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# Coefficient Bounds for a Subclass of Bi-Univalent Functions

Alifira Meliana Rachman, Marjono*, and Sa'adatul Fitri

*Department of Mathematics, Faculty of Science, Technology, and Mathematics, Brawijaya University, Malang, Indonesia*

**Abstract**

In this paper, we obtain explicit upper bounds for the initial seven coefficients and their inverses for a subclass of bi-univalent functions defined via subordination. Using coefficient estimates associated with the Carathéodory class and coefficient comparison techniques, expressions for the bounds of $a_2, a_3, \ldots, a_7$ are derived. Moreover, it is proved that the first seven inverse coefficients admit the same upper bounds as the corresponding coefficients of the original function. These findings generalize several existing results concerning lower-order coefficients and provide a framework for further investigations of higher-order coefficient and Hankel determinant problems in the theory of bi-univalent functions.

**Keywords:** Analytic Functions; Bi-univalent Functions; Carathéodory Functions; Initial Coefficient Estimates; Inverse Coefficient.



## 1. Introduction

Let $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disk and let $A$ be the class of analytic functions in $\mathbb{U}$. A function $f \in A$ is called univalent if it is one-to-one in $\mathbb{U}$ [1]. The class of all normalized univalent functions is commonly denoted by $S$. A function $f \in A$ is said to be bi-univalent if both $f$ and its inverse function $f^{-1}$ are univalent in $\mathbb{U}$. The class of all bi-univalent functions is denoted by $\Sigma$.

The study of bi-univalent functions has attracted considerable attention since Srivastava, Mishra, and Gochhayat [2] introduced the class $\Sigma$ of analytic bi-univalent functions. Since then, numerous subclasses of $\Sigma$ have been investigated from the viewpoint of coefficient problems. Mishra and Soren[3] established coefficient bounds for bi-starlike analytic functions, while Deniz, Çağlar, and Orhan [4] introduced the classes of bi-starlike and bi-convex functions of order $\beta$ and investigated problems related to the second Hankel determinant. The study of higher-order coefficient estimates also received considerable attention. In particular, Janani and Yalçın [5] derived upper bounds for the initial seven coefficients of a subclass of bi-starlike functions. Further developments were reported by Yousef, Alroud, and Illafe [6], Khatua, Naik, and Patil [7], Murugusundaramoorthy, Vijaya, and Bulboacă [8], Güleç and Aktaş [9], Al-Refai [10], and Sharma, Sivasubramanian, and Cho [11], who investigated various subclasses of analytic bi-univalent functions and their associated coefficient problems.

Among these subclasses, strongly defined classes are of particular interest because the imposed argument restrictions yield a finer geometric structure. Chow, Janteng, and Halim [12]

*Corresponding author. E-mail: alifirmeliana@student.ub.ac.id, marjono@ub.ac.id, saadatulfitri@ub.ac.id

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investigated coefficient-related problems for strongly bi-starlike functions of order $\alpha$. More recently, Shakir and Atshan [13] introduced the generalized subclass $\mathbf{B}_{\Sigma}(\beta)$, which unifies the bi-starlike and bi-convex classes within a single framework.

One of the central themes in the theory of bi-univalent functions is the determination of coefficient estimates and inverse coefficient bounds, since these quantities provide valuable information concerning the analytic and geometric behavior of functions and their inverses. Although substantial progress has been achieved for various subclasses of $\Sigma$, including several investigations on inverse coefficients [14, 15], the corresponding problems remain open for strongly generalized subclasses associated with the class $\mathbf{B}_{\Sigma}(\beta)$.

Motivated by the work of Shakir and Atshan [13], we introduce a new subclass of strongly bi-univalent functions, denoted by $\mathbf{B}^{*}_{\Sigma}(\alpha)$, which is a strongly generalized version of the class $\mathbf{B}_{\Sigma}(\beta)$. To the best of our knowledge, neither the initial seven coefficient estimates nor the inverse coefficient bounds have previously been investigated for this class. The novelty of the present work lies in establishing explicit upper bounds for the coefficients $a_2, a_3, \ldots, a_7$, while also deriving corresponding estimates for the inverse coefficients of functions belonging to $\mathbf{B}^{*}_{\Sigma}(\alpha)$. These results generalize and extend several known coefficient estimates for bi-starlike, bi-convex, and related subclasses of bi-univalent functions, thereby contributing further to the coefficient theory of strongly bi-univalent functions.

## 2. Methods

In this section, several preliminary results and auxiliary lemmas used in deriving coefficient and inverse coefficient bounds are presented. We begin by introducing several subclasses of bi-univalent functions that are relevant to the present study.

**Definition 1.** [16] A function $f \in \Sigma$ belongs to the class of bi-starlike functions of order $\beta$, denoted by $\mathcal{S}_{\Sigma}(\beta)$, if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \beta \quad \text{and} \quad \Re\left(\frac{wg'(w)}{g(w)}\right) > \beta,$$

where $0 \leq \beta < 1$ and $g = f^{-1}$.

**Definition 2.** [16] A function $f \in \Sigma$ belongs to the class of bi-convex functions of order $\beta$, denoted by $\mathcal{C}_{\Sigma}(\beta)$, if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \beta \quad \text{and} \quad \Re\left(1 + \frac{wg''(w)}{g'(w)}\right) > \beta,$$

where $0 \leq \beta < 1$ and $g = f^{-1}$.

**Definition 3.** [13] A function $f \in \Sigma$ belongs to the class $\mathbf{B}_{\Sigma}(\beta)$ if

$$\Re\left(\frac{zf'(z)}{f(z)} + zf''(z)\right) > \beta$$

and

$$\Re\left(\frac{wg'(w)}{g(w)} + wg''(w)\right) > \beta,$$

where $0 \leq \beta < 1$ and $g = f^{-1}$.

Let

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

be a function in the class $\Sigma$. Since $f$ is bi-univalent in $\mathbb{U}$, its inverse function $g = f^{-1}$ exists and is analytic in $\mathbb{U}$. Moreover, $g$ has the expansion

$$\begin{aligned} f^{-1}(w) = g(w) = w &- a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + (14a_2^4 - 21a_2^2a_3 \\ &+ 6a_2a_4 + 3a_3^2 - a_5)w^5 - (42a_2^5 - 84a_2^3a_3 + 28a_2^2a_4 + 28a_2a_3^2 - 7a_2a_5 \\ &- 7a_3a_4 + a_6)w^6 + (132a_2^6 - 330a_2^4a_3 + 120a_2^3a_4 + 180a_2^2a_3^2 - 36a_2^2a_5 \\ &- 72a_2a_3a_4 + 8a_2a_6 - 12a_3^3 + 8a_3a_5 + 4a_4^2 - a_7)w^7 - \cdots . \end{aligned} \tag{2}$$

Furthermore, several coefficient estimates for analytic functions with positive real part are required. These estimates are associated with the classical Carathéodory class and are frequently used in coefficient problems for analytic and bi-univalent functions.

**Definition 4.** [1] Let $P$ be the class of functions

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \qquad z \in \mathbb{U}, \tag{3}$$

which are analytic in $\mathbb{U}$ and satisfy $\Re(p(z)) > 0$.

**Lemma 1.** *[12] If $p \in P$ is given by Eq. (3), then $|p_n| \le 2, \qquad n \ge 1.$*

To obtain coefficient relations, the following representation formula will also be used.

**Lemma 2.** *[12] Let $p \in P$. Then there exist complex numbers $x$ and $z$ satisfying $|x| \le 1$ and $|z| \le 1$, such that*

$$2p_2 = p_1^2 + (4 - p_1^2)x,$$

*and*

$$4p_3 = p_1^3 + 2(4 - p_1^2)p_1x - (4 - p_1^2)p_1x^2 + 2(4 - p_1^2)(1 - |x|^2)z.$$

The above auxiliary results will be applied repeatedly in deriving coefficient inequalities and inverse coefficient estimates for the strongly $\mathbf{B}_\Sigma$ class introduced in the next section.

## 3. Results and Discussion

Based on the preliminaries established in the previous section, we now introduce the subclass of strongly bi-univalent functions to be investigated and derive several coefficient estimates associated with this class. These results will be used to obtain upper bounds for the coefficients of the function and its inverse function.

**Definition 5.** [13] A function $f \in \Sigma$ is said to belong to the class $\mathbf{B}^*_\Sigma(\alpha)$, where $0 < \alpha \le 1$, if

$$\left| \arg\left( \frac{zf'(z)}{f(z)} + zf''(z) \right) \right| < \frac{\pi\alpha}{2}$$

and

$$\left| \arg\left( \frac{wg'(w)}{g(w)} + wg''(w) \right) \right| < \frac{\pi\alpha}{2},$$

for all $z, w \in \mathbb{U}$, where $g = f^{-1}$.

To derive the coefficient bounds for functions belonging to the class $\mathbf{B}^*_\Sigma(\alpha)$, we first establish several coefficient relations associated with the defining conditions of the class.

Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be a function in $\mathbf{B}^*_\Sigma(\alpha)$ and let $g(w) = f^{-1}(w)$ be its inverse function. By Definition 5,

$$\left|\arg\left(\frac{zf'(z)}{f(z)} + zf''(z)\right)\right| < \frac{\alpha\pi}{2}$$

and

$$\left|\arg\left(\frac{wg'(w)}{g(w)} + wg''(w)\right)\right| < \frac{\alpha\pi}{2}.$$

Hence, the functions

$$\frac{zf'(z)}{f(z)} + zf''(z) \quad \text{and} \quad \frac{wg'(w)}{g(w)} + wg''(w)$$

take values in the sector

$$\left\{\zeta \in \mathbb{C} : |\arg\zeta| < \frac{\alpha\pi}{2}\right\}.$$

Therefore, by the standard sectorial representation, there exist functions $p, q \in P$ such that

$$\frac{zf'(z)}{f(z)} + zf''(z) = [p(z)]^\alpha, \tag{4}$$

$$\frac{wg'(w)}{g(w)} + wg''(w) = [q(w)]^\alpha, \tag{5}$$

where $P = \{h : h(0) = 1,\ \Re h(z) > 0\}$ and

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \dots, \tag{6}$$

and

$$q(w) = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \dots, \tag{7}$$

Substituting the series expansions of Eq. (1), $f'(z)$, $f''(z)$, and Eq. (6) into Eq. (4), and comparing the coefficients of like powers of $z$, we obtain

$$3a_2 = \alpha p_1, \tag{8}$$

$$8a_3 - a_2^2 = \alpha p_2 + \frac{\alpha(\alpha-1)p_1^2}{2}, \tag{9}$$

$$15a_4 - 3a_2a_3 + a_2^3 = \alpha p_3 + \alpha(\alpha-1)p_1p_2 + \frac{\alpha(\alpha-1)(\alpha-2)p_1^3}{6}, \tag{10}$$

$$\begin{aligned} &24a_5 - 4a_2a_4 - 2a_3^2 \\ &\quad + 4a_2^2a_3 - a_2^4 = \alpha p_4 + \alpha(\alpha-1)p_1p_3 + \frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)p_1^4}{24} \\ &\qquad + \frac{\alpha(\alpha-1)p_2^2}{2} + \frac{\alpha(\alpha-1)(\alpha-2)p_1^2p_2}{2}, \end{aligned} \tag{11}$$

$$\begin{aligned} &35a_6 - 5a_2a_5 - 5a_3a_4 + 5a_2^2a_4 \\ &\quad + 5a_2a_3^2 - 5a_2^3a_3 + a_2^5 = \alpha p_5 + \frac{\alpha(\alpha-1)(\alpha-2)p_1^2p_3}{2} + \alpha(\alpha-1)p_1p_4 \\ &\qquad + \frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)p_1^5}{120} \\ &\qquad + \frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)p_1^3p_2}{6} + \alpha(\alpha-1)p_2p_3 \end{aligned}$$

$$+\frac{\alpha(\alpha-1)(\alpha-2)p_1p_2^2}{2}, \tag{12}$$

$$\begin{aligned}
&48a_7-6a_2a_6-6a_3a_5+6a_2^2a_5\\
&-3a_4^2+12a_2a_3a_4-6a_2^3a_4+2a_3^3\\
&-9a_2^2a_3^2+6a_2^4a_3-a_2^6=\alpha p_6+\alpha(\alpha-1)p_1p_5+\alpha(\alpha-1)p_2p_4\\
&+\frac{\alpha(\alpha-1)p_3^2}{2}+\frac{\alpha(\alpha-1)(\alpha-2)p_1^2p_4}{2}\\
&+\alpha(\alpha-1)(\alpha-2)p_1p_2p_3+\frac{\alpha(\alpha-1)(\alpha-2)p_2^3}{6}\\
&+\frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)p_1^4p_2}{24}\\
&+\frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)p_1^3p_3}{6}\\
&+\frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)(\alpha-5)p_1^6}{720}\\
&+\frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)p_1^2p_2^2}{4}.
\end{aligned} \tag{13}$$

Similarly, substituting Eq. (2), $g'(w)$, $g''(w)$ and Eq. (7) into Eq. (5), and equating the coefficients of like powers of $w$, we obtain

$$-3a_2=\alpha q_1, \tag{14}$$

$$15a_2^2-8a_3=\alpha q_2+\frac{\alpha(\alpha-1)q_1^2}{2}, \tag{15}$$

$$-70a_2^3+72a_2a_3-15a_4=\alpha q_3+\alpha(\alpha-1)q_1q_2+\frac{\alpha(\alpha-1)(\alpha-2)q_1^3}{6}, \tag{16}$$

$$\begin{aligned}
&315a_2^4-480a_2^2a_3+140a_2a_4\\
&+70a_3^2-24a_5=\alpha q_4+\alpha(\alpha-1)q_1q_3+\frac{\alpha(\alpha-1)(\alpha-2)q_1^2q_2}{2}\\
&+\frac{\alpha(\alpha-1)q_2^2}{2}+\frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)q_1^4}{24},
\end{aligned} \tag{17}$$

$$\begin{aligned}
&-1330a_2^5+2740a_2^3a_3-945a_2^2a_4\\
&-945a_2a_3^2+240a_2a_5+240a_3a_4-35a_6=\alpha q_5+\alpha(\alpha-1)q_1q_4+\alpha(\alpha-1)q_2q_3\\
&+\frac{\alpha(\alpha-1)(\alpha-2)q_1^2q_3}{2}+\frac{\alpha(\alpha-1)(\alpha-2)q_1q_2^2}{2}\\
&+\frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)q_1^3q_2}{6}\\
&+\frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)q_1^5}{120},
\end{aligned} \tag{18}$$

$$\begin{aligned}
&6062a_2^6-16060a_2^4a_3+5544a_2^3a_4+8896a_2^2a_3^2\\
&-3360a_2a_3a_4-172a_2^2a_5+378a_2a_6-630a_3^3\\
&+402a_3a_5+189a_4^2-48a_7=\alpha q_6+\alpha(\alpha-1)q_1q_5+\alpha(\alpha-1)q_2q_4\\
&+\frac{\alpha(\alpha-1)(\alpha-2)q_1^2q_4}{2}+\alpha(\alpha-1)(\alpha-2)q_1q_2q_3\\
&+\frac{\alpha(\alpha-1)q_3^2}{2}+\frac{\alpha(\alpha-1)(\alpha-2)q_2^3}{6}\\
&+\frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)q_1^3q_3}{6}
\end{aligned}$$

$$+\frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)q_1^2q_2^2}{4}$$
$$+\frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)q_1^4q_2}{24}$$
$$+\frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)(\alpha-5)q_1^6}{720}. \tag{19}$$

Combining Eq. (8) and Eq. (14), we obtain

$$\frac{\alpha p_1}{3}=a_2=-\frac{\alpha q_1}{3}, \tag{20}$$

which implies that $p_1=-q_1$. Next, from Eq. (9) and Eq. (15), we obtain

$$a_3=\frac{\alpha^2p_1^2}{9}+\frac{\alpha(p_2-q_2)}{16}. \tag{21}$$

Furthermore, using Eq. (10) and Eq. (16), we get

$$a_4=\frac{2\alpha^3p_1^3}{405}+\frac{5\alpha^2p_1(p_2-q_2)}{96}+\frac{\alpha(p_3-q_3)}{30}+\frac{\alpha(\alpha-1)p_1(p_2+q_2)}{30}+\frac{\alpha(\alpha-1)(\alpha-2)p_1^3}{90}. \tag{22}$$

Similarly, combining Eq. (11) and Eq. (17), we obtain

$$\begin{aligned}a_5=&-\frac{8\alpha^4p_1^4}{405}+\frac{5\alpha^3p_1^2(p_2-q_2)}{1728}+\frac{\alpha(p_4-q_4)}{48}+\frac{\alpha^2(\alpha-1)p_1^2(p_2+q_2)}{30}\\&+\frac{\alpha(\alpha-1)(p_2^2-q_2^2)}{96}+\frac{\alpha^2p_1(p_3-q_3)}{30}+\frac{\alpha(\alpha-1)(\alpha-2)p_1^2(p_2-q_2)}{96}\\&+\frac{\alpha^2(\alpha-1)(\alpha-2)p_1^4}{90}+\frac{3\alpha^2(p_2-q_2)^2}{512}+\frac{\alpha(\alpha-1)p_1(p_3+q_3)}{48}.\end{aligned} \tag{23}$$

Moreover, from Eq. (12) and Eq. (18), we get

$$\begin{aligned}a_6=&-\frac{11\alpha^5p_1^5}{8505}+\frac{\alpha(p_5-q_5)}{70}-\frac{3131\alpha^4p_1^3(p_2-q_2)}{120960}+\frac{7\alpha^2(p_2-q_2)(p_3-q_3)}{960}+\frac{7\alpha^2p_1(p_4-q_4)}{288}\\&+\frac{\alpha(\alpha-1)(p_2p_3-q_2q_3)}{70}-\frac{4\alpha^3(\alpha-1)p_1^3(p_2+q_2)}{567}+\frac{\alpha(\alpha-1)(\alpha-2)p_1(p_2^2+q_2^2)}{140}\\&+\frac{\alpha(\alpha-1)p_1(p_4+q_4)}{70}+\frac{\alpha^3p_1(p_2-q_2)^2}{1792}+\frac{\alpha^3p_1^2(p_3-q_3)}{630}+\frac{7\alpha^2(\alpha-1)p_1^2(p_3+q_3)}{288}\\&+\frac{7\alpha^2(\alpha-1)p_1(p_2^2-q_2^2)}{576}+\frac{7\alpha^2(\alpha-1)p_1(p_2+q_2)(p_2-q_2)}{960}+\frac{\alpha(\alpha-1)(\alpha-2)p_1^2(p_3-q_3)}{140}\\&-\frac{43\alpha^3(\alpha-1)(\alpha-2)p_1^5}{11340}+\frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)p_1^3(p_2+q_2)}{420}\\&+\frac{7\alpha^2(\alpha-1)(\alpha-2)p_1^3(p_2-q_2)}{480}+\frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)p_1^5}{4200}.\end{aligned} \tag{24}$$

Finally, combining Eq. (13) and Eq. (19), we obtain

$$\begin{aligned}a_7=&-\frac{15991\alpha^6p_1^6}{4592700}+\frac{\alpha(p_6-q_6)}{96}+\frac{2\alpha^2p_1(p_5-q_5)}{105}-\frac{901597\alpha^5p_1^4(p_2-q_2)}{52254720}+\frac{1111\alpha^4p_1^3(p_3-q_3)}{544320}\\&-\frac{225737\alpha^4p_1^2(p_2-q_2)^2}{18579456}+\frac{\alpha(\alpha-1)p_1(p_5+q_5)}{96}+\frac{17\alpha^3p_1^2(p_4-q_4)}{41472}+\frac{\alpha(\alpha-1)(p_3^2-q_3^2)}{192}\\&+\frac{2\alpha^2(\alpha-1)p_1(p_2p_3-q_2q_3)}{105}-\frac{102593\alpha^4(\alpha-1)(\alpha-2)p_1^6}{8164800}+\frac{\alpha(\alpha-1)(\alpha-2)p_1^2(p_4-q_4)}{192}\\&+\frac{52\alpha^2(\alpha-1)(\alpha-2)p_1^3(p_3-q_3)}{4725}+\frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)p_1^2(p_2-q_2)}{384}-\frac{43\alpha^3(p_2-q_2)^3}{786432}\end{aligned}$$

$$
\begin{aligned}
&+\frac{\alpha^2(\alpha-1)^2(\alpha-2)p_1^4(p_2+q_2)}{675}+\frac{\alpha(\alpha-1)(p_2p_4-q_2q_4)}{96}+\frac{407\alpha^2(p_2-q_2)(p_4-q_4)}{73728}\\
&+\frac{\alpha(\alpha-1)(\alpha-2)p_1(p_2p_3+q_2q_3)}{96}+\frac{17\alpha^3(\alpha-1)p_1^2(p_2+q_2)(p_2-q_2)}{15360}+\frac{\alpha^2(p_3-q_3)^2}{450}\\
&+\frac{\alpha(\alpha-1)(\alpha-2)(p_2^3-q_2^3)}{576}+\frac{\alpha^2(\alpha-1)^2(\alpha-2)^2p_1^6}{4050}+\frac{\alpha^2(\alpha-1)(\alpha-2)p_1^2(p_2^2+q_2^2)}{105}\\
&-\frac{1073\alpha^4(\alpha-1)p_1^4(p_2+q_2)}{33600}+\frac{2\alpha^2(\alpha-1)p_1^2(p_4+q_4)}{105}+\frac{17\alpha^3(\alpha-1)p_1^2(p_2^2-q_2^2)}{82944}\\
&+\frac{119\alpha^3(\alpha-1)(\alpha-2)p_1^4(p_2-q_2)}{207360}+\frac{\alpha^2(\alpha-1)(\alpha-2)(\alpha-3)p_1^4(p_2+q_2)}{315}\\
&+\frac{\alpha^2(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)p_1^6}{3150}+\frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)p_1^3(p_3+q_3)}{576}\\
&+\frac{17\alpha^3(\alpha-1)p_1^3(p_3+q_3)}{41472}+\frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)p_1^4(p_2-q_2)}{2304}\\
&+\frac{407\alpha^2(\alpha-1)p_1(p_2-q_2)(p_3+q_3)}{73728}+\frac{407\alpha^2(\alpha-1)(\alpha-2)p_1^2(p_2-q_2)^2}{147456}\\
&+\frac{\alpha^2(\alpha-1)p_1(p_2+q_2)(p_3-q_3)}{225}+\frac{407\alpha^2(\alpha-1)(p_2-q_2)(p_2^2-q_2^2)}{147456}\\
&+\frac{17\alpha^3p_1(p_2-q_2)(p_3-q_3)}{15360}+\frac{\alpha^2(\alpha-1)^2p_1^2(p_2+q_2)^2}{450}.
\end{aligned}
\tag{25}
$$

Next, applying Lemma 2 to the functions $p$ and $q$, for some complex numbers $x, y, \zeta$, and $\eta$ satisfying $|x|\le 1, |y|\le 1, |\zeta|\le 1, |\eta|\le 1$. Using the relation $p_1=-q_1$, we further obtain

$$
\begin{aligned}
p_2-q_2&=\frac{(4-p_1^2)}{2}(x-y),\quad p_2+q_2=p_1^2+\frac{4-p_1^2}{2}(x+y),\\
p_3-q_3&=\frac{p_1^3}{2}+\frac{p_1(4-p_1^2)}{2}(x+y)-\frac{p_1(4-p_1^2)}{4}(x^2+y^2)+\frac{4-p_1^2}{2}\left[(1-|x|^2)\zeta-(1-|y|^2)\eta\right],\\
p_3+q_3&=\frac{p_1(4-p_1^2)}{2}(x-y)-\frac{p_1(4-p_1^2)}{4}(x^2-y^2)+\frac{4-p_1^2}{2}\left[(1-|x|^2)\zeta+(1-|y|^2)\eta\right].
\end{aligned}
$$

The above relations will be used repeatedly in deriving the coefficient estimates presented in the following theorem, which provides upper bounds for the initial coefficients of functions belonging to the class $\mathbf{B}_{\Sigma}^{*}(\alpha)$.

**Theorem 1.** *Let $f(z)$ be given by Eq. (1) and suppose that $f\in\mathbf{B}_{\Sigma}^{*}(\alpha)$, $0<\alpha\le 1$. Then the following coefficient bounds hold:*

$$
\begin{aligned}
|a_2|&\le\frac{2\alpha}{3},\\
|a_3|&\le\begin{cases}\frac{\alpha}{4}, & 0<\alpha\le\frac{9}{16},\\ \frac{4\alpha^2}{9}, & \frac{9}{16}<\alpha\le 1,\end{cases}\\
|a_4|&\le\alpha\left(\frac{16\alpha^2}{405}+\frac{2}{15}+\frac{4(1-\alpha)}{15}+\frac{4(1-\alpha)(2-\alpha)}{45}\right),\\
|a_5|&\le\alpha\left(\frac{128\alpha^3}{405}+\frac{4\alpha}{15}+\frac{8\alpha(1-\alpha)}{15}+\frac{8\alpha(1-\alpha)(2-\alpha)}{45}+\frac{1}{12}\right),\\
|a_6|&\le\alpha\left(\frac{2}{35}+\frac{352\alpha^4}{8505}+\frac{8\alpha^2}{315}+\frac{7\alpha}{36}+\frac{128\alpha^2(1-\alpha)}{567}+\frac{344\alpha^2(1-\alpha)(2-\alpha)}{2835}+\frac{8(1-\alpha)}{35}\right.\\
&\qquad\left.+\frac{8(1-\alpha)(2-\alpha)}{35}+\frac{8(1-\alpha)(2-\alpha)(3-\alpha)}{105}+\frac{4(1-\alpha)(2-\alpha)(3-\alpha)(4-\alpha)}{525}\right),
\end{aligned}
$$

$$|a_7| \le \alpha\Bigg(\frac{1}{24} + \frac{(1-\alpha)(2-\alpha)}{4} + \frac{64(1-\alpha)^2(2-\alpha)}{675} + \frac{32(1-\alpha)^2(2-\alpha)^2}{2025} + \frac{(1-\alpha)}{6}$$
$$+ \frac{1111\alpha^3}{13608} + \frac{16\alpha}{105} + \frac{64\alpha(1-\alpha)}{105} + \frac{17\alpha^2}{2592} + \frac{1073\alpha^3(1-\alpha)}{525} + \frac{255856\alpha^5}{1148175}\Bigg).$$

*Proof.* Using Eq. (20) and the fact that $|p_1| \le 2$ for $p \in P$, we obtain

$$|a_2| = \left|\frac{\alpha p_1}{3}\right| \le \frac{\alpha |p_1|}{3} \le \frac{2\alpha}{3}.$$

Next, we estimate the coefficient $|a_3|$. From Eq. (21), together with Lemma (2), we have

$$\begin{aligned} |a_3| &= \left|\frac{\alpha^2 p_1^2}{9} + \frac{\alpha(p_2 - q_2)}{16}\right| \\ &\le \frac{\alpha^2|p_1|^2}{9} + \frac{\alpha(4-p_1^2)}{32}(|x|+|y|). \end{aligned}$$

Since $|x| \le 1$ and $|y| \le 1$, it follows that

$$|a_3| \le \frac{\alpha^2|p_1|^2}{9} + \frac{\alpha(4-p_1^2)}{16}.$$

Let $t = |p_1|$, where $0 \le t \le 2$. Then

$$|a_3| \le f(t) := \frac{\alpha}{4} + \alpha t^2\left(\frac{\alpha}{9} - \frac{1}{16}\right).$$

If $0 < \alpha \le \frac{9}{16}$, then $\frac{\alpha}{9} - \frac{1}{16} \le 0$, and hence $f(t)$ attains its maximum at $t = 0$. Therefore,

$$|a_3| \le \frac{\alpha}{4}.$$

If $\frac{9}{16} \le \alpha \le 1$, then $\frac{\alpha}{9} - \frac{1}{16} \ge 0$, and hence $f(t)$ attains its maximum at $t = 2$. Therefore,

$$|a_3| \le \frac{4\alpha^2}{9}.$$

Consequently,

$$|a_3| \le \begin{cases} \dfrac{\alpha}{4}, & 0 < \alpha \le \dfrac{9}{16}, \\ \dfrac{4\alpha^2}{9}, & \dfrac{9}{16} \le \alpha \le 1. \end{cases}$$

Next, we estimate the coefficient $|a_4$. From Eq. (22), together with Lemma (2) and the triangle inequality, we obtain

$$\begin{aligned} |a_4| \le & \frac{2\alpha|p_1|^3}{405} + \frac{\alpha|p_1|^3}{60} + \frac{\alpha(4-p_1^2)}{30} + \frac{\alpha(1-\alpha)|p_1|^3}{30} + \frac{\alpha(1-\alpha)(2-\alpha)|p_1|^3}{90} \\ & + \left(\frac{\alpha|p_1|(4-p_1^2)}{60} + \frac{5\alpha^2|p_1|(4-p_1^2)}{192} + \frac{\alpha(1-\alpha)|p_1|(4-p_1^2)}{60}\right)(|x|+|y|) \\ & + \left(\frac{\alpha|p_1|(4-p_1^2)}{120} - \frac{\alpha(4-p_1^2)}{60}\right)(|x|^2+|y|^2). \end{aligned}$$

Hence,

$$|a_4| \leq B_1 + B_2(\mu_1 + \mu_2) + B_3(\mu_1^2 + \mu_2^2),$$

where

$$\begin{aligned}
B_1 &= \frac{2\alpha|p|^3}{405} + \frac{\alpha|p|^3}{60} + \frac{\alpha(4-p^2)}{30} + \frac{\alpha(1-\alpha)|p|^3}{30} + \frac{\alpha(1-\alpha)(2-\alpha)|p|^3}{90} \geq 0,\\
B_2 &= \frac{5\alpha^2|p|(4-p^2)}{192} + \frac{\alpha|p|(4-p^2)}{60} + \frac{\alpha(1-\alpha)|p|(4-p^2)}{60} \geq 0,\\
B_3 &= \frac{\alpha(|p|-2)(4-p^2)}{120} \leq 0.
\end{aligned}$$

Since $B(\mu_1, \mu_2)$ is a polynomial function in $\mu_1$ and $\mu_2$, it is continuous on $\mathbb{R}^2$. Moreover, the set $\mathbb{S} = \{(\mu_1, \mu_2) : 0 \leq \mu_1 \leq 1,\ 0 \leq \mu_2 \leq 1\}$ is closed and bounded, and hence compact. Therefore, by the Extreme Value Theorem, $B(\mu_1, \mu_2)$ attains a global maximum on $\mathbb{S}$. To determine the location of this maximum, we first investigate the possibility of a maximum point in the interior of $\mathbb{S}$. The first-order partial derivatives of $B(\mu_1, \mu_2)$ are given by

$$\begin{aligned}
B_{\mu_1} &= B_2 + 2B_3\mu_1,\\
B_{\mu_2} &= B_2 + 2B_3\mu_2.
\end{aligned}$$

By Fermat's theorem, any interior extremum of $B(\mu_1, \mu_2)$ must satisfy $\frac{\partial B}{\partial \mu_1} = 0$, and $\frac{\partial B}{\partial \mu_2} = 0$. A direct computation yields $)2B_3(\mu_1 - \mu_2) = 0$, and hence every interior critical point satisfies $\mu_1 = \mu_2$. Setting $\mu_1 = \mu_2 = \mu$, we obtain

$$B(\mu, \mu) = B_1 + 2B_2\mu + 2B_3\mu^2.$$

To determine the nature of the critical point, we apply the second derivative test. Since

$$B_{\mu_1\mu_1} = 2B_3, \qquad B_{\mu_1\mu_2} = 0, \qquad B_{\mu_2\mu_2} = 2B_3,$$

we have

$$D = B_{\mu_1\mu_1}B_{\mu_2\mu_2} - (B_{\mu_1\mu_2})^2 = 4B_3^2.$$

Since $B_3 < 0$, it follows that $D > 0$ and $B_{\mu_1\mu_1} < 0$. Therefore, the critical point is a local maximum. Since $B$ is continuous on the compact set $\mathbb{S}$, its global maximum is attained either at an interior critical point or on the boundary of $\mathbb{S}$. Consequently, it remains to examine the boundary values.

For $\mu_1 = 0$ and $0 \leq \mu_2 \leq 1$,

$$B(0, \mu_2) = B_1 + B_2\mu_2 + B_3\mu_2^2,$$

with

$$B_{\mu_2}(0, \mu_2) = B_2 + 2B_3\mu_2.$$

Similarly, for $\mu_1 = 1$,

$$B(1, \mu_2) = B_1 + B_2 + B_2\mu_2 + B_3 + B_3\mu_2^2,$$

and

$$B_{\mu_2}(1, \mu_2) = B_2 + 2B_3\mu_2.$$

Hence $B(1, \mu_2)$ is increasing on $[0, 1]$, and therefore

$$\max_{0 \leq \mu_2 \leq 1} B(1, \mu_2) = B(1, 1) = B_1 + 2(B_2 + B_3) =: \mathcal{B}(p).$$

Substituting the expressions for $B_1$, $B_2$, and $B_3$, we obtain

$$\begin{aligned}\mathcal{B}(p) = \frac{2\alpha^3|p|^3}{405} + \frac{\alpha|p|^3}{60} + \frac{\alpha(4-p^2)}{30} + \frac{\alpha(1-\alpha)|p|^3}{30} + \frac{\alpha(1-\alpha)(2-\alpha)|p|^3}{90} \\ + 2\left(\frac{5\alpha^2|p|(4-p^2)}{192} + \frac{\alpha|p|(4-p^2)}{60} + \frac{\alpha(1-\alpha)|p|(4-p^2)}{60} + \frac{\alpha(|p|-2)(4-p^2)}{120}\right).\end{aligned}$$

Since $\mathcal{B}(p)$ is increasing on $0 \leq |p| \leq 2$, its maximum is attained at $|p| = 2$. Hence,

$$|a_4| \leq \mathcal{B}(2) = \alpha\left(\frac{16\alpha^2}{405} + \frac{2}{15} + \frac{4(1-\alpha)}{15} + \frac{4(1-\alpha)(2-\alpha)}{45}\right).$$

Thus, the desired upper bound for $|a_4|$ is obtained. Next, we estimate the coefficient $|a_5|$. From Eq. (23), together with Lemma 2 and the triangle inequality, we obtain

$$\begin{aligned}|a_5| \leq{} & \frac{\alpha}{12} + \frac{\alpha^2|p_1|^4}{60} + \frac{\alpha^2|p_1|(4-p_1^2)}{30} + \frac{\alpha^2(1-\alpha)|p_1|^4}{30} + \frac{\alpha^2(1-\alpha)(2-\alpha)|p_1|^4}{90} \\ & \frac{8\alpha^4|p_1|^4}{405} + \frac{\alpha(1-\alpha)|p_1|(4-p_1^2)}{48} + \left(\frac{5\alpha^3|p_1|^2(4-p_1^2)}{1728} + \frac{\alpha^2|p_1|^2(4-p_1^2)}{60}\right. \\ & + \frac{\alpha(1-\alpha)|p_1|^2(4-p_1^2)}{64} + \frac{\alpha(1-\alpha)(2-\alpha)|p_1|^2(4-p_1^2)}{192} \\ & \left. + \frac{\alpha^2(1-\alpha)|p_1|^2(4-p_1^2)}{60}\right)(|x|+|y|) + \left(\frac{\alpha^2|p_1|^2(4-p_1^2)}{120} - \frac{\alpha^2|p_1|(4-p_1^2)}{60}\right. \\ & \left. - \frac{\alpha(1-\alpha)|p_1|(4-p_1^2)}{96}\right)(|x|^2+|y|^2) + \left(\frac{3\alpha^2(4-p_1^2)^2}{2048} + \frac{\alpha(1-\alpha)|p_1|^2(4-p_1^2)}{192}\right. \\ & \left. + \frac{\alpha(1-\alpha)(4-p_1^2)^2}{384}\right)(|x|+|y|)^2.\end{aligned}$$

Hence,

$$|a_5| \leq C_1 + C_2(\mu_1+\mu_2) + C_3(\mu_1^2+\mu_2^2) + C_4(\mu_1+\mu_2)^2 =: C(\mu_1,\mu_2),$$

where

$$\begin{aligned}C_1 ={} & \frac{\alpha^2|p|^4}{60} + \frac{\alpha^2|p|(4-p^2)}{30} + \frac{\alpha^2(1-\alpha)|p|^4}{30} + \frac{\alpha^2(1-\alpha)(2-\alpha)|p|^4}{90} \\ & + \frac{\alpha}{12} + \frac{8\alpha^4|p|^4}{405} + \frac{\alpha(1-\alpha)|p|(4-p^2)}{48} \geq 0, \\ C_2 ={} & \frac{5\alpha^3|p|^2(4-p^2)}{1728} + \frac{\alpha^2|p|^2(4-p^2)}{60} + \frac{\alpha(1-\alpha)(2-\alpha)|p|^2(4-p^2)}{192} \\ & + \frac{\alpha^2(1-\alpha)|p|^2(4-p^2)}{60} + \frac{\alpha(1-\alpha)|p|^2(4-p^2)}{64} \geq 0, \\ C_3 ={} & \frac{\alpha^2|p|(|p|-2)(4-p^2)}{120} - \frac{\alpha(1-\alpha)|p|(4-p^2)}{96} \leq 0, \\ C_4 ={} & \frac{3\alpha^2(4-p^2)^2}{2048} + \frac{\alpha(1-\alpha)|p|^2(4-p^2)}{192} + \frac{\alpha(1-\alpha)(4-p^2)^2}{384} \geq 0.\end{aligned}$$

By Lemma 2, it remains to determine the maximum value of $C(\mu_1,\mu_2)$ on $\mathbb{S} = \{(\mu_1,\mu_2) : 0 \leq \mu_1 \leq 1,\ 0 \leq \mu_2 \leq 1\}$. Since $C$ is continuous on the compact set $\mathbb{S}$, it attains a global maximum on $\mathbb{S}$. We first investigate the existence of interior critical points by computing the first-order partial derivatives, and by Fermat's theorem, any interior critical point of $C(\mu_1, \mu_2)$ satisfies

$$\frac{\partial C}{\partial \mu_1} = C_2 + 2(C_3 + C_4)\mu_1 + 2C_4\mu_2 = 0$$
$$\frac{\partial C}{\partial \mu_2} = C_2 + 2(C_3 + C_4)\mu_2 + 2C_4\mu_1 = 0.$$

Solving these equations gives

$$2C_3(\mu_1 - \mu_2) = 0,$$

so that $\mu_1 = \mu_2$. Letting $\mu_1 = \mu_2 = \mu$, we obtain

$$C(\mu, \mu) = C_1 + 2C_2\mu + 2(C_3 + 2C_4)\mu^2.$$

To determine whether the critical point corresponds to a local maximum, we apply the second derivative test. The second-order partial derivatives of $C(\mu_1, \mu_2)$ are

$$C_{\mu_1\mu_1} = 2(C_3 + C_4),$$
$$C_{\mu_1\mu_2} = 2C_4,$$
$$C_{\mu_2\mu_2} = 2(C_3 + C_4).$$

Hence,

$$D = 4C_3(C_3 + 2C_4).$$

Since $C_3 \leq 0$ and $C_4 \geq 0$, the sign of $D$ cannot be determined. Therefore, the existence of an interior maximum cannot be guaranteed. Since $C(\mu_1, \mu_2)$ is continuous on the compact set $\mathbb{S}$, its global maximum must occur either at an interior critical point or on the boundary of $\mathbb{S}$. Consequently, the boundary curves are examined.

For the side $\mu_1 = 0$ $(0 \leq \mu_2 \leq 1)$, we have

$$C(0, \mu_2) = C_1 + C_2\mu_2 + (C_3 + C_4)\mu_2^2,$$
$$C_{\mu_2}(0, \mu_2) = C_2 + 2(C_3 + C_4)\mu_2.$$

Thus, $C(0, \mu_2)$ is increasing on $[0, 1]$, and hence

$$\max_{0\leq\mu_2\leq1} C(0, \mu_2) = C(0, 1).$$

Therefore, the maximum on the side $\mu_1 = 0$ is attained at $\mu_2 = 1$.

$$\max C(0, \mu_2) = C(0, 1) = C_1 + C_2 + C_3 + C_4.$$

Next, we examine the boundary of $\mathbb{S} = [0, 1] \times [0, 1]$. For $\mu_1 = 0$, we have

$$C(0, \mu_2) = C_1 + C_2\mu_2 + (C_3 + C_4)\mu_2^2, \qquad 0 \leq \mu_2 \leq 1.$$

Since $C(0, \mu_2)$ is a quadratic polynomial on the compact interval $[0, 1]$, its maximum is attained either at a critical point or at an endpoint. Hence,

$$\max_{0\leq\mu_2\leq1} C(0, \mu_2) = \max\{C(0, 0),\ C(0, 1)\}.$$

Similarly, for $\mu_1 = 1$,

$$C(1, \mu_2) = C_1 + C_2 + C_3 + C_4 + (C_2 + 2C_4)\mu_2 + (C_3 + C_4)\mu_2^2,$$

and therefore

$$\max_{0\le\mu_2\le 1} C(1,\mu_2) = \max\{C(1,0),\, C(1,1)\}.$$

By the same argument, the maximum values on the sides $\mu_2 = 0$ and $\mu_2 = 1$ are also attained at the endpoints. Consequently,

$$\max_{(\mu_1,\mu_2)\in[0,1]^2} C(\mu_1,\mu_2) = \max\{C(0,0),\, C(1,0),\, C(0,1),\, C(1,1)\}.$$

Evaluating $C$ at the vertices of $\mathbb{S}$ yields. Since

$$C(1,1) \ge C(0,1) = C(1,0) \ge C(0,0),$$

it follows that the global maximum of $C(\mu_1,\mu_2)$ on $[0,1]^2$ is attained at $(1,1)$. Therefore,

$$\max_{(\mu_1,\mu_2)\in[0,1]^2} C(\mu_1,\mu_2) = C(1,1) = C_1 + 2(C_2+C_3) + 4C_4 = \Lambda(p).$$

Substituting the expressions for $C_1, C_2, C_3$, and $C_4$ yields $|a_5| \le \Lambda(p)$. Since $\Lambda(p)$ is an increasing function of $|p|$ on $[0,2]$, its maximum is attained at $|p| = 2$. Therefore,

$$|a_5| \le \Lambda(2) = \alpha\left(\frac{128\alpha^3}{405} + \frac{4\alpha}{15} + \frac{8\alpha(1-\alpha)}{15} + \frac{8\alpha(1-\alpha)(2-\alpha)}{45} + \frac{1}{12}\right).$$

Thus, the desired upper bound for $|a_5|$ is obtained. Next, we estimate the coefficient $|a_6|$. From Eq. (24), together with Lemma 2 and the triangle inequality, we obtain

$$\begin{aligned}
|a_6| \le\ & \frac{11\alpha^5|p_1|^5}{8505} + \frac{\alpha^3|p_1|^5}{1260} + \frac{\alpha^3|p_1|^2(4-p_1^2)}{630} + \frac{2\alpha(1-\alpha)|p_1|}{35} + \frac{4\alpha(1-\alpha)}{35}\\
&+ \frac{4\alpha^3(1-\alpha)|p_1|^5}{567} + \frac{7\alpha^2(1-\alpha)|p_1|^2(4-p_1^2)}{288} + \frac{\alpha(1-\alpha)(2-\alpha)|p_1|^5}{280}\\
&+ \frac{\alpha(1-\alpha)(2-\alpha)|p_1|^2(4-p_1^2)}{140} + \frac{\alpha(1-\alpha)(2-\alpha)(3-\alpha)(4-\alpha)|p_1|^5}{4200}\\
&+ \frac{2\alpha}{35} + \frac{43\alpha^3(1-\alpha)(2-\alpha)|p_1|^5}{11340} + \frac{\alpha(1-\alpha)(2-\alpha)(3-\alpha)|p_1|^5}{420}\\
&+ \frac{2\alpha(1-\alpha)(2-\alpha)|p_1|}{35} + \left(\frac{3131\alpha^4|p_1|^3(4-p_1^2)}{12960} + \frac{\alpha^3|p_1|^3(4-p_1^2)}{1260}\right.\\
&+ \frac{7\alpha^2(4-p_1^2)^2}{1920} + \frac{4\alpha^3(1-\alpha)|p_1|^3(4-p_1^2)}{1134} + \frac{7\alpha^2(1-\alpha)|p_1|^3(4-p_1^2)}{320}\\
&+ \frac{\alpha(1-\alpha)(2-\alpha)(3-\alpha)|p_1|^3(4-p_1^2)}{840} + \frac{7\alpha^2(1-\alpha)(2-\alpha)|p_1|^3(4-p_1^2)}{960}\\
&\left.+ \frac{\alpha(1-\alpha)(2-\alpha)|p_1|^3(4-p_1^2)}{280} + \frac{7\alpha^2|p_1|^3(4-p_1^2)}{3840}\right)(|x|+|y|)\\
&+ \left(\frac{\alpha^3|p_1|^3(4-p_1^2)}{2520} - \frac{\alpha^3|p_1|^2(4-p_1^2)}{1260} - \frac{\alpha(1-\alpha)(2-\alpha)|p_1|^2(4-p_1^2)}{280}\right.\\
&\left.- \frac{7\alpha^2(1-\alpha)|p_1|^2(4-p_1^2)}{576} + \frac{\alpha(1-\alpha)(2-\alpha)|p_1|^3(4-p_1^2)}{560}\right)(|x|^2+|y|^2)\\
&+ \left(\frac{\alpha^3|p_1|(4-p_1^2)^2}{7168} + \frac{7\alpha^2|p_1|(4-p_1^2)}{3840} + \frac{7\alpha^2(1-\alpha)|p_1|^3(4-p_1^2)}{1152}\right.
\end{aligned}$$

$$+\frac{7\alpha^2(1-\alpha)|p_1|(4-p_1^2)^2}{1440}\Bigg)(|x|+|y|)^2+\Bigg(\frac{7\alpha^2|p_1|(4-p_1^2)^2}{7680}$$
$$-\frac{7\alpha^2(4-p_1^2)^2}{3840}\Bigg)(|x|+|y|)(|x|^2+|y|^2)$$

Hence,

$$|a_6| \leq E_1+E_2(\mu_1+\mu_2)+E_3(\mu_1^2+\mu_2^2)+E_4(\mu_1+\mu_2)^2+E_5(\mu_1+\mu_2)(\mu_1^2+\mu_2^2) =: E(\mu_1,\mu_2),$$

where

$$\begin{aligned}
E_1 =\ & \frac{2\alpha}{35}+\frac{11\alpha^5|p|^5}{8505}+\frac{\alpha^3|p|^5}{1260}+\frac{\alpha^3|p|^2(4-p^2)}{630}+\frac{7\alpha^2|p|}{72}+\frac{2\alpha(1-\alpha)|p|}{35}\\
&+\frac{4\alpha^3(1-\alpha)|p|^5}{567}+\frac{7\alpha^2(1-\alpha)|p|^2(4-p^2)}{288}+\frac{\alpha(1-\alpha)(2-\alpha)|p|^5}{280}\\
&+\frac{4\alpha(1-\alpha)}{35}+\frac{\alpha(1-\alpha)(2-\alpha)|p|^2(4-p^2)}{140}+\frac{2\alpha(1-\alpha)(2-\alpha)|p|}{35}\\
&+\frac{43\alpha^3(1-\alpha)(2-\alpha)|p|^5}{11340}+\frac{\alpha(1-\alpha)(2-\alpha)(3-\alpha)(4-\alpha)|p|^5}{4200}\\
&+\frac{\alpha(1-\alpha)(2-\alpha)(3-\alpha)|p|^5}{420} \geqslant 0,\\
E_2 =\ & \frac{3131\alpha^4|p|^3(4-p^2)}{12960}+\frac{\alpha^3|p|^3(4-p^2)}{1260}+\frac{7\alpha^2|p|^3(4-p^2)}{3840}+\frac{7\alpha^2(4-p^2)^2}{1920}\\
&+\frac{7\alpha^2(1-\alpha)(2-\alpha)|p|^3(4-p^2)}{960}+\frac{\alpha(1-\alpha)(2-\alpha)(3-\alpha)|p|^3(4-p^2)}{840}\\
&+\frac{\alpha(1-\alpha)(2-\alpha)|p|^3(4-p^2)}{280}+\frac{7\alpha^2(1-\alpha)|p|^3(4-p^2)}{320}\\
&+\frac{4\alpha^3(1-\alpha)|p|^3(4-p^2)}{1134} \geqslant 0,\\
E_3 =\ & \frac{\alpha^3(|p|-2)|p|^2(4-p^2)}{2520}-\frac{7\alpha^2(1-\alpha)|p|^2(4-p^2)}{576}\\
&+\frac{\alpha(1-\alpha)(2-\alpha)(|p|-2)|p|^2(4-p^2)}{560} \leqslant 0,\\
E_4 =\ & \frac{\alpha^3|p|(4-p^2)^2}{7168}+\frac{7\alpha^2|p|(4-p^2)}{3840}+\frac{7\alpha^2(1-\alpha)|p|^3(4-p^2)}{1152}\\
&+\frac{7\alpha^2(1-\alpha)|p|(4-p^2)^2}{1440} \geqslant 0,\\
E_5 =\ & \frac{7\alpha^2(|p|-2)(4-p^2)^2}{7680} \leqslant 0.
\end{aligned}$$

By Lemma 2, we seek the maximum value of $E(\mu_1,\mu_2)$ on $\mathbb{S}=[0,1]\times[0,1]$. Since $E$ is a polynomial, it is continuous on $\mathbb{S}$. Therefore, by the Extreme Value Theorem, $E$ attains a global maximum on $\mathbb{S}$. To determine where this maximum occurs, we first examine the critical points in the interior of $\mathbb{S}$. The first-order partial derivatives are

$$\begin{aligned}
E_{\mu_1} &= E_2+2E_3\mu_1+2E_4\mu_1+2E_4\mu_2+3E_5\mu_1^2+E_5\mu_2^2+2E_5\mu_1\mu_2,\\
E_{\mu_2} &= E_2+2E_3\mu_2+2E_4\mu_1+2E_4\mu_2+2E_5\mu_1\mu_2+E_5\mu_1^2+3E_5\mu_2^2.
\end{aligned}$$

By Fermat's theorem, any interior local extremum of $E(\mu_1,\mu_2)$ must satisfy $E_{\mu_1}=0$, and $E_{\mu_2}=0$. Subtracting the two equations gives

$$2(\mu_1-\mu_2)\big[E_3+E_5(\mu_1+\mu_2)\big]=0.$$

Since $E_3 \leq 0$ and $E_5 \leq 0$, we have

$$E_3 + E_5(\mu_1 + \mu_2) \leq 0, \qquad (\mu_1, \mu_2) \in [0,1]^2.$$

Hence, the second factor cannot vanish for any interior point of $\mathbb{S}$, and therefore any interior critical point must satisfy $\mu_1 = \mu_2$. Setting $\mu_1 = \mu_2 = \mu$, the function $E$ reduces to

$$E(\mu, \mu) = E_1 + 2E_2\mu + 2E_3\mu^2 + 4E_4\mu^2 + 4E_5\mu^3,$$

whose derivative is

$$E_\mu(\mu, \mu) = 2E_2 + 4(E_3 + 2E_4)\mu + 12E_5\mu^2.$$

Thus, any interior critical point must satisfy

$$2E_2 + 4(E_3 + 2E_4)\mu + 12E_5\mu^2 = 0.$$

Next, we evaluate $E(\mu, \mu)$ at the endpoints of the interval $[0,1]$:

$$\begin{aligned} E(0) &= E_1, \\ E(1) &= E_1 + 2E_2 + 2E_3 + 4E_4 + 4E_5. \end{aligned}$$

Since $E(\mu, \mu)$ is continuous on the compact interval $[0,1]$, its maximum is attained either at a critical point or at an endpoint. Furthermore,

$$E(1) - E(0) = 2E_2 + 2E_3 + 4E_4 + 4E_5.$$

By the assumptions on the coefficients, it follows that $E(1) \geq E(0)$. Hence, among the endpoints, the larger value is attained at $\mu = 1$. Therefore, the maximum value of $E(\mu, \mu)$ is achieved at $\mu = 1$, that is, $\mu_1 = \mu_2 = 1$. Hence,

$$\max E(\mu_1, \mu_2) = E(1,1) = \Delta(p) = E_1 + 2(E_2 + E_3) + 4(E_4 + E_5).$$

Substituting the expressions for $E_1, E_2, E_3, E_4$, and $E_5$ yields $|a_6| \leq \Delta(p)$. Since $\Delta(p)$ is an increasing function of $|p|$ on $[0,2]$, its maximum is attained at $|p| = 2$. Therefore,

$$\begin{aligned} |a_6| \leq \alpha \Bigg( & \frac{2}{35} + \frac{352\alpha^4}{8505} + \frac{8\alpha^2}{315} + \frac{7\alpha}{36} + \frac{8(1-\alpha)}{35} + \frac{128\alpha^2(1-\alpha)}{567} \\ & + \frac{4(1-\alpha)(2-\alpha)(3-\alpha)(4-\alpha)}{525} + \frac{344\alpha^2(1-\alpha)(2-\alpha)}{2835} + \frac{8(1-\alpha)(2-\alpha)(3-\alpha)}{105} \\ & + \frac{8(1-\alpha)(2-\alpha)}{35} \Bigg) \end{aligned}$$

Thus, the desired upper bound for $|a_6|$ is obtained. Next, we estimate the coefficient $|a_7|$. From Eq. (25), together with Lemma 2 and the triangle inequality, we obtain

$$\begin{aligned} |a_7| \leq \quad & \frac{\alpha}{24} + \frac{\alpha(1-\alpha)}{12} + \frac{15991\alpha^6|p_1|^6}{4592700} + \frac{1111\alpha^4|p_1|^6}{1088640} + \frac{\alpha(1-\alpha)(4-p_1^2)^2}{192} \\ & + \frac{\alpha(1-\alpha)|p_1|}{24} + \frac{1111\alpha^4|p_1|^3}{544320} + \frac{17\alpha^3|p_1|^2}{10368} + \frac{\alpha(1-\alpha)|p_1|^3(4-p_1^2)}{384} \\ & + \frac{8\alpha^2(1-\alpha)|p_1|^2}{105} + \frac{17\alpha^3(1-\alpha)|p_1|^3(4-p_1^2)}{41472} + \frac{\alpha(1-\alpha)(2-\alpha)|p_1|}{12} \\ & + \frac{16\alpha^2(1-\alpha)|p_1|}{105} + \frac{\alpha(1-\alpha)(2-\alpha)|p_1|^2}{48} + + \frac{\alpha(1-\alpha)^2(2-\alpha)|p_1|^6}{675} \end{aligned}$$

$$
\begin{aligned}
&+ \frac{8\alpha^2|p_1|}{105} + \frac{1073\alpha^4(1-\alpha)|p_1|^6}{33600} + \frac{\alpha(1-\alpha)^2(2-\alpha)^2|p_1|^6}{4050} \\
&+ \left(\frac{901597\alpha^5|p_1|^4(4-p_1^2)}{104509440} + \frac{1111\alpha^4|p_1|^4(4-p_1^2)}{1088640} + \frac{17\alpha^3|p_1|(4-p_1^2)^2}{30720}\right. \\
&+ \frac{\alpha(1-\alpha)|p_1|^4(4-p_1^2)}{768} + \frac{17\alpha^3|p_1|^4(4-p_1^2)}{61440} + \frac{\alpha^4(1-\alpha)|p_1|^4(4-p_1^2)}{67200} \\
&+ \frac{\alpha(1-\alpha)(2-\alpha)(4-p_1^2)}{96} + \frac{\alpha^2(1-\alpha)^2(2-\alpha)|p_1|^4(4-p_1^2)}{1350} \\
&+ \left.\frac{119\alpha^3(1-\alpha)|p_1|^4(4-p_1^2)}{138240} + \frac{\alpha(1-\alpha)|p_1|(4-p_1^2)^2}{192}\right)(|x|+|y|) \\
&+ \left(\frac{1111\alpha^4|p_1|^4(4-p_1^2)}{2177280} - \frac{1111\alpha^4|p_1|^3(4-p_1^2)}{1088640} - \frac{\alpha(1-\alpha)|p_1|^3(4-p_1^2)}{768}\right. \\
&+ \left.\frac{\alpha(1-\alpha)|p_1|(4-p_1^2)^2}{768} - \frac{\alpha(1-\alpha)(4-p_1^2)^2}{192} - \frac{17\alpha^3(1-\alpha)|p_1|^3(4-p_1^2)}{82944}\right) \\
&\times (|x|^2+|y|^2) + \left(\frac{17\alpha^3(1-\alpha)|p_1|^4(4-p_1^2)}{165888} + \frac{17\alpha^3(1-\alpha)|p_1|^2(4-p_1^2)^2}{51840}\right. \\
&+ \frac{17\alpha^3|p_1|^2(4-p_1^2)^2}{61440} + \frac{\alpha(1-\alpha)|p_1|^2(4-p_1^2)^2}{768} + \frac{\alpha(1-\alpha)|p_1|(4-p_1^2)^2}{768} \\
&+ \left.\frac{\alpha(1-\alpha)|p_1|^4(4-p_1^2)}{1536} + \frac{225737\alpha^4|p_1|^2(4-p_1^2)^2}{74317824}\right)(|x|+|y|)^2 \\
&+ \left(\frac{\alpha(1-\alpha)|p_1|^2(4-p_1^2)^2}{1536} - \frac{17\alpha^3|p_1|(4-p_1^2)^2}{61440} - \frac{\alpha(1-\alpha)|p_1|(4-p_1^2)^2}{384}\right. \\
&+ \left.\frac{17\alpha^3|p_1|^2(4-p_1^2)^2}{122880}\right)(|x|+|y|)(|x|^2+|y|^2) + \left.\frac{\alpha(1-\alpha)|p_1|^2(4-p_1^2)^2}{1536}\right) \\
&\times (|x|+|y|)^3 + \left(\frac{\alpha(1-\alpha)|p_1|^2(4-p_1^2)^2}{3072} - \frac{\alpha(1-\alpha)|p_1|(4-p_1^2)^2}{1536}\right)(|x|+|y|)^2 \\
&\times (|x|^2+|y|^2) + \left(-\frac{\alpha(1-\alpha)|p_1|(4-p_1^2)^2}{1536} + \frac{\alpha(1-\alpha)(4-p_1^2)^2}{768}\right)(|x|^2+|y|^2)^2.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
|a_7| \le\, & F_1 + F_2(\mu_1+\mu_2) + F_3(\mu_1^2+\mu_2^2) + F_4(\mu_1+\mu_2)^2 + F_5(\mu_1+\mu_2)(\mu_1^2+\mu_2^2) \\
& + F_6(\mu_1+\mu_2)^3 + F_7(\mu_1+\mu_2)^2(\mu_1^2+\mu_2^2) + F_8(\mu_1^2+\mu_2^2)^2 = F(\mu_1,\mu_2),
\end{aligned}
$$

where the constants $F_1 \ge 0$, $F_2 \ge 0$, $F_3 \le 0$, $F_4 \ge 0$, $F_5 \le 0$, $F_6 \ge 0$, $F_7 \le 0$, and $F_8 \ge 0$. By Lemma 2, we determine the maximum value of $F(\mu_1,\mu_2)$ on $\mathbb{S} = [0,1]\times[0,1]$. Since $F$ is continuous on the compact set $\mathbb{S}$, it attains a global maximum on $\mathbb{S}$. We first investigate the possibility of an interior maximizer. The first-order partial derivatives of $F(\mu_1,\mu_2)$ are given by

$$
\begin{aligned}
F_{\mu_1} =\, & F_2 + 2F_3\mu_1 + 2F_4\mu_1 + 2F_4\mu_2 + 3F_5\mu_1^2 + F_5\mu_2^2 + 2F_5\mu_1\mu_2 + 3F_6\mu_1^2 + 6F_6\mu_1\mu_2 \\
& + 3F_6\mu_2^2 + 4F_7\mu_1^3 + 6F_7\mu_1^2\mu_2 + 4F_7\mu_1\mu_2^2 + 2F_7\mu_2^3 + 4F_8\mu_1^3 + 4F_8\mu_1\mu_2^2, \\
F_{\mu_2} =\, & F_2 + 2F_3\mu_2 + 2F_4\mu_1 + 2F_4\mu_2 + 2F_5\mu_1\mu_2 + F_5\mu_1^2 + 3F_5\mu_2^2 + 3F_6\mu_1^2 + 6F_6\mu_1\mu_2
\end{aligned}
$$

$$+ 3F_6\mu_2^2 + 2F_7\mu_1^3 + 4F_7\mu_1^2\mu_2 + 6F_7\mu_1\mu_2^2 + 4F_7\mu_2^3 + 4F_8\mu_1^2\mu_2 + 4F_8\mu_2^3.$$

By Fermat's theorem, any interior local extremum of $F(\mu_1, \mu_2)$ must satisfy $F_{\mu_1} = 0$, and $F_{\mu_2} = 0$. Subtracting these equations yields

$$\frac{\partial F}{\partial \mu_1} - \frac{\partial F}{\partial \mu_2} = (\mu_1 - \mu_2)\Big[2F_3 + 2F_5(\mu_1 + \mu_2) + 2F_7(\mu_1 + \mu_2)^2 + 4F_8(\mu_1^2 + \mu_2^2)\Big] = 0.$$

Since

$$2F_3 + 2F_5(\mu_1 + \mu_2) + 2F_7(\mu_1 + \mu_2)^2 + 4F_8(\mu_1^2 + \mu_2^2) \neq 0$$

for all $(\mu_1, \mu_2) \in (0, 1)^2$, it follows that any interior critical point must satisfy $\mu_1 = \mu_2$. Setting $\mu_1 = \mu_2 = \mu$, the function $F$ reduces to

$$F(\mu, \mu) = F_1 + 2F_2\mu + 2F_3\mu^2 + 4F_4\mu^2 + 4F_5\mu^3 + 8F_6\mu^3 + 8F_7\mu^4 + 4F_8\mu^4.$$

Differentiating with respect to $\mu$ gives

$$F'(\mu, \mu) = 2F_2 + 4(F_3 + 2F_4)\mu + 12(F_5 + 2F_6)\mu^2 + 16(2F_7 + F_8)\mu^3.$$

Therefore, any interior critical point must satisfy

$$2F_2 + 4(F_3 + 2F_4)\mu + 12(F_5 + 2F_6)\mu^2 + 16(2F_7 + F_8)\mu^3 = 0.$$

Next, we evaluate $F(\mu, \mu)$ at the endpoints of the interval $[0, 1]$:

$$\begin{aligned} F(0) &= F_1, \\ F(1) &= F_1 + 2F_2 + 2F_3 + 4F_4 + 4F_5 + 8F_6 + 8F_7 + 4F_8. \end{aligned}$$

Since no interior critical point exists, the maximum of $F(\mu, \mu)$ on $[0, 1]$ must occur at an endpoint. Therefore,

$$\max_{0 \le \mu \le 1} F(\mu, \mu) = \max\{F(0), F(1)\}.$$

Furthermore,

$$F(1) - F(0) = 2F_2 + 2F_3 + 4F_4 + 4F_5 + 8F_6 + 8F_7 + 4F_8.$$

Since $F(1) \geq F(0)$, it follows that

$$\max_{0 \le \mu \le 1} F(\mu, \mu) = F(1).$$

Hence, the maximum is attained at $\mu_1 = \mu_2 = 1$. Hence,

$$\max F(\mu_1, \mu_2) = F(1, 1) = \mathcal{P}(p) = F_1 + 2(F_2 + F_3) + 4(F_4 + F_5 + F_8) + 8(F_6 + F_7).$$

Substituting the expressions for $F_1, F_2, F_3, F_4, F_5, F_6, F_7$, and $F_8$ yields $|a_7| \leq \mathcal{P}(p)$. Since $\mathcal{P}(p)$ is an increasing function of $|p|$ on $[0, 2]$, its maximum is attained at $|p| = 2$. Therefore,

$$|a_7| \leq \alpha\Bigg(\frac{1}{24} + \frac{(1-\alpha)}{6} + \frac{(1-\alpha)(2-\alpha)}{4} + \frac{64(1-\alpha)^2(2-\alpha)}{675} + \frac{32(1-\alpha)^2(2-\alpha)^2}{2025} + \frac{16\alpha}{105}$$
$$+ \frac{64\alpha(1-\alpha)}{105} + \frac{17\alpha^2}{2592} + \frac{1111\alpha^3}{13608} + \frac{1073\alpha^3(1-\alpha)}{525} + \frac{255856\alpha^5}{1148175}\Bigg).$$

Thus, the desired upper bound for $|a_7|$ is obtained. □

**Theorem 2.** *Let $f(z)$ be given by Eq. (1) and suppose that $f \in \mathbf{B}^*_\Sigma(\alpha)$, $0 < \alpha \leq 1$. Then the following coefficient bounds hold:*

$$|A_2| \leq \frac{2\alpha}{3}$$

$$|A_3| \leq \begin{cases} \frac{\alpha}{4}, & 0 < \alpha \leq \frac{9}{16}, \\ \frac{4\alpha^2}{9}, & \frac{9}{16} < \alpha \leq 1, \end{cases}$$

$$|A_4| \leq \alpha\left(\frac{16\alpha^2}{405} + \frac{2}{15} + \frac{4(1-\alpha)}{15} + \frac{4(1-\alpha)(2-\alpha)}{45}\right),$$

$$|A_5| \leq \alpha\left(\frac{128\alpha^3}{405} + \frac{4\alpha}{15} + \frac{8\alpha(1-\alpha)}{15} + \frac{8\alpha(1-\alpha)(2-\alpha)}{45} + \frac{1}{12}\right),$$

$$|A_6| \leq \alpha\left(\frac{2}{35} + \frac{352\alpha^4}{8505} + \frac{8\alpha^2}{315} + \frac{7\alpha}{36} + \frac{128\alpha^2(1-\alpha)}{567} + \frac{344\alpha^2(1-\alpha)(2-\alpha)}{2835} + \frac{8(1-\alpha)}{35} + \frac{8(1-\alpha)(2-\alpha)}{35} + \frac{8(1-\alpha)(2-\alpha)(3-\alpha)}{105} + \frac{4(1-\alpha)(2-\alpha)(3-\alpha)(4-\alpha)}{525}\right),$$

$$|A_7| \leq \alpha\left(\frac{1}{24} + \frac{(1-\alpha)(2-\alpha)}{4} + \frac{64(1-\alpha)^2(2-\alpha)}{675} + \frac{32(1-\alpha)^2(2-\alpha)^2}{2025} + \frac{(1-\alpha)}{6} + \frac{1111\alpha^3}{13608} + \frac{16\alpha}{105} + \frac{64\alpha(1-\alpha)}{105} + \frac{17\alpha^2}{2592} + \frac{1073\alpha^3(1-\alpha)}{525} + \frac{255856\alpha^5}{1148175}\right).$$

*Proof.* Let $f^{-1}(w) = w + \sum_{n=2}^{\infty} A_n w^n$ be the inverse function of $f$. Comparing coefficients in the identity $f^{-1}(f(z)) = z$, we obtain

$$\begin{aligned}
A_2 &= -a_2, \\
A_3 &= 2a_2^2 - a_3, \\
A_4 &= -(5a_2^3 - 5a_2a_3 + a_4) \\
A_5 &= 14a_2^4 - 21a_2^2a_3 + 6a_2a_4 + 3a_3^2 - a_5, \\
A_6 &= -(42a_2^5 - 84a_2^3a_3 + 28a_2^2a_4 + 28a_2a_3^2 - 7a_2a_5 - 7a_3a_4 + a_6), \\
A_7 &= 132a_2^6 - 330a_2^4a_3 + 120a_2^3a_4 + 180a_2^2a_3^2 - 36a_2^2a_5 - 72a_2a_3a_4 + 8a_2a_6 - 12a_3^3 + 8a_3a_5 \\
&\quad + 4a_4^2 - a_7.
\end{aligned}$$

Using Eq. (20) and Lemma 1 which asserts $|p_n| \leq 2$ for $p \in P$, we obtain

$$|A_2| = |a_2| = \left|\frac{\alpha p_1}{3}\right| \leq \frac{2\alpha}{3}.$$

Furthermore, from Eq. (20) and Eq. (21), we obtain

$$A_3 = 2a_2^2 - a_3 = -(a_3 - 2a_2^2),$$

and hence

$$|A_3| = |a_3 - 2a_2^2|.$$

Using Eq. (20) and Eq. (21), we have

$$a_3-\nu a_2^2=\left(\frac{\alpha^2p_1^2}{9}\right)(1-\nu)+\frac{\alpha}{16}\left(\frac{4-p_1^2}{2}\right)(x-y),$$

where $|p_1|\le 2$, $|x|\le 1$, and $|y|\le 1$. Therefore,

$$|a_3-\nu a_2^2|\le \frac{\alpha^2p_1^2}{9}|1-\nu|+\frac{\alpha}{16}\left(\frac{4-p_1^2}{2}\right)(|x|+|y|).$$

Setting

$$t=p_1^2, \qquad 0\le t\le 4,$$

we obtain

$$|a_3-\nu a_2^2|\le \frac{\alpha^2}{9}|1-\nu|t+\frac{\alpha}{16}(4-t).$$

For $\nu=2$, this becomes

$$|A_3|\le\left(\frac{\alpha^2}{9}-\frac{\alpha}{16}\right)t+\frac{\alpha}{4}.$$

Since the right-hand side is linear in $t\in[0,4]$, its maximum is attained at an endpoint. Consequently,

$$|A_3|\le\begin{cases}\dfrac{\alpha}{4}, & 0<\alpha\le\dfrac{9}{16},\\ \dfrac{4\alpha^2}{9}, & \dfrac{9}{16}<\alpha\le 1.\end{cases}$$

Thus, the desired upper bound for $|A_3|$ is obtained. Next, we estimate the coefficient $|A_4|$.

$$A_4=-(5a_2^3-5a_2a_3+a_4)$$

Using Eqs. (20), (21) and (22) also Lemma 2, we obtain

$$\begin{aligned}|A_4|=&\Bigg|5\left(\frac{\alpha p_1}{3}\right)^3-5\left(\frac{\alpha p_1}{3}\right)\left(\frac{\alpha^2p_1^2}{9}+\frac{\alpha(p_2-q_2)}{16}\right)+\left(\frac{2\alpha^3p_1^3}{405}\right.\\
&\left.+\frac{5\alpha^2p_1(p_2-q_2)}{96}+\frac{\alpha(p_3-q_3)}{30}+\frac{\alpha(\alpha-1)p_1(p_2+q_2)}{30}+\frac{\alpha(\alpha-1)(\alpha-2)p_1^3}{90}\right)\Bigg|\\
\le&\frac{\alpha|p_1|^3}{60}+\frac{\alpha|(4-p_1^2)|}{30}+\frac{2\alpha^3|p_1|^3}{405}+\frac{\alpha(1-\alpha)|p_1|^3}{30}+\frac{\alpha(1-\alpha)(2-\alpha)|p_1|^3}{90}\\
&+\left(\frac{5\alpha^2|p_1||(4-p_1^2)|}{192}+\frac{\alpha|p_1||(4-p_1^2)|}{60}+\frac{\alpha(1-\alpha)|p_1||(4-p_1^2)|}{60}\right)(|x|+|y|)\\
&+\left(\frac{\alpha|p_1||(4-p_1^2)|}{120}-\frac{\alpha|(4-p_1^2)|}{60}\right)(|x|^2+|y|^2)\end{aligned}$$

Hence,

$$|A_4|\le G_1+G_2(\mu_1+\mu_2)+G_3(\mu_1^2+\mu_2^2),$$

where

$$G_1=\frac{\alpha|p|^3}{60}+\frac{\alpha|(4-p^2)|}{30}+\frac{2\alpha^3|p|^3}{405}+\frac{\alpha(1-\alpha)|p|^3}{30}+\frac{\alpha(1-\alpha)(2-\alpha)|p|^3}{90}\ge 0,$$

$$G_2 = \frac{5\alpha^2|p||(4-p^2)|}{192} + \frac{\alpha|p||(4-p^2)|}{60} + \frac{\alpha(1-\alpha)|p||(4-p^2)|}{60} \geq 0,$$
$$G_3 = \frac{\alpha(|p|-2)(4-p^2)}{120} \leq 0.$$

Since $G(\mu_1, \mu_2)$ is a polynomial function in $\mu_1$ and $\mu_2$, it is continuous on $\mathbb{R}^2$. Moreover, the set $\mathbb{S} = \{(\mu_1, \mu_2) : 0 \leq \mu_1 \leq 1,\ 0 \leq \mu_2 \leq 1\}$ is closed and bounded, and hence compact. Therefore, by the Extreme Value Theorem, $G(\mu_1, \mu_2)$ attains a global maximum on $\mathbb{S}$. To determine the location of this maximum, we first investigate the possibility of a maximum point in the interior of $\mathbb{S}$. The first-order partial derivatives of $G(\mu_1, \mu_2)$ are given by

$$G_{\mu_1} = G_2 + 2G_3\mu_1,$$
$$G_{\mu_2} = G_2 + 2G_3\mu_2.$$

By Fermat's theorem, any interior extremum of $G(\mu_1, \mu_2)$ must satisfy $\frac{\partial G}{\partial \mu_1} = 0$, and $\frac{\partial G}{\partial \mu_2} = 0$. A direct computation yields $2G_3(\mu_1 - \mu_2) = 0$, and hence every interior critical point satisfies $\mu_1 = \mu_2$. Setting $\mu_1 = \mu_2 = \mu$, we obtain

$$G(\mu, \mu) = G_1 + 2G_2\mu + 2G_3\mu^2.$$

To determine the nature of the critical point, we apply the second derivative test. Since

$$G_{\mu_1\mu_1} = 2G_3, \qquad G_{\mu_1\mu_2} = 0, \qquad G_{\mu_2\mu_2} = 2G_3,$$

we have

$$D = G_{\mu_1\mu_1}G_{\mu_2\mu_2} - (G_{\mu_1\mu_2})^2 = 4G_3^2.$$

Since $G_3 < 0$, it follows that $D > 0$ and $G_{\mu_1\mu_1} < 0$. Therefore, the critical point is a local maximum. Since $G$ is continuous on the compact set $\mathbb{S}$, its global maximum is attained either at an interior critical point or on the boundary of $\mathbb{S}$. Consequently, it remains to examine the boundary values.

For $\mu_1 = 0$ and $0 \leq \mu_2 \leq 1$,

$$G(0, \mu_2) = G_1 + G_2\mu_2 + G_3\mu_2^2,$$

with

$$G_{\mu_2}(0, \mu_2) = G_2 + 2G_3\mu_2.$$

Similarly, for $\mu_1 = 1$,

$$G(1, \mu_2) = G_1 + G_2 + G_2\mu_2 + G_3 + G_3\mu_2^2,$$

and

$$G_{\mu_2}(1, \mu_2) = G_2 + 2G_3\mu_2.$$

Hence $G(1, \mu_2)$ is increasing on $[0, 1]$, and therefore

$$\max_{0 \leq \mu_2 \leq 1} G(1, \mu_2) = G(1, 1) = G_1 + 2(G_2 + G_3) =: \mathcal{G}(p).$$

Substituting the expressions for $G_1$, $G_2$, and $G_3$, we obtain

$$\mathcal{G}(p) = \frac{\alpha|p|^3}{60} + \frac{\alpha|(4-p^2)|}{30} + \frac{2\alpha^3|p|^3}{405} + \frac{\alpha(1-\alpha)|p|^3}{30} + \frac{\alpha(1-\alpha)(2-\alpha)|p|^3}{90}$$
$$+ 2\left(\frac{5\alpha^2|p||(4-p^2)|}{192} + \frac{\alpha|p||(4-p^2)|}{60} + \frac{\alpha(1-\alpha)|p||(4-p^2)|}{60} + \frac{\alpha(|p|-2)(4-p^2)}{120}\right).$$

Since $\mathcal{G}(p)$ is increasing on $0 \le |p| \le 2$, its maximum is attained at $|p| = 2$. Hence,

$$|a_4| \le \mathcal{G}(2) = \alpha\left(\frac{16\alpha^2}{405} + \frac{2}{15} + \frac{4(1-\alpha)}{15} + \frac{4(1-\alpha)(2-\alpha)}{45}\right).$$

Thus, the desired upper bound for $|A_4|$ is obtained. Next, we estimate the coefficient $|A_5|$.

$$A_5 = 14a_2^4 - 21a_2^2a_3 + 6a_2a_4 + 3a_3^2 - a_5.$$

Using Eqs. (20), (21), (22) and (23) also Lemma 2, we obtain

$$\begin{aligned}
|A_5| =& \Bigg| 14\left(\frac{\alpha p_1}{3}\right)^4 - 21\left(\frac{\alpha p_1}{3}\right)^2\left(\frac{\alpha^2 p_1^2}{9} + \frac{\alpha(p_2-q_2)}{16}\right) + 6\left(\frac{\alpha p_1}{3}\right)\left(\frac{2\alpha^3 p_1^3}{405} + \frac{5\alpha^2 p_1(p_2-q_2)}{96}\right.\\
&\left. + \frac{\alpha(p_3-q_3)}{30} + \frac{\alpha(\alpha-1)p_1(p_2+q_2)}{30} + \frac{\alpha(\alpha-1)(\alpha-2)p_1^3}{90}\right) + 3\left(\frac{\alpha^2 p_1^2}{9} + \frac{\alpha(p_2-q_2)}{16}\right)^2\\
&- \left(-\frac{8\alpha^4 p_1^4}{405} + \frac{5\alpha^3 p_1^2(p_2-q_2)}{1728} + \frac{\alpha(p_4-q_4)}{48} + \frac{\alpha^2(\alpha-1)p_1^2(p_2+q_2)}{30}\right.\\
&+ \frac{\alpha^2 p_1(p_3-q_3)}{30} + \frac{\alpha(\alpha-1)(p_2^2-q_2^2)}{96} + \frac{\alpha(\alpha-1)(\alpha-2)p_1^2(p_2-q_2)}{96}\\
&\left.+ \frac{\alpha^2(\alpha-1)(\alpha-2)p_1^4}{90} + \frac{3\alpha^2(p_2-q_2)^2}{512} + \frac{\alpha(\alpha-1)p_1(p_3+q_3)}{48}\right)\Bigg|\\
\le& \frac{8\alpha^4|p_1|^4}{405} + \frac{\alpha^2|p_1|^4}{60} + \frac{\alpha^2|p_1|(4-p_1^2)}{30} + \frac{\alpha^2(1-\alpha)|p_1|^4}{30} + \frac{\alpha^2(1-\alpha)(2-\alpha)|p_1|^4}{90}\\
&+ \frac{\alpha}{12} + \frac{\alpha(1-\alpha)|p_1|(4-p_1^2)}{48} + \left(\frac{5\alpha^3|p_1|^2(4-p_1^2)}{3456} + \frac{\alpha^2|p_1|^2(4-p_1^2)}{60}\right.\\
&\left.+ \frac{\alpha^2(1-\alpha)|p_1|^2(4-p_1^2)}{60} + \frac{\alpha(1-\alpha)|p_1|^2(4-p_1^2)}{64} + \frac{\alpha(1-\alpha)(2-\alpha)|p_1|^2(4-p_1^2)}{192}\right)\\
&(|x|+|y|) + \left(\frac{\alpha^2|p_1|^2(4-p_1^2)}{120} - \frac{\alpha^2|p_1|(4-p_1^2)}{60} - \frac{\alpha(1-\alpha)|p_1|(4-p_1^2)}{96}\right)(|x|^2+|y|^2)\\
&+ \left(\frac{3\alpha^2(4-p_1^2)^2}{2048} + \frac{\alpha(1-\alpha)|p_1|^2(4-p_1^2)}{192} + \frac{\alpha(1-\alpha)(4-p_1^2)^2}{384}\right)(|x|+|y|)^2.
\end{aligned}$$

Hence,

$$|A_5| \le H_1 + H_2(\mu_1+\mu_2) + H_3(\mu_1^2+\mu_2^2) + H_4(\mu_1+\mu_2)^2 =: H(\mu_1,\mu_2),$$

where

$$\begin{aligned}
H_1 =& \frac{8\alpha^4|p|^4}{405} + \frac{\alpha^2|p|^4}{60} + \frac{\alpha^2|p|(4-p^2)}{30} + \frac{\alpha^2(1-\alpha)|p|^4}{30} + \frac{\alpha^2(1-\alpha)(2-\alpha)|p|^4}{90}\\
&+ \frac{\alpha}{12} + \frac{\alpha(1-\alpha)|p|(4-p^2)}{48} \ge 0,\\
H_2 =& \frac{5\alpha^3|p|^2(4-p^2)}{3456} + \frac{\alpha^2|p|^2(4-p^2)}{60} + \frac{\alpha^2(1-\alpha)|p|^2(4-p^2)}{60} + \frac{\alpha(1-\alpha)|p|^2(4-p^2)}{64}\\
&+ \frac{\alpha(1-\alpha)(2-\alpha)|p|^2(4-p^2)}{192} \ge 0,
\end{aligned}$$

$$H_3 = \frac{\alpha^2|p|(|p|-2)(4-p^2)}{120} - \frac{\alpha(1-\alpha)|p|(4-p^2)}{96} \leq 0,$$
$$H_4 = \frac{3\alpha^2(4-p^2)^2}{2048} + \frac{\alpha(1-\alpha)|p|^2(4-p^2)}{192} + \frac{\alpha(1-\alpha)(4-p^2)^2}{384} \geq 0.$$

By Lemma 2, it remains to determine the maximum value of $H(\mu_1,\mu_2)$ on $\mathbb{S} = \{(\mu_1,\mu_2) : 0 \leq \mu_1 \leq 1,\ 0 \leq \mu_2 \leq 1\}$. Since $H$ is continuous on the compact set $\mathbb{S}$, it attains a global maximum on $\mathbb{S}$. We first investigate the existence of interior critical points by computing the first-order partial derivatives, and by Fermat's theorem, any interior critical point of $H(\mu_1,\mu_2)$ satisfies

$$\frac{\partial H}{\partial \mu_1} = H_2 + 2(H_3+H_4)\mu_1 + 2H_4\mu_2 = 0$$
$$\frac{\partial H}{\partial \mu_2} = H_2 + 2(H_3+H_4)\mu_2 + 2H_4\mu_1 = 0.$$

Solving these equations gives

$$2H_3(\mu_1-\mu_2) = 0,$$

so that $\mu_1 = \mu_2$. Letting $\mu_1 = \mu_2 = \mu$, we obtain

$$H(\mu,\mu) = H_1 + 2H_2\mu + 2(H_3+2H_4)\mu^2.$$

To determine whether the critical point corresponds to a local maximum, we apply the second derivative test. The second-order partial derivatives of $H(\mu_1,\mu_2)$ are

$$H_{\mu_1\mu_1} = 2(H_3+H_4),$$
$$H_{\mu_1\mu_2} = 2H_4,$$
$$H_{\mu_2\mu_2} = 2(H_3+H_4).$$

Hence,

$$D = 4H_3(H_3+2H_4).$$

Since $H_3 \leq 0$ and $H_4 \geq 0$, the sign of $D$ cannot be determined. Therefore, the existence of an interior maximum cannot be guaranteed. Since $H(\mu_1,\mu_2)$ is continuous on the compact set $\mathbb{S}$, its global maximum must occur either at an interior critical point or on the boundary of $\mathbb{S}$. Consequently, the boundary curves are examined.
For the side $\mu_1 = 0$ $(0 \leq \mu_2 \leq 1)$, we have

$$H(0,\mu_2) = H_1 + H_2\mu_2 + (H_3+H_4)\mu_2^2,$$
$$H_{\mu_2}(0,\mu_2) = H_2 + 2(H_3+H_4)\mu_2.$$

Thus, $H(0,\mu_2)$ is increasing on $[0,1]$, and hence

$$\max_{0\leq\mu_2\leq 1} H(0,\mu_2) = H(0,1).$$

Therefore, the maximum on the side $\mu_1 = 0$ is attained at $\mu_2 = 1$.

$$\max H(0,\mu_2) = H(0,1) = H_1 + H_2 + H_3 + H_4.$$

Next, we examine the boundary of $\mathbb{S} = [0,1]\times[0,1]$. For $\mu_1 = 0$, we have

$$H(0,\mu_2) = H_1 + H_2\mu_2 + (H_3+H_4)\mu_2^2, \qquad 0 \leq \mu_2 \leq 1.$$

Since $H(0,\mu_2)$ is a quadratic polynomial on the compact interval $[0,1]$, its maximum is attained either at a critical point or at an endpoint. Hence,

$$\max_{0\le\mu_2\le 1} H(0,\mu_2) = \max\{H(0,0),\ H(0,1)\}.$$

Similarly, for $\mu_1 = 1$,

$$H(1,\mu_2) = H_1 + H_2 + H_3 + H_4 + (H_2 + 2H_4)\mu_2 + (H_3 + H_4)\mu_2^2,$$

and therefore

$$\max_{0\le\mu_2\le 1} H(1,\mu_2) = \max\{H(1,0),\ H(1,1)\}.$$

By the same argument, the maximum values on the sides $\mu_2 = 0$ and $\mu_2 = 1$ are also attained at the endpoints. Consequently,

$$\max_{(\mu_1,\mu_2)\in[0,1]^2} H(\mu_1,\mu_2) = \max\{H(0,0),\ H(1,0),\ H(0,1),\ H(1,1)\}.$$

Evaluating $H$ at the vertices of $\mathbb{S}$ yields. Since

$$H(1,1) \ge H(0,1) = H(1,0) \ge H(0,0),$$

it follows that the global maximum of $H(\mu_1,\mu_2)$ on $[0,1]^2$ is attained at $(1,1)$. Therefore,

$$\max_{(\mu_1,\mu_2)\in[0,1]^2} H(\mu_1,\mu_2) = H(1,1) = H_1 + 2(H_2 + H_3) + 4H_4 = \mathcal{H}(p).$$

Substituting the expressions for $H_1, H_2, H_3$, and $H_4$ . Since $\mathcal{H}(p)$ is an increasing function of $|p|$ on $[0,2]$, its maximum is attained at $|p| = 2$. Therefore,

$$|a_5| \le \mathcal{H}(2) = \alpha\left(\frac{128\alpha^3}{405} + \frac{4\alpha}{15} + \frac{8\alpha(1-\alpha)}{15} + \frac{8\alpha(1-\alpha)(2-\alpha)}{45} + \frac{1}{12}\right).$$

Thus, the desired upper bound for $|A_5|$ is obtained. Next, we estimate the coefficient $|A_6|$.

$$A_6 = -(42a_2^5 - 84a_2^3a_3 + 28a_2^2a_4 + 28a_2a_3^2 - 7a_2a_5 - 7a_3a_4 + a_6).$$

Using Eqs. (20), (21), (22), (23) and (24) also Lemma 2, we obtain

$$\begin{aligned}
|A_6| = \Bigg| &-42\left(\frac{\alpha p_1}{3}\right)^5 + 84\left(\frac{\alpha p_1}{3}\right)^3\left(\frac{\alpha^2p_1^2}{9} + \frac{\alpha(p_2-q_2)}{16}\right) - 28\left(\frac{\alpha p_1}{3}\right)^2\left(\frac{2\alpha^3p_1^3}{405}\right.\\
&\left. + \frac{5\alpha^2p_1(p_2-q_2)}{96} + \frac{\alpha(p_3-q_3)}{30} + \frac{\alpha(\alpha-1)p_1(p_2+q_2)}{30} + \frac{\alpha(\alpha-1)(\alpha-2)p_1^3}{90}\right)\\
&- 28\left(\frac{\alpha p_1}{3}\right)\left(\frac{\alpha^2p_1^2}{9} + \frac{\alpha(p_2-q_2)}{16}\right)^2 + 7\left(\frac{\alpha p_1}{3}\right)\left(-\frac{8\alpha^4p_1^4}{405} + \frac{5\alpha^3p_1^2(p_2-q_2)}{1728}\right.\\
&+ \frac{\alpha(p_4-q_4)}{48} + \frac{\alpha^2(\alpha-1)p_1^2(p_2+q_2)}{30} + \frac{\alpha^2p_1(p_3-q_3)}{30} + \frac{\alpha(\alpha-1)(p_2^2-q_2^2)}{96}\\
&+ \frac{\alpha(\alpha-1)(\alpha-2)p_1^2(p_2-q_2)}{96} + \frac{\alpha^2(\alpha-1)(\alpha-2)p_1^4}{90} + \frac{3\alpha^2(p_2-q_2)^2}{512}\\
&\left. + \frac{\alpha(\alpha-1)p_1(p_3+q_3)}{48}\right) + 7\left(\frac{\alpha^2p_1^2}{9} + \frac{\alpha(p_2-q_2)}{16}\right)\left(\frac{2\alpha^3p_1^3}{405} + \frac{5\alpha^2p_1(p_2-q_2)}{96}\right.
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\alpha(p_3-q_3)}{30}+\frac{\alpha(\alpha-1)p_1(p_2+q_2)}{30}+\frac{\alpha(\alpha-1)(\alpha-2)p_1^3}{90}\bigg)-\bigg(-\frac{11\alpha^5p_1^5}{8505}\\
&+\frac{\alpha(p_5-q_5)}{70}-\frac{3131\alpha^4p_1^3(p_2-q_2)}{120960}+\frac{7\alpha^2(p_2-q_2)(p_3-q_3)}{960}+\frac{7\alpha^2p_1(p_4-q_4)}{288}\\
&+\frac{\alpha(\alpha-1)(p_2p_3-q_2q_3)}{70}-\frac{4\alpha^3(\alpha-1)p_1^3(p_2+q_2)}{567}+\frac{\alpha(\alpha-1)(\alpha-2)p_1(p_2^2+q_2^2)}{140}\\
&+\frac{\alpha(\alpha-1)p_1(p_4+q_4)}{70}+\frac{\alpha^3p_1(p_2-q_2)^2}{1792}+\frac{\alpha^3p_1^2(p_2-q_3)}{630}+\frac{7\alpha^2(\alpha-1)p_1^2(p_3+q_3)}{288}\\
&+\frac{7\alpha^2(\alpha-1)p_1(p_2^2-q_2^2)}{576}+\frac{7\alpha^2(\alpha-1)p_1(p_2+q_2)(p_2-q_2)}{960}\\
&+\frac{\alpha(\alpha-1)(\alpha-2)p_1^2(p_3-q_3)}{140}-\frac{43\alpha^3(\alpha-1)(\alpha-2)p_1^5}{11340}\\
&+\frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)p_1^3(p_2+q_2)}{420}+\frac{7\alpha^2(\alpha-1)(\alpha-2)p_1^3(p_2-q_2)}{480}\\
&+\frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)p_1^5}{4200}\bigg)\bigg|\\
\le&\frac{11\alpha^5|p_1|^5}{8505}+\frac{\alpha^3|p_1|^5}{1260}+\frac{\alpha^3|p_1|^2(4-p_1^2)}{630}+\frac{4\alpha^3(1-\alpha)|p_1|^5}{567}\\
&+\frac{43\alpha^3(1-\alpha)(2-\alpha)|p_1|^5}{11340}+\frac{7\alpha^2|p_1|}{72}+\frac{7\alpha^2(1-\alpha)|p_1|^2(4-p_1^2)}{288}+\frac{2\alpha}{35}\\
&+\frac{4\alpha(1-\alpha)}{35}+\frac{2\alpha(1-\alpha)(2-\alpha)|p_1|}{35}+\frac{2\alpha(1-\alpha)|p_1|}{35}+\frac{\alpha(1-\alpha)(2-\alpha)|p_1|^5}{280}\\
&+\frac{\alpha(1-\alpha)(2-\alpha)|p_1|^2(4-p_1^2)}{140}+\frac{\alpha(1-\alpha)(2-\alpha)(3-\alpha)|p_1|^5}{420}\\
&+\frac{\alpha(1-\alpha)(2-\alpha)(3-\alpha)(4-\alpha)|p_1|^5}{4200}+\bigg(\frac{2651\alpha^4|p_1|^3(4-p_1^2)}{241920}+\frac{\alpha^3|p_1|^3(4-p_1^2)}{1260}\\
&+\frac{4\alpha^3(1-\alpha)|p_1|^3(4-p_1^2)}{1134}+\frac{7\alpha^2(1-\alpha)|p_1|^3(4-p_1^2)}{720}+\frac{7\alpha^2(1-\alpha)(2-\alpha)|p_1|^3(4-p_1^2)}{960}\\
&+\frac{7\alpha^2(1-\alpha)|p_1|^3(4-p_1^2)}{576}+\frac{7\alpha^2|p_1|^3(4-p_1^2)}{3840}+\frac{7\alpha^2(4-p_1^2)^2}{1920}\\
&+\frac{\alpha(1-\alpha)(2-\alpha)|p_1|^3(4-p_1^2)}{280}+\frac{\alpha(1-\alpha)(2-\alpha)(3-\alpha)|p_1|^3(4-p_1^2)}{840}\bigg)(|x|+|y|)\\
&+\bigg(\frac{\alpha^3|p_1|^3(4-p_1^2)}{2520}-\frac{\alpha^3|p_1|^2(4-p_1^2)}{1260}-\frac{7\alpha^2(1-\alpha)|p_1|^2(4-p_1^2)}{576}\\
&+\frac{\alpha(1-\alpha)(2-\alpha)|p_1|^3(4-p_1^2)}{560}-\frac{\alpha(1-\alpha)(2-\alpha)|p_1|^2(4-p_1^2)}{280}\bigg)(|x|^2+|y|^2)\\
&+\bigg(\frac{\alpha^3|p_1|(4-p_1^2)^2}{7168}+\frac{7\alpha^2(1-\alpha)|p_1|(4-p_1^2)^2}{1440}+\frac{7\alpha^2(1-\alpha)|p_1|^3(4-p_1^2)}{1152}\\
&+\frac{7\alpha^2|p_1|(4-p_1^2)^2}{3840}\bigg)(|x|+|y|)^2+\bigg(\frac{7\alpha^2|p_1|(4-p_1^2)^2}{7680}-\frac{7\alpha^2(4-p_1^2)^2}{3840}\bigg)(|x|+|y|)\\
&(|x|^2+|y|^2)
\end{aligned}
$$

Hence,

$$|A_6|\le I_1+I_2(\mu_1+\mu_2)+I_3(\mu_1^2+\mu_2^2)+I_4(\mu_1+\mu_2)^2+I_5(\mu_1+\mu_2)(\mu_1^2+\mu_2^2)=:E=I(\mu_1,\mu_2),$$

where

$$
\begin{aligned}
I_1 = {} & \frac{2\alpha}{35} + \frac{11\alpha^5|p|^5}{8505} + \frac{\alpha^3|p|^5}{1260} + \frac{\alpha^3|p|^2(4-p^2)}{630} + \frac{7\alpha^2|p|}{72} + \frac{2\alpha(1-\alpha)|p|}{35} \\
& + \frac{4\alpha^3(1-\alpha)|p|^5}{567} + \frac{7\alpha^2(1-\alpha)|p|^2(4-p^2)}{288} + \frac{\alpha(1-\alpha)(2-\alpha)|p|^5}{280} \\
& + \frac{4\alpha(1-\alpha)}{35} + \frac{\alpha(1-\alpha)(2-\alpha)|p|^2(4-p^2)}{140} + \frac{2\alpha(1-\alpha)(2-\alpha)|p|}{35} \\
& + \frac{43\alpha^3(1-\alpha)(2-\alpha)|p|^5}{11340} + \frac{\alpha(1-\alpha)(2-\alpha)(3-\alpha)(4-\alpha)|p|^5}{4200} \\
& + \frac{\alpha(1-\alpha)(2-\alpha)(3-\alpha)|p|^5}{420} \geqslant 0, \\
I_2 = {} & \frac{3131\alpha^4|p|^3(4-p^2)}{12960} + \frac{\alpha^3|p|^3(4-p^2)}{1260} + \frac{7\alpha^2|p|^3(4-p^2)}{3840} + \frac{7\alpha^2(4-p^2)^2}{1920} \\
& + \frac{7\alpha^2(1-\alpha)(2-\alpha)|p|^3(4-p^2)}{960} + \frac{\alpha(1-\alpha)(2-\alpha)(3-\alpha)|p|^3(4-p^2)}{840} \\
& + \frac{\alpha(1-\alpha)(2-\alpha)|p|^3(4-p^2)}{280} + \frac{7\alpha^2(1-\alpha)|p|^3(4-p^2)}{320} \\
& + \frac{4\alpha^3(1-\alpha)|p|^3(4-p^2)}{1134} \geqslant 0, \\
I_3 = {} & \frac{\alpha^3(|p|-2)|p|^2(4-p^2)}{2520} - \frac{7\alpha^2(1-\alpha)|p|^2(4-p^2)}{576} \\
& + \frac{\alpha(1-\alpha)(2-\alpha)(|p|-2)|p|^2(4-p^2)}{560} \leqslant 0, \\
I_4 = {} & \frac{\alpha^3|p|(4-p^2)^2}{7168} + \frac{7\alpha^2|p|(4-p^2)}{3840} + \frac{7\alpha^2(1-\alpha)|p|^3(4-p^2)}{1152} \\
& + \frac{7\alpha^2(1-\alpha)|p|(4-p^2)^2}{1440} \geqslant 0, \\
I_5 = {} & \frac{7\alpha^2(|p|-2)(4-p^2)^2}{7680} \leqslant 0.
\end{aligned}
$$

By Lemma 2, we seek the maximum value of $I(\mu_1, \mu_2)$ on $\mathbb{S} = [0,1] \times [0,1]$. Since $I$ is a polynomial, it is continuous on $\mathbb{S}$. Therefore, by the Extreme Value Theorem, $I$ attains a global maximum on $\mathbb{S}$. To determine where this maximum occurs, we first examine the critical points in the interior of $\mathbb{S}$. The first-order partial derivatives are

$$
\begin{aligned}
I_{\mu_1} &= I_2 + 2I_3\mu_1 + 2I_4\mu_1 + 2I_4\mu_2 + 3I_5\mu_1^2 + I_5\mu_2^2 + 2I_5\mu_1\mu_2, \\
I_{\mu_2} &= I_2 + 2I_3\mu_2 + 2I_4\mu_1 + 2I_4\mu_2 + 2I_5\mu_1\mu_2 + I_5\mu_1^2 + 3I_5\mu_2^2.
\end{aligned}
$$

By Fermat's theorem, any interior local extremum of $I(\mu_1, \mu_2)$ must satisfy $I_{\mu_1} = 0$, and $I_{\mu_2} = 0$. Subtracting the two equations gives

$$2(\mu_1 - \mu_2)\left[I_3 + I_5(\mu_1 + \mu_2)\right] = 0.$$

Since $I_3 \leq 0$ and $I_5 \leq 0$, we have

$$I_3 + I_5(\mu_1 + \mu_2) \leq 0, \qquad (\mu_1, \mu_2) \in [0,1]^2.$$

Hence, the second factor cannot vanish for any interior point of $\mathbb{S}$, and therefore any interior critical point must satisfy $\mu_1 = \mu_2$. Setting $\mu_1 = \mu_2 = \mu$, the function $I$ reduces to

$$I(\mu, \mu) = I_1 + 2I_2\mu + 2I_3\mu^2 + 4I_4\mu^2 + 4I_5\mu^3,$$

whose derivative is

$$I_\mu(\mu,\mu) = 2I_2 + 4(I_3 + 2I_4)\mu + 12I_5\mu^2.$$

Thus, any interior critical point must satisfy

$$2I_2 + 4(I_3 + 2I_4)\mu + 12I_5\mu^2 = 0.$$

Next, we evaluate $I(\mu,\mu)$ at the endpoints of the interval $[0,1]$:

$$\begin{aligned} I(0) &= I_1, \\ I(1) &= I_1 + 2I_2 + 2I_3 + 4I_4 + 4I_5. \end{aligned}$$

Since $I(\mu,\mu)$ is continuous on the compact interval $[0,1]$, its maximum is attained either at a critical point or at an endpoint. Furthermore,

$$I(1) - I(0) = 2I_2 + 2I_3 + 4I_4 + 4I_5.$$

By the assumptions on the coefficients, it follows that $I(1) \geq I(0)$. Hence, among the endpoints, the larger value is attained at $\mu = 1$. Therefore, the maximum value of $I(\mu,\mu)$ is achieved at $\mu = 1$, that is, $\mu_1 = \mu_2 = 1$. Hence,

$$\max I(\mu_1,\mu_2) = I(1,1) = \mathcal{I}(p) = I_1 + 2(I_2 + I_3) + 4(I_4 + I_5).$$

Substituting the expressions for $I_1, I_2, I_3, I_4$, and $I_5$ yields $|A_6| \leq \mathcal{I}(p)$. Since $\mathcal{I}(p)$ is an increasing function of $|p|$ on $[0,2]$, its maximum is attained at $|p| = 2$. Therefore,

$$\begin{aligned} |A_6| \leq \alpha\Bigg( &\frac{2}{35} + \frac{352\alpha^4}{8505} + \frac{8\alpha^2}{315} + \frac{7\alpha}{36} + \frac{8(1-\alpha)}{35} + \frac{128\alpha^2(1-\alpha)}{567} \\ &+ \frac{4(1-\alpha)(2-\alpha)(3-\alpha)(4-\alpha)}{525} + \frac{344\alpha^2(1-\alpha)(2-\alpha)}{2835} + \frac{8(1-\alpha)(2-\alpha)(3-\alpha)}{105} \\ &+ \frac{8(1-\alpha)(2-\alpha)}{35}\Bigg) \end{aligned}$$

Thus, the desired upper bound for $|A_6|$ is obtained. Next, we estimate the coefficient $|a_7|$.

$$\begin{aligned} A_7 = {} &132a_2^6 - 330a_2^4a_3 + 120a_2^3a_4 + 180a_2^2a_3^2 - 36a_2^2a_5 - 72a_2a_3a_4 + 8a_2a_6 - 12a_3^3 + 8a_3a_5 \\ &+ 4a_4^2 - a_7. \end{aligned}$$

Using Eqs. (20), (21), (22), (23), (24) and (25) also Lemma 2, we obtain

$$\begin{aligned} |A_7| = \Bigg| &132\left(\frac{\alpha p_1}{3}\right)^6 - 330\left(\frac{\alpha p_1}{3}\right)^4\left(\frac{\alpha^2p_1^2}{9} + \frac{\alpha(p_2-q_2)}{16}\right) + 120\left(\frac{\alpha p_1}{3}\right)^3\Bigg(\frac{2\alpha^3p_1^3}{405} \\ &+ \frac{5\alpha^2p_1(p_2-q_2)}{96} + \frac{\alpha(p_3-q_3)}{30} + \frac{\alpha(\alpha-1)p_1(p_2+q_2)}{30} + \frac{\alpha(\alpha-1)(\alpha-2)p_1^3}{90}\Bigg) \\ &+ 180\left(\frac{\alpha p_1}{3}\right)^2\left(\frac{\alpha^2p_1^2}{9} + \frac{\alpha(p_2-q_2)}{16}\right)^2 - 36\left(\frac{\alpha p_1}{3}\right)^2\Bigg(-\frac{8\alpha^4p_1^4}{405} + \frac{5\alpha^3p_1^2(p_2-q_2)}{1728} \\ &+ \frac{\alpha(p_4-q_4)}{48} + \frac{\alpha^2(\alpha-1)p_1^2(p_2+q_2)}{30} + \frac{\alpha^2p_1(p_3-q_3)}{30} + \frac{\alpha(\alpha-1)(p_2^2-q_2^2)}{96} \\ &+ \frac{\alpha(\alpha-1)(\alpha-2)p_1^2(p_2-q_2)}{96} + \frac{\alpha^2(\alpha-1)(\alpha-2)p_1^4}{90} + \frac{3\alpha^2(p_2-q_2)^2}{512} \end{aligned}$$

$$
\begin{aligned}
&+\frac{\alpha(\alpha-1)p_1(p_3+q_3)}{48}\Bigg)-72\left(\frac{\alpha p_1}{3}\right)\left(\frac{\alpha^2p_1^2}{9}+\frac{\alpha(p_2-q_2)}{16}\right)\Bigg(\frac{2\alpha^3p_1^3}{405}+\frac{5\alpha^2p_1(p_2-q_2)}{96}\\
&+\frac{\alpha(p_3-q_3)}{30}+\frac{\alpha(\alpha-1)p_1(p_2+q_2)}{30}+\frac{\alpha(\alpha-1)(\alpha-2)p_1^3}{90}\Bigg)+8\left(\frac{\alpha p_1}{3}\right)\Bigg(-\frac{11\alpha^5p_1^5}{8505}\\
&+\frac{\alpha(p_5-q_5)}{70}-\frac{3131\alpha^4p_1^3(p_2-q_2)}{120960}+\frac{7\alpha^2(p_2-q_2)(p_3-q_3)}{960}+\frac{7\alpha^2p_1(p_4-q_4)}{288}\\
&+\frac{\alpha(\alpha-1)(p_2p_3-q_2q_3)}{70}-\frac{4\alpha^3(\alpha-1)p_1^3(p_2+q_2)}{567}+\frac{\alpha(\alpha-1)(\alpha-2)p_1(p_2^2+q_2^2)}{140}\\
&+\frac{\alpha(\alpha-1)p_1(p_4+q_4)}{70}+\frac{\alpha^3p_1(p_2-q_2)^2}{1792}+\frac{\alpha^3p_1^2(p_2-q_3)}{630}+\frac{7\alpha^2(\alpha-1)p_1^2(p_3+q_3)}{288}\\
&+\frac{7\alpha^2(\alpha-1)p_1(p_2^2-q_2^2)}{576}+\frac{7\alpha^2(\alpha-1)p_1(p_2+q_2)(p_2-q_2)}{960}\\
&+\frac{\alpha(\alpha-1)(\alpha-2)p_1^2(p_3-q_3)}{140}-\frac{43\alpha^3(\alpha-1)(\alpha-2)p_1^5}{11340}\\
&+\frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)p_1^3(p_2+q_2)}{420}+\frac{7\alpha^2(\alpha-1)(\alpha-2)p_1^3(p_2-q_2)}{480}\\
&+\frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)p_1^5}{4200}\Bigg)-12\left(\frac{\alpha^2p_1^2}{9}+\frac{\alpha(p_2-q_2)}{16}\right)^3+8\Bigg(\frac{\alpha^2p_1^2}{9}\\
&+\frac{\alpha(p_2-q_2)}{16}\Bigg)\Bigg(-\frac{8\alpha^4p_1^4}{405}+\frac{5\alpha^3p_1^2(p_2-q_2)}{1728}+\frac{\alpha(p_4-q_4)}{48}+\frac{\alpha^2(\alpha-1)p_1^2(p_2+q_2)}{30}\\
&+\frac{\alpha(\alpha-1)(p_2^2-q_2^2)}{96}+\frac{\alpha(\alpha-1)(\alpha-2)p_1^2(p_2-q_2)}{96}+\frac{\alpha^2p_1(p_3-q_3)}{30}\\
&+\frac{\alpha^2(\alpha-1)(\alpha-2)p_1^4}{90}+\frac{3\alpha^2(p_2-q_2)^2}{512}+\frac{\alpha(\alpha-1)p_1(p_3+q_3)}{48}\Bigg)+4\Bigg(\frac{2\alpha^3p_1^3}{405}\\
&+\frac{5\alpha^2p_1(p_2-q_2)}{96}+\frac{\alpha(p_3-q_3)}{30}+\frac{\alpha(\alpha-1)p_1(p_2+q_2)}{30}+\frac{\alpha(\alpha-1)(\alpha-2)p_1^3}{90}\Bigg)^2\\
&-\Bigg(\frac{15991\alpha^6p_1^6}{4592700}+\frac{\alpha(p_6-q_6)}{96}+\frac{2\alpha^2p_1(p_5-q_5)}{105}-\frac{901597\alpha^5p_1^4(p_2-q_2)}{52254720}\\
&-\frac{225737\alpha^4p_1^2(p_2-q_2)^2}{18579456}+\frac{1111\alpha^4p_1^3(p_3-q_3)}{544320}+\frac{\alpha(\alpha-1)p_1(p_5+q_5)}{96}\\
&+\frac{17\alpha^3p_1^2(p_4-q_4)}{41472}+\frac{\alpha(\alpha-1)(p_3^2-q_3^2)}{192}+\frac{2\alpha^2(\alpha-1)p_1(p_2p_3-q_2q_3)}{105}\\
&-\frac{102593\alpha^4(\alpha-1)(\alpha-2)p_1^6}{8164800}+\frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)p_1^2(p_2-q_2)}{384}\\
&+\frac{52\alpha^2(\alpha-1)(\alpha-2)p_1^3(p_3-q_3)}{4725}+\frac{\alpha(\alpha-1)(\alpha-2)p_1(p_2p_3+q_2q_3)}{96}\\
&+\frac{17\alpha^3(\alpha-1)p_1^2(p_2+q_2)(p_2-q_2)}{15360}+\frac{119\alpha^3(\alpha-1)(\alpha-2)p_1^4(p_2-q_2)}{207360}\\
&+\frac{\alpha(\alpha-1)(\alpha-2)p_1^2(p_4-q_4)}{192}+\frac{\alpha^2(\alpha-1)(\alpha-2)(\alpha-3)p_1^4(p_2+q_2)}{315}\\
&+\frac{\alpha^2(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)p_1^6}{3150}+\frac{\alpha^2(\alpha-1)(\alpha-2)p_1^2(p_2^2+q_2^2)}{105}\\
&+\frac{\alpha(\alpha-1)(\alpha-2)(p_3^2-q_3^2)}{576}+\frac{17\alpha^3(\alpha-1)p_1^3(p_3+q_3)}{41472}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)p_1^3(p_3+q_3)}{576}+\frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)p_1^4(p_2-q_2)}{2304}\\
&-\frac{1073\alpha^4(\alpha-1)p_1^4(p_2+q_2)}{33600}+\frac{407\alpha^2(\alpha-1)p_1(p_2-q_2)(p_3+q_3)}{73728}\\
&+\frac{2\alpha^2(\alpha-1)p_1^2(p_4+q_4)}{105}+\frac{17\alpha^3(\alpha-1)p_1^2(p_2^2-q_2^2)}{82944}-\frac{13\alpha^3(p_2-q_2)^3}{786432}\\
&+\frac{407\alpha^2(\alpha-1)(\alpha-2)p_1^2(p_2-q_2)^2}{147456}+\frac{\alpha^2(\alpha-1)^2(\alpha-2)p_1^4(p_2+q_2)}{675}\\
&+\frac{\alpha(\alpha-1)(p_2p_4-q_2q_4)}{96}+\frac{407\alpha^2(p_2-q_2)(p_4-q_4)}{73728}\Bigg)+\frac{\alpha^2(p_3-q_3)^2}{450}\\
&+\frac{\alpha^2(\alpha-1)p_1(p_2+q_2)(p_3-q_3)}{225}+\frac{107\alpha^2(\alpha-1)(p_2-q_2)(p_2^2-q_2^2)}{147456}\\
&+\frac{17\alpha^3p_1(p_2-q_2)(p_3-q_3)}{15360}+\frac{\alpha^2(\alpha-1)^2(\alpha-2)^2p_1^6}{4050}+\frac{\alpha^2(\alpha-1)^2p_1^2(p_2+q_2)^2}{450}\Bigg)\Bigg|\\
\leq\;&\frac{\alpha}{24}+\frac{\alpha(1-\alpha)}{12}+\frac{15991\alpha^6|p_1|^6}{4592700}+\frac{1111\alpha^4|p_1|^6}{1088640}+\frac{\alpha(1-\alpha)(4-p_1^2)^2}{192}\\
&+\frac{\alpha(1-\alpha)|p_1|}{24}+\frac{1111\alpha^4|p_1|^3}{544320}+\frac{17\alpha^3|p_1|^2}{10368}+\frac{\alpha(1-\alpha)|p_1|^3(4-p_1^2)}{384}\\
&+\frac{8\alpha^2(1-\alpha)|p_1|^2}{105}+\frac{17\alpha^3(1-\alpha)|p_1|^3(4-p_1^2)}{41472}+\frac{\alpha(1-\alpha)(2-\alpha)|p_1|}{12}\\
&+\frac{16\alpha^2(1-\alpha)|p_1|}{105}+\frac{\alpha(1-\alpha)(2-\alpha)|p_1|^2}{48}++\frac{\alpha(1-\alpha)^2(2-\alpha)|p_1|^6}{675}\\
&+\frac{8\alpha^2|p_1|}{105}+\frac{1073\alpha^4(1-\alpha)|p_1|^6}{33600}+\frac{\alpha(1-\alpha)^2(2-\alpha)^2|p_1|^6}{4050}\\
&+\Bigg(\frac{901597\alpha^5|p_1|^4(4-p_1^2)}{104509440}+\frac{1111\alpha^4|p_1|^4(4-p_1^2)}{1088640}+\frac{17\alpha^3|p_1|(4-p_1^2)^2}{30720}\\
&+\frac{\alpha(1-\alpha)|p_1|^4(4-p_1^2)}{768}+\frac{17\alpha^3|p_1|^4(4-p_1^2)}{61440}+\frac{\alpha^4(1-\alpha)|p_1|^4(4-p_1^2)}{67200}\\
&+\frac{\alpha(1-\alpha)(2-\alpha)(4-p_1^2)}{96}+\frac{\alpha^2(1-\alpha)^2(2-\alpha)|p_1|^4(4-p_1^2)}{1350}\\
&+\frac{119\alpha^3(1-\alpha)|p_1|^4(4-p_1^2)}{138240}+\frac{\alpha(1-\alpha)|p_1|(4-p_1^2)^2}{192}\Bigg)(|x|+|y|)\\
&+\Bigg(\frac{1111\alpha^4|p_1|^4(4-p_1^2)}{2177280}-\frac{1111\alpha^4|p_1|^3(4-p_1^2)}{1088640}-\frac{\alpha(1-\alpha)|p_1|^3(4-p_1^2)}{768}\\
&+\frac{\alpha(1-\alpha)|p_1|(4-p_1^2)^2}{768}-\frac{\alpha(1-\alpha)(4-p_1^2)^2}{192}-\frac{17\alpha^3(1-\alpha)|p_1|^3(4-p_1^2)}{82944}\Bigg)\\
&\times(|x|^2+|y|^2)+\Bigg(\frac{17\alpha^3(1-\alpha)|p_1|^4(4-p_1^2)}{165888}+\frac{17\alpha^3(1-\alpha)|p_1|^2(4-p_1^2)^2}{51840}\\
&+\frac{17\alpha^3|p_1|^2(4-p_1^2)^2}{61440}+\frac{\alpha(1-\alpha)|p_1|^2(4-p_1^2)^2}{768}+\frac{\alpha(1-\alpha)|p_1|(4-p_1^2)^2}{768}\\
&+\frac{\alpha(1-\alpha)|p_1|^4(4-p_1^2)}{1536}+\frac{225737\alpha^4|p_1|^2(4-p_1^2)^2}{74317824}\Bigg)(|x|+|y|)^2
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\frac{\alpha(1-\alpha)|p_1|^2(4-p_1^2)^2}{1536}-\frac{17\alpha^3|p_1|(4-p_1^2)^2}{61440}-\frac{\alpha(1-\alpha)|p_1|(4-p_1^2)^2}{384}\right.\\
&\left.+\frac{17\alpha^3|p_1|^2(4-p_1^2)^2}{122880}\right)(|x|+|y|)(|x|^2+|y|^2)+\frac{\alpha(1-\alpha)|p_1|^2(4-p_1^2)^2}{1536}\Bigg)\\
&\times(|x|+|y|)^3+\left(\frac{\alpha(1-\alpha)|p_1|^2(4-p_1^2)^2}{3072}-\frac{\alpha(1-\alpha)|p_1|(4-p_1^2)^2}{1536}\right)(|x|+|y|)^2\\
&\times(|x|^2+|y|^2)+\left(-\frac{\alpha(1-\alpha)|p_1|(4-p_1^2)^2}{1536}+\frac{\alpha(1-\alpha)(4-p_1^2)^2}{768}\right)(|x|^2+|y|^2)^2.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
|A_7|\le{}& J_1+J_2(\mu_1+\mu_2)+J_3(\mu_1^2+\mu_2^2)+J_4(\mu_1+\mu_2)^2+J_5(\mu_1+\mu_2)(\mu_1^2+\mu_2^2)+J_6(\mu_1+\mu_2)^3\\
&+J_7(\mu_1+\mu_2)^2(\mu_1^2+\mu_2^2)+J_8(\mu_1^2+\mu_2^2)^2=J(\mu_1,\mu_2),
\end{aligned}
$$

where the constants $J_1\ge 0,\ J_2\ge 0,\ J_3\le 0,\ J_4\ge 0,\ J_5\le 0,\ J_6\ge 0,\ J_7\le 0$, and $J_8\ge 0$. By Lemma 2, we determine the maximum value of $J(\mu_1,\mu_2)$ on $\mathbb{S}=[0,1]\times[0,1]$. Since $J$ is continuous on the compact set $\mathbb{S}$, it attains a global maximum on $\mathbb{S}$. We first investigate the possibility of an interior maximizer. The first-order partial derivatives of $J(\mu_1,\mu_2)$ are given by

$$
\begin{aligned}
J_{\mu_1}={}&J_2+2J_3\mu_1+2J_4\mu_1+2J_4\mu_2+3J_5\mu_1^2+J_5\mu_2^2+2J_5\mu_1\mu_2+3J_6\mu_1^2+6J_6\mu_1\mu_2\\
&+3J_6\mu_2^2+4J_7\mu_1^3+6J_7\mu_1^2\mu_2+4J_7\mu_1\mu_2^2+2J_7\mu_2^3+4J_8\mu_1^3+4J_8\mu_1\mu_2^2,\\
J_{\mu_2}={}&J_2+2J_3\mu_2+2J_4\mu_1+2J_4\mu_2+2J_5\mu_1\mu_2+J_5\mu_1^2+3J_5\mu_2^2+3J_6\mu_1^2+6J_6\mu_1\mu_2\\
&+3J_6\mu_2^2+2J_7\mu_1^3+4J_7\mu_1^2\mu_2+6J_7\mu_1\mu_2^2+4J_7\mu_2^3+4J_8\mu_1^2\mu_2+4J_8\mu_2^3.
\end{aligned}
$$

By Fermat's theorem, any interior local extremum of $J(\mu_1,\mu_2)$ must satisfy $J_{\mu_1}=0$, and $J_{\mu_2}=0$. Subtracting these equations yields

$$
\frac{\partial J}{\partial\mu_1}-\frac{\partial J}{\partial\mu_2}=(\mu_1-\mu_2)\Big[2J_3+2J_5(\mu_1+\mu_2)+2J_7(\mu_1+\mu_2)^2+4J_8(\mu_1^2+\mu_2^2)\Big]=0.
$$

Since

$$
2J_3+2J_5(\mu_1+\mu_2)+2J_7(\mu_1+\mu_2)^2+4J_8(\mu_1^2+\mu_2^2)\neq 0
$$

for all $(\mu_1,\mu_2)\in(0,1)^2$, it follows that any interior critical point must satisfy $\mu_1=\mu_2$. Setting $\mu_1=\mu_2=\mu$, the function $J$ reduces to

$$
J(\mu,\mu)=J_1+2J_2\mu+2J_3\mu^2+4J_4\mu^2+4J_5\mu^3+8J_6\mu^3+8J_7\mu^4+4J_8\mu^4.
$$

Differentiating with respect to $\mu$ gives

$$
J'(\mu,\mu)=2J_2+4(J_3+2J_4)\mu+12(J_5+2J_6)\mu^2+16(2J_7+J_8)\mu^3.
$$

Therefore, any interior critical point must satisfy

$$
2J_2+4(J_3+2J_4)\mu+12(J_5+2J_6)\mu^2+16(2J_7+J_8)\mu^3=0.
$$

Next, we evaluate $J(\mu,\mu)$ at the endpoints of the interval $[0,1]$:

$$
\begin{aligned}
J(0)&=J_1,\\
J(1)&=J_1+2J_2+2J_3+4J_4+4J_5+8J_6+8J_7+4J_8.
\end{aligned}
$$

Since no interior critical point exists, the maximum of $J(\mu,\mu)$ on $[0,1]$ must occur at an endpoint. Therefore,

$$\max_{0\le\mu\le1} J(\mu,\mu) = \max\{J(0), J(1)\}.$$

Furthermore,

$$J(1) - J(0) = 2J_2 + 2J_3 + 4J_4 + 4J_5 + 8J_6 + 8J_7 + 4J_8.$$

Since $J(1) \ge J(0)$, it follows that

$$\max_{0\le\mu\le1} J(\mu,\mu) = J(1).$$

Hence, the maximum is attained at $\mu_1 = \mu_2 = 1$. Hence,

$$\max J(\mu_1,\mu_2) = J(1,1) = \mathcal{P}(p) = J_1 + 2(J_2 + J_3) + 4(J_4 + J_5 + J_8) + 8(J_6 + J_7).$$

Substituting the expressions for $J_1, J_2, J_3, J_4, J_5, J_6, J_7$, and $J_8$ yields $|A_7| \le \mathcal{J}(p)$. Since $\mathcal{J}(p)$ is an increasing function of $|p|$ on $[0,2]$, its maximum is attained at $|p| = 2$. Therefore,

$$|A_7| \le \alpha\Bigg(\frac{1}{24} + \frac{(1-\alpha)}{6} + \frac{(1-\alpha)(2-\alpha)}{4} + \frac{64(1-\alpha)^2(2-\alpha)}{675} + \frac{32(1-\alpha)^2(2-\alpha)^2}{2025} + \frac{16\alpha}{105}$$
$$+ \frac{64\alpha(1-\alpha)}{105} + \frac{17\alpha^2}{2592} + \frac{1111\alpha^3}{13608} + \frac{1073\alpha^3(1-\alpha)}{525} + \frac{255856\alpha^5}{1148175}\Bigg).$$

Thus, the desired upper bound for $|A_7|$ is obtained. □

By specializing the parameter $\alpha$ to 1 in Theorem 1, the bounds for the initial coefficients reduce to those given in the following corollary.

**Corollary 1.** *Let $f(z)$ given by Eq. (1) be in the class $\mathbf{B}\Sigma(\alpha)$, then*

$$|a_2| \le \frac{2}{3},$$
$$|a_3| \le \frac{4}{9},$$
$$|a_4| \le \frac{14}{81},$$
$$|a_5| \le \frac{1079}{1620},$$
$$|a_6| \le \frac{10831}{34020},$$
$$|a_7| \le \frac{18557687}{36741600}.$$

## 4. Conclusion

In this paper, we established upper bounds for the initial seven coefficients of functions belonging to the considered subclass of bi-univalent functions. By employing the corresponding subordination conditions together with coefficient estimates associated with the Carathéodory class, explicit bounds for the coefficients $a_2, a_3, \ldots, a_7$ were derived.

Furthermore, analogous estimates were obtained for the first seven coefficients of the inverse function. It was shown that the inverse coefficients satisfy the same upper bounds as the corresponding coefficients of the original function. These results extend several existing studies in the literature, which mainly focus on lower-order coefficients, by providing estimates up to the seventh coefficient and their inverses.

It is expected that the techniques developed in this work may be useful in investigating higher-order coefficients, inverse coefficient problems, and related Hankel determinant problems for other subclasses of bi-univalent functions.

## CRediT Authorship Contribution Statement

**Alifira Meliana Rachman:** Conceptualization, Methodology, Formal Analysis, Investigation, Writing–Original Draft Preparation. **Marjono Marjono:** Conceptualization, Methodology, Validation, Supervision, Writing–Review & Editing. **Saadatul Fitri:** Formal Analysis, Validation, Writing–Review & Editing, Supervision.

## Declaration of Generative AI and AI-assisted Technologies

ChatGPT (OpenAI) was used solely for language editing, grammar checking, and improving the clarity of the manuscript. The authors reviewed and verified all mathematical results, proofs, and conclusions presented in this manuscript and take full responsibility for the content of the work.

## Declaration of Competing Interest

The authors declare no competing interests.

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## Data and Code Availability

No datasets were generated or analyzed during the current study. No custom code was developed for this research. All results were obtained through theoretical mathematical analysis. Therefore, data and code sharing is not applicable to this study.

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