcal{I}_{*}^{\alpha;w} f(x))\|_{X_w^p(a,b)} \\
&= \left(\int_a^b |w(x)\ln({}_{a+}^{H}\mathcal{I}_{*}^{\alpha;w} f(x))|^p \frac{dx}{x} \right)^{1/p} \\
&= \left(\int_a^b \left| w(x)\frac{w^{-1}(x)}{\Gamma(\alpha)} \int_a^x \frac{w(t)}{t}\left(\ln\frac{x}{t}\right)^{\alpha-1} \ln(f(t))dt \right|^p \frac{dx}{x} \right)^{1/p} \\
&= \frac{1}{\Gamma(\alpha)} \left(\int_a^b \left| \int_a^x \frac{w(t)}{t}\left(\ln\frac{x}{t}\right)^{\alpha-1} \ln(f(t))dt \right|^p \frac{dx}{x} \right)^{1/p} \\
&\leq \frac{1}{\Gamma(\alpha)} \left(\int_a^b \left[\left(\int_a^x \left(\ln\frac{x}{t}\right)^{\frac{(\alpha-1)p}{p-1}} \frac{dt}{t} \right)^{\frac{p-1}{p}} \left(\int_a^x |w(t)\ln(f(t))|^p \frac{dt}{t} \right)^{\frac{1}{p}} \right]^p \frac{dx}{x} \right)^{1/p}
\end{aligned}$$

$$= \frac{1}{\Gamma(\alpha)} \left(\int_a^b \left(\int_a^x \left(\ln \frac{x}{t} \right)^{\frac{(\alpha-1)p}{p-1}} \frac{dt}{t} \right)^{p-1} \left(\int_a^x |w(t) \ln(f(t))|^p \frac{dt}{t} \right) \frac{dx}{x} \right)^{1/p}$$

$$= \frac{1}{\Gamma(\alpha)} \left(\frac{p-1}{p\alpha - 1} \right)^{\frac{p-1}{p}} \left(\int_a^b \ln \left(\frac{x}{a} \right)^{p\alpha-1} \left(\int_a^x |w(t) \ln(f(t))|^p \frac{dt}{t} \right) \frac{dx}{x} \right)^{1/p},$$

then using Fubini's theorem, we get

$$\begin{aligned}
&\| \ln({}_{a+}^{H}\mathcal{I}_{*}^{\alpha;w} f(x)) \|_{X_w^p(a,b)} \\
&\leq \frac{1}{\Gamma(\alpha)} \left(\frac{p-1}{p\alpha - 1} \right)^{\frac{p-1}{p}} \left(\int_a^b \ln \left(\frac{x}{a} \right)^{p\alpha-1} \frac{dx}{x} \right)^{1/p} \left(\int_a^x |w(t) \ln(f(t))|^p \frac{dt}{t} \right)^{1/p} \\
&= \frac{\| \ln(f(x)) \|_{X_w^p(a,b)}}{\Gamma(\alpha)} \left(\frac{p-1}{p\alpha - 1} \right)^{\frac{p-1}{p}} \left(\int_a^b \ln \left(\frac{x}{a} \right)^{p\alpha-1} \frac{dx}{x} \right)^{1/p} \\
&= \frac{\| \ln(f(x)) \|_{X_w^p(a,b)}}{\Gamma(\alpha)} \left(\frac{p-1}{p\alpha - 1} \right)^{\frac{p-1}{p}} \frac{1}{(p\alpha)^{1/p}} \ln \left(\frac{b}{a} \right)^{\alpha} \\
&= \frac{\| \ln(f(x)) \|_{X_w^p(a,b)} (p-1)^{\frac{p-1}{p}} \ln \left(\frac{b}{a} \right)^{\alpha}}{\Gamma(\alpha)(p\alpha)^{1/p}(p\alpha - 1)^{\frac{p-1}{p}}}.
\end{aligned}$$

For the case $p = \infty$, we similarly obtain

$$\begin{aligned}
|w(x) \ln({}_{a+}^{H}\mathcal{I}_{*}^{\alpha;w} f(x))| &= \left| w(x) \frac{w^{-1}(x)}{\Gamma(\alpha)} \int_a^x \frac{w(t)}{t} \left(\ln \frac{x}{t} \right)^{\alpha-1} \ln(f(t)) dt \right| \\
&= \left| \frac{1}{\Gamma(\alpha)} \int_a^x \frac{w(t)}{t} \left(\ln \frac{x}{t} \right)^{\alpha-1} \ln(f(t)) dt \right| \\
&= \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln \frac{x}{t} \right)^{\alpha-1} |w(t) \ln(f(t))| \frac{dt}{t} \\
&\leq \frac{\| \ln(f(x)) \|_{X_w^\infty(a,b)}}{\Gamma(\alpha)} \int_a^x \left(\ln \frac{x}{t} \right)^{\alpha-1} \frac{dt}{t} \\
&= \frac{\| \ln(f(x)) \|_{X_w^\infty(a,b)}}{\Gamma(\alpha+1)} \left(\ln \frac{x}{a} \right)^{\alpha}.
\end{aligned}$$

Then

$$\begin{aligned}
\| \ln({}_{a+}^{H}\mathcal{I}_{*}^{\alpha;w} f(x)) \|_{X_w^\infty(a,b)} &= ess \sup_{a \leq x \leq b} |w(x) \ln({}_{a+}^{H}\mathcal{I}_{*}^{\alpha;w} f(x))| \\
&\leq \frac{\| \ln(f(x)) \|_{X_w^\infty(a,b)}}{\Gamma(\alpha+1)} ess \sup_{a \leq x \leq b} \left(\ln \frac{x}{a} \right)^{\alpha} \\
&\leq \frac{\| \ln(f(x)) \|_{X_w^\infty(a,b)}}{\Gamma(\alpha+1)} \left(\ln \frac{b}{a} \right)^{\alpha}.
\end{aligned}$$

□

Finally, we present the following theorem, which establishes the semigroup property satisfied by the multiplicative weighted Hadamard fractional integral operators.

Theorem 7. *Let $f : [a, b] \to \mathbb{R}^+$ which is continuous function and w is a strictly positive measurable function on $[a, b]$. Then, for all $x \in [a, b]$ and $m, n > 0$,*

$$\left({}_{a+}^{H}\mathcal{I}_{*}^{m;w} \; {}_{a+}^{H}\mathcal{I}_{*}^{n;w} \right) f(x) = \left({}_{a+}^{H}\mathcal{I}_{*}^{(m+n);w} \right) f(x).$$

Proof. Using Definition 5,

$$\begin{aligned}
\left({}_{a+}^{H}\mathcal{I}_{*}^{m;w} \; {}_{a+}^{H}\mathcal{I}_{*}^{n;w} \right) f(x) &= \left({}_{a+}^{H}\mathcal{I}_{*}^{m;w} \right) \left[{}_{a+}^{H}\mathcal{I}_{*}^{n;w} f(x) \right] \\
&= \exp\left\{ \frac{w^{-1}(x)}{\Gamma(m)} \int_a^x \left(\ln \frac{x}{t} \right)^{m-1} w(t) \ln({}^{H}\mathcal{I}_{a+;w}^{n} f)(t) \frac{dt}{t} \right\} \\
&= \exp\left\{ \frac{w^{-1}(x)}{\Gamma(m)\Gamma(n)} \int_a^x \int_a^t \left(\ln \frac{x}{t} \right)^{m-1} \left(\ln \frac{t}{s} \right)^{n-1} w(s) \ln(f(s)) \frac{ds}{s} \frac{dt}{t} \right\}.
\end{aligned}$$

By Theorem 5, the integral is finite. Therefore, by Fubini's theorem, we change of order of integration,

$$= \exp\left\{ \frac{w^{-1}(x)}{\Gamma(m)\Gamma(n)} \int_a^x w(s) \ln(f(s)) \left[\int_s^x \left(\ln \frac{x}{t} \right)^{m-1} \left(\ln \frac{t}{s} \right)^{n-1} \frac{dt}{t} \right] \frac{ds}{s} \right\}. \tag{2}$$

First, we evaluate the inner integral in (2) using a change of variables,

$$u = \frac{\ln(t/s)}{\ln(x/s)}, \qquad 0 \le u \le 1,$$

since $m, n > 0$, we obtain

$$\begin{aligned}
\int_s^x \left(\ln \frac{x}{t} \right)^{m-1} \left(\ln \frac{t}{s} \right)^{n-1} \frac{dt}{t} &= \left(\ln \frac{x}{s} \right)^{m+n-1} \int_0^1 (1-u)^{m-1} u^{n-1} du \\
&= \left(\ln \frac{x}{s} \right)^{m+n-1} \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.
\end{aligned} \tag{3}$$

By substituting (3) into (2), it follows that

$$\begin{aligned}
\left({}_{a+}^{H}\mathcal{I}_{*}^{m;w} \; {}_{a+}^{H}\mathcal{I}_{*}^{n;w} \right) f(x) &= \exp\left\{ \frac{w^{-1}(x)}{\Gamma(m)\Gamma(n)} \int_a^x w(s) \ln(f(s)) \left[\left(\ln \frac{x}{s} \right)^{m+n-1} \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \right] \frac{ds}{s} \right\} \\
&= \exp\left\{ \frac{w^{-1}(x)}{\Gamma(m+n)} \int_a^x \left(\ln \frac{x}{s} \right)^{m+n-1} w(s) \ln(f(s)) \frac{ds}{s} \right\} \\
&= \left({}_{a+}^{H}\mathcal{I}_{*}^{m+n;w} f \right)(x).
\end{aligned}$$

□

4. Conclusion

Based on these results, we establish that the proposed weighted operator is well-defined within the considered function space and exhibits several fundamental analytical properties. In particular, the operator is shown to possess the properties of linearity, semigroup structure, and continuity. We also established the boundedness of the logarithm of the image of the operator in the weighted space. These properties indicate that the introduced operator provides a consistent extension of the classical weighted Hadamard fractional integral operator into the multiplicative setting. Furthermore, the obtained results demonstrate that the operator preserves the essential characteristics required for further theoretical analysis and potential applications in fractional calculus and related mathematical models.

CRediT Authorship Contribution Statement

Almeer Naufal Hakim: Conceptualization, Methodology, Formal Analysis, Writing–Original Draft. **Marjono Marjono:** Conceptualization, Supervision, Formal Analysis, Funding Acquisition. **Corina Karim:** Conceptualization, Supervision, Formal Analysis, Funding Acquisition, Writing–Review & Editing.

Declaration of Generative AI and AI-assisted technologies

Generative AI tool such as ChatGPT were used for language refinement and grammatical editing. All mathematical results, proofs, and conclusions presented were conducted entirely by the author.

Declaration of Competing Interest

The authors declare no competing interests.

Ethics Approval and Consent to Participate

Ethics approval was not required for this study.

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Data and Code Availability

Data sharing is not applicable to this study, as no datasets were generated or analyzed, and no code was developed.

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