



Packing Chromatic Number of Graph Families Constructed from the Fan Graph

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Abstract

Packing coloring is particularly sensitive to graph distances, since even a simple structural modification may change whether a color can be reused. This paper investigates the packing chromatic number of several constructions derived from the fan graph, namely the subdivision fan $S(F_n)$, the umbrella graph $U_{r,s}$, two barbell-type fan constructions, and the generalized q -fan chain $CF_m(q)$ formed by sequentially joining the centers of q copies of F_m . The known packing chromatic number of the ordinary fan is used only as a reference result. By combining explicit packing colorings with lower bound arguments based on independence numbers, graph diameters, color class capacities, and cross-component distances, we prove that $\chi_\rho(S(F_n)) = 4$ for $n \geq 2$ and determine the exact packing chromatic numbers of the umbrella and both barbell-type constructions. For $CF_m(q)$, we establish general lower and upper bounds for all $m, q \geq 2$ by showing that a color $\ell \geq 2$ can occur on at most $\lceil q/(\ell - 1) \rceil$ vertices. Moreover, when $m \geq 2q - 2$, the bounds coincide and yield $\chi_\rho(CF_m(q)) = q(\lfloor \frac{m}{2} \rfloor + 2) - \sum_{t=1}^{q-1} \lceil \frac{q}{t} \rceil$. These results show that the packing chromatic behavior of fan-derived graphs is governed by the distance structure created by the underlying graph operation, particularly the extent to which low colors can be reused across different fan copies.

Keywords: Packing chromatic number; Fan graph; Subdivision graph; Fan chain; Distance-constrained coloring.

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1. Introduction

Packing coloring is a distance-constrained form of vertex coloring in which the admissibility of repeated colors is governed by the metric structure of a graph. More precisely, vertices assigned color i must be separated by a distance greater than i . This requirement makes packing coloring highly sensitive to structural changes, since even a modest graph operation may create shorter routes between vertices or, conversely, increase their separation sufficiently to permit color reuse. The concept was introduced by Goddard *et al.* under the name broadcast coloring [1] and has since developed into an established topic in distance-based graph coloring [2]. Its computational difficulty is already evident on restricted graph classes, including trees [3], while further complexity aspects have been investigated in [4]. These features make packing coloring particularly suitable for examining how graph operations reshape distance constraints.

The fan graph provides a compact yet structurally rich setting for this purpose. For $F_n = K_1 + P_n$, every path vertex is adjacent to a common central vertex, so distances that would

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otherwise be determined by the path may be substantially shortened through the center. Packing colorings of fan-related graphs have therefore been investigated under several constructions. Circular fans were considered in [5], while packing chromatic numbers of particular fan- and wheel-related graph families were studied in [6]. Other transformations associated with fan graphs, including middle, total, central, and line graphs, were examined from the packing-coloring perspective in [7]. The packing chromatic number of the ordinary fan also follows from the more general result for graph joins established in [8]. Together, these studies show that fan-based structures provide a natural framework for investigating the interaction between graph operations and distance-dependent coloring.

Edge subdivision offers one way to alter this interaction. By inserting a new vertex into each edge, subdivision changes many pairwise distances while preserving the incidence pattern represented by the original graph. General results on packing chromatic numbers of subdivision graphs were obtained in [9], while the effect of local graph modifications on packing coloring was studied further in [10]. More specialized investigations include subdivisions of cubic graphs and finite super subdivisions [11, 12]. The standard subdivision graph $S(G)$ is described, for example, in [13]. Subdivisions of fan graphs have also appeared in the study of other graph invariants, including cycle-antimagic labeling [14]. From a packing-coloring perspective, the essential issue is how the additional vertices modify the distance restrictions that determine the minimum number of colors.

A different metric effect arises from attaching paths or coupling several fan structures. Umbrella-type graphs combine a fan with an attached path, and related constructions have been investigated, for example, through partition dimension [15]. Amalgamation and barbell-type constructions have likewise appeared in studies of packing and locating colorings [16, 17]. Such connections are particularly relevant to packing coloring because the admissibility of repeated colors depends not only on the structure of each component but also on the position of the connecting edge. A color that cannot occur twice within a single fan may still be reused in another fan copy when the resulting cross-component distance is sufficiently large. Thus, the geometry of the connection becomes an essential part of the coloring problem.

An even broader distance structure arises when several fan copies are linked sequentially. The generalized q -fan chain $CF_m(q)$ consists of q copies of F_m whose central vertices are joined consecutively to form a path. Equivalently, this construction can be viewed as the rooted product $P_q \circ_u F_m$, where u denotes the central vertex of F_m . The resulting structure introduces an additional level of separation between fan copies, which directly affects the admissibility of repeated colors. In particular, increasing the distance between two copies along the central path creates greater separation between vertices assigned the same color. This observation leads naturally to spacing conditions on color occurrences across the fan indices and, consequently, to bounds on the corresponding color class sizes. The packing-coloring analysis of $CF_m(q)$ developed here uses this distance structure to derive general bounds and to identify conditions under which the packing chromatic number can be determined exactly.

This paper investigates the packing chromatic number of the subdivision fan graph $S(F_n)$, the umbrella graph $U_{r,s}$, two barbell-type fan constructions $B_{u_i}(F_m)$ and $B_{v_{i,1}}(F_r)$, and the generalized fan chain $CF_m(q)$. The analysis combines explicit packing colorings with lower bound arguments based on independence numbers, graph diameter, color class capacities, and distances between vertices belonging to different fan copies. For the generalized fan chain, the separation of fan indices provides a systematic way to control repeated colors, leading to general lower and upper bounds and an exact value under an appropriate relation between m and q . These results highlight how subdivision, path attachment, barbell coupling, and sequential fan connections affect packing chromatic behavior through the distance structures they create.

2. Methods

This study employs a theoretical and combinatorial approach to determine packing chromatic numbers and related bounds for fan-derived graph constructions. Each construction is specified in terms of its vertex sets, connecting edges, and graph parameters before the corresponding packing-coloring analysis is carried out. The known packing chromatic number of the ordinary fan F_n , obtained from the general result for graph joins [8], is used where appropriate as a reference in establishing bounds for the derived constructions.

The analysis is driven primarily by the metric structure of each graph. Distances are examined both within individual fan components and between vertices belonging to different components. This distinction is essential because a color that cannot be repeated within a single fan may still be admissible in another component when the connecting structure provides sufficient separation. Upper bounds are therefore obtained through explicit packing colorings. Whenever two distinct vertices $x, y \in V(G)$ receive the same color ℓ , their positions are checked against the packing condition $d_G(x, y) \geq \ell + 1$. Particular attention is given to repeated low colors, whose admissibility is especially sensitive to the distances created by the graph operation under consideration.

Lower bounds are derived from structural restrictions on the color classes. Depending on the graph, the arguments use subgraph monotonicity [2], independence numbers, graph diameters, and capacity bounds for individual colors. Color 1 is controlled through independent set arguments, whereas colors $\ell \geq 2$ are restricted by their required pairwise distances. These restrictions are then combined to bound the number of vertices that can share a color and to account directly for possible color reuse across different components.

For the generalized fan chain $CF_m(q)$, the fan copies are indexed according to the path induced by their central vertices. This indexing translates the packing requirement for a repeated color into a minimum separation between the indices of the fan copies in which that color appears. The resulting spacing condition yields an upper bound on the size of each color class and forms the basis of the general lower bound argument. A corresponding upper bound is obtained from an explicit coloring constructed by assigning repeated colors to suitably spaced fan copies. Under an additional relation between m and q , the lower and upper bounds coincide, yielding an exact value of the packing chromatic number.

Small parameter cases are treated separately whenever the general structural argument does not apply directly. In such cases, explicit color assignments are combined with matching lower bound arguments. Accordingly, an exact packing chromatic number is stated only when a valid packing coloring and a corresponding lower bound have both been established. All results reported in this study are therefore supported by explicit distance verification and combinatorial arguments.

3. Results and Discussion

This section develops the packing-coloring results for the fan-derived constructions considered in this study. We begin by fixing the notation and recalling the preliminary facts used throughout the proofs. Particular attention is given to distances between vertices belonging to different fan copies, since these distances determine whether a color may be reused across components.

Throughout the paper, all graphs are finite, simple, and connected. For a graph G , its vertex set and edge set are denoted by $V(G)$ and $E(G)$, respectively, and its order is $|V(G)|$. The distance between two vertices $x, y \in V(G)$ is denoted by $d_G(x, y)$. When the underlying graph is clear from the context, we simply write $d(x, y)$. The diameter of G is defined by $\text{diam}(G) = \max\{d_G(x, y) : x, y \in V(G)\}$. A path and a cycle of order n are denoted by P_n and C_n , respectively. The symbols $\lfloor x \rfloor$ and $\lceil x \rceil$ denote the floor and ceiling of a real number x .

For two vertex-disjoint graphs G and H , their join $G + H$ is obtained from their disjoint union by adding every edge xy with $x \in V(G)$ and $y \in V(H)$. The fan graph F_n is the join $F_n = K_1 + P_n$, where K_1 is the one-vertex complete graph [6]. We write $V(F_n) = \{u, v_1, v_2, \dots, v_n\}$, where u is

the center of the fan and $v_1v_2 \cdots v_n$ is the underlying path. Thus, $uv_j \in E(F_n)$, $1 \leq j \leq n$, and $v_jv_{j+1} \in E(F_n)$, $1 \leq j \leq n-1$.

An independent set of G is a set of pairwise nonadjacent vertices. The maximum cardinality of an independent set of G is called its *independence number* and is denoted by $\alpha(G)$. We shall use the elementary identity $\alpha(P_n) = \lceil \frac{n}{2} \rceil$. Indeed, choosing alternate vertices gives an independent set of this size. Conversely, grouping consecutive vertices of the path into adjacent pairs, with one possible unpaired vertex when n is odd, shows that no larger independent set exists.

Definition 1. [2] Let G be a connected graph. A packing k -coloring of G is a mapping $c : V(G) \rightarrow \{1, 2, \dots, k\}$ such that whenever two distinct vertices $x, y \in V(G)$ satisfy $c(x) = c(y) = \ell$, one has $d_G(x, y) \geq \ell + 1$. The smallest positive integer k for which such a coloring exists is called the *packing chromatic number* of G and is denoted by $\chi_\rho(G)$.

For a packing coloring c , the color class corresponding to color ℓ is denoted by $C_\ell = \{v \in V(G) : c(v) = \ell\}$. We shall use the following standard facts.

Lemma 1. [2] If H is a subgraph of a graph G , then $\chi_\rho(H) \leq \chi_\rho(G)$.

Lemma 2. [1, 2] For $n \geq 2$,

$$\chi_\rho(P_n) = \begin{cases} 2, & 2 \leq n \leq 3, \\ 3, & n \geq 4. \end{cases}$$

Since $F_n = K_1 + P_n$, its packing chromatic number is already contained in the more general result for graph joins established in [8].

Lemma 3. [8] For $n \geq 2$,

$$\chi_\rho(F_n) = \left\lfloor \frac{n}{2} \right\rfloor + 2.$$

The following elementary observation will be used repeatedly in the lower bound arguments.

Lemma 4. Let c be a packing coloring of a graph G . Then C_1 is an independent set and hence $|C_1| \leq \alpha(G)$. Moreover, if $X \subseteq V(G)$ satisfies $d_G(x, y) \leq D$ for every pair of distinct vertices $x, y \in X$, then $|C_\ell \cap X| \leq 1$ for every $\ell \geq D$.

Proof. If two distinct vertices belong to C_1 , the packing condition requires their distance to be at least 2. Thus no two vertices of C_1 are adjacent, so C_1 is independent and $|C_1| \leq \alpha(G)$. Now let $\ell \geq D$. If two distinct vertices $x, y \in X$ both received color ℓ , then $d_G(x, y) \geq \ell + 1 \geq D + 1$, contradicting the assumption $d_G(x, y) \leq D$. Hence at most one vertex of X can receive color ℓ . $\square$

We first consider edge subdivision. The standard subdivision operation is used in the following form.

Definition 2. [13] Let $G = (V, E)$ be a connected graph. The subdivision graph $S(G)$ is obtained by replacing every edge $xy \in E(G)$ with the path $xw_{xy}y$, where w_{xy} is a new vertex associated with the edge xy .

For F_n , let x_i denote the subdivision vertex inserted into the edge uv_i , for $1 \leq i \leq n$, and let y_i denote the subdivision vertex inserted into v_iv_{i+1} , for $1 \leq i \leq n-1$.

Theorem 1. For every $n \geq 2$, $\chi_\rho(S(F_n)) = 4$.

Proof. We first establish the lower bound. Since $n \geq 2$, the vertices u, v_1, v_2 form a triangle in F_n . After subdivision, its three edges form the cycle $u x_1 v_1 y_1 v_2 x_2 u$, which is isomorphic to C_6 . We claim that C_6 admits no packing 3-coloring. Suppose otherwise. Since $\alpha(C_6) = 3$, we have $|C_1| \leq 3$. Two vertices receiving color 2 must be at distance at least 3. On C_6 , color 2 can therefore occur on at most two vertices. Moreover, $\text{diam}(C_6) = 3$, so color 3, whose repeated occurrences would require distance at least 4, can occur at most once. Consequently, $|C_1| + |C_2| + |C_3| \leq 3 + 2 + 1 = 6$. A packing 3-coloring of all six vertices would force equality in each of these bounds. In particular, $|C_1| = 3$ and $|C_2| = 2$. Every independent set of order 3 in C_6 consists of alternating vertices. The three remaining vertices are then pairwise at distance 2, so color 2 cannot occur twice among them. This contradiction shows that $\chi_\rho(C_6) \geq 4$. Since C_6 is a subgraph of $S(F_n)$, Lemma 1 gives $\chi_\rho(S(F_n)) \geq 4$. For the upper bound, define $c(x_i) = 1$, $1 \leq i \leq n$, and $c(y_i) = 1$, $1 \leq i \leq n - 1$. All subdivision vertices form an independent set because every edge of $S(F_n)$ joins an original vertex of F_n to a subdivision vertex. Hence color 1 is valid. For the original path vertices, define

$$c(v_i) = \begin{cases} 2, & i \text{ odd}, \\ 3, & i \text{ even}, \end{cases}$$

and set $c(u) = 4$. For distinct path vertices v_i and v_j , $d_{S(F_n)}(v_i, v_j) = \min\{2|i - j|, 4\}$. Indeed, a shortest route either follows the subdivided path or passes through the two subdivided spokes incident with u . If i and j have the same parity, then $|i - j| \geq 2$, and consequently $d_{S(F_n)}(v_i, v_j) = 4$. Thus repeated occurrences of color 2 are separated by at least 3, while repeated occurrences of color 3 are separated by at least 4. Color 4 occurs only at u . Hence c is a packing 4-coloring and $\chi_\rho(S(F_n)) \leq 4$. Combining the two bounds proves the theorem. $\square$

The structure of a subdivision fan and the coloring used in the upper-bound construction are illustrated in Fig. 1.

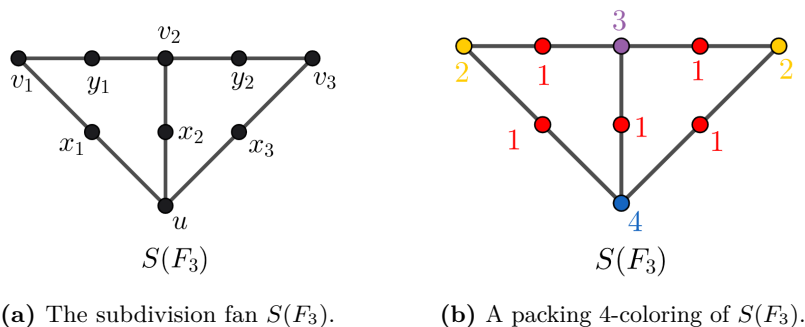


Fig. 1: Illustration of the subdivision fan $S(F_3)$ and the packing coloring used in Theorem 1. The left panel shows the subdivided graph, while the right panel displays a valid packing 4-coloring.

We next consider the umbrella graph. The fan-plus-path structure follows the type of umbrella construction considered in [15]; however, the convention used here is stated explicitly so that the meaning of the parameter s is unambiguous.

Definition 3. [15] Let $r, s \geq 1$. Let F_r have center u and path vertices $v_1, \dots, v_r$, and let $P_s = w_1 w_2 \dots w_s$ be a path of order s . The umbrella graph $U_{r,s}$ is obtained by identifying w_1 with u . Thus the attached path has s vertices, one of which is the center of the fan.

Theorem 2. For $r, s \geq 1$,

$$\chi_\rho(U_{r,s}) = \begin{cases} 2, & r = 1 \text{ and } s \leq 2, \\ 3, & r = 1 \text{ and } s \geq 3, \\ \left\lfloor \frac{r}{2} \right\rfloor + 2, & r \geq 2. \end{cases}$$

Proof. Suppose first that $r = 1$. Since $F_1 \cong K_2$ and $w_1 = u$, the graph $U_{1,s}$ is isomorphic to P_{s+1} . The first two cases therefore follow immediately from Lemma 2. Now assume $r \geq 2$ and set $a = \lfloor \frac{r}{2} \rfloor$. Since F_r is a subgraph of $U_{r,s}$, Lemmas 1 and 3 give $\chi_\rho(U_{r,s}) \geq \chi_\rho(F_r) = a + 2$. We construct a packing coloring attaining this bound. On the fan part, set $c(v_{2j-1}) = 1$, $1 \leq j \leq \lceil \frac{r}{2} \rceil$, and $c(v_{2j}) = j + 1$, $1 \leq j \leq a$, together with $c(u) = a + 2$. The vertices receiving color 1 are pairwise nonadjacent, while every color from 2 to $a + 1$ occurs only once in the fan. Starting at w_2 , assign the periodic pattern 1, 2, 1, 3, 1, 2, 1, 3, ... More precisely, for each integer $h \geq 0$, whenever the corresponding indices do not exceed s , set $c(w_{4h+2}) = 1$, $c(w_{4h+3}) = 2$, $c(w_{4h+4}) = 1$, and $c(w_{4h+5}) = 3$. Within the attached path, consecutive occurrences of color 1 are at distance 2, whereas consecutive occurrences of colors 2 and 3 are at distance 4. It remains to check colors occurring both in the fan and on the attached path. If w_2 exists, then for every odd-indexed fan vertex, $d(v_{2j-1}, w_2) = 2$. If w_3 exists, the fan vertex receiving color 2 is v_2 , and $d(v_2, w_3) = 3$. If a tail vertex receiving color 3 exists, its first possible occurrence is w_5 . When $r = 2$ or $r = 3$, we have $a = 1$, so the fan center receives color 3, and $d(u, w_5) = 4$. When $r \geq 4$, the fan vertex receiving color 3 is v_4 , and $d(v_4, w_5) = d(v_4, u) + d(u, w_5) = 1 + 4 = 5$. Every later occurrence of color 3 on the attached path is farther from the fan. No color greater than 3 is used on the attached path. Therefore all repeated colors satisfy the packing condition, and the coloring uses only $a + 2$ colors. Hence $\chi_\rho(U_{r,s}) \leq a + 2$. Together with the lower bound, this gives $\chi_\rho(U_{r,s}) = a + 2$ for $r \geq 2$. $\square$

An instance of the umbrella construction and its packing coloring is shown in Fig. 2.

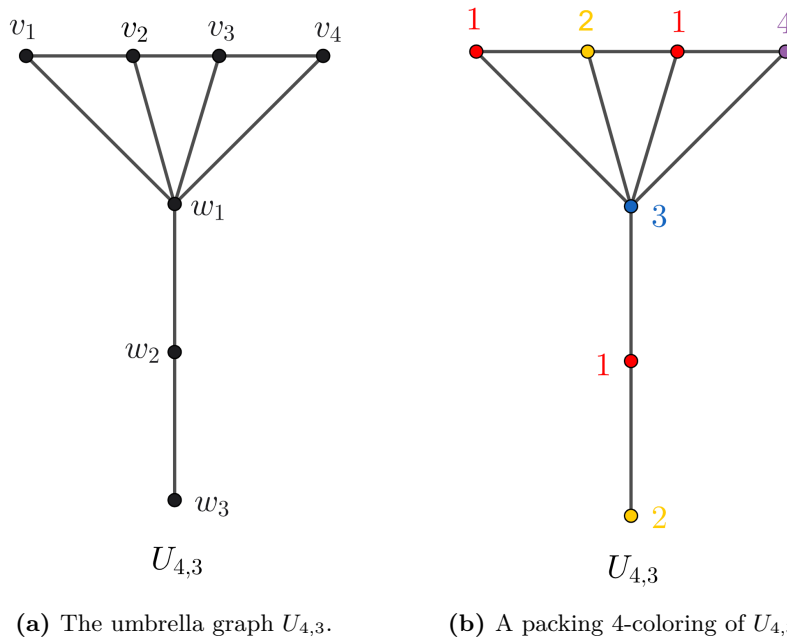


Fig. 2: The umbrella graph $U_{4,3}$ under the path-order convention used in this paper and a packing coloring attaining $\chi_\rho(U_{4,3}) = 4$.

We now consider two different ways of coupling two copies of a fan. The first construction joins their centers. Barbell- and amalgamation-type graph constructions have appeared in related coloring studies [16, 17]. The precise fan-based construction considered here is defined as follows.

Definition 4. Let $m \geq 2$. Take two vertex-disjoint copies $F_m^{(1)}$ and $F_m^{(2)}$ of F_m . For $i \in \{1, 2\}$, let u_i be the center of $F_m^{(i)}$ and let $v_{i,1}, v_{i,2}, \dots, v_{i,m}$ be its path vertices. The barbell fan graph type I, denoted by $B_{u_i}(F_m)$, is obtained by adding the edge $u_1 u_2$. Thus the notation $B_{u_i}(F_m)$ records that the two copies are joined through their central vertices.

Theorem 3. For every $m \geq 2$, $\chi_\rho(B_{u_i}(F_m)) = 2 \lfloor \frac{m}{2} \rfloor + 2$.

Proof. Set $a = \lfloor \frac{m}{2} \rfloor$. We first construct a packing coloring using $2a + 2$ colors. For $i \in \{1, 2\}$, assign $c(v_{i,2j-1}) = 1$, $1 \leq j \leq \lfloor \frac{m}{2} \rfloor$. These vertices are pairwise nonadjacent. Next set $c(v_{1,2}) = c(v_{2,2}) = 2$. Since the only connection between the two fan copies is the edge $u_1 u_2$, $d(v_{1,2}, v_{2,2}) = 1 + 1 + 1 = 3$, so the repeated use of color 2 is valid. Assign $c(u_1) = 3$, $c(u_2) = 4$. For every integer j with $2 \leq j \leq a$, define $c(v_{1,2j}) = 2j + 1$, $c(v_{2,2j}) = 2j + 2$. When $a = 1$, this last range is empty. All vertices have now been colored. Color 1 is assigned to an independent set, color 2 occurs twice at distance 3, and every color at least 3 occurs exactly once. Therefore $\chi_\rho(B_{u_i}(F_m)) \leq 2a + 2$. For the lower bound, first determine the independence number. Let I be an independent set. Within $F_m^{(i)}$, if $u_i \in I$, then none of its m path vertices belongs to I , so this fan contributes only one vertex. If $u_i \notin I$, at most $\alpha(P_m) = \lfloor \frac{m}{2} \rfloor$ path vertices can be selected. Hence $|I| \leq 2 \lfloor \frac{m}{2} \rfloor$. Equality is attained by taking all odd-indexed path vertices in both fans. Consequently, $\alpha(B_{u_i}(F_m)) = 2 \lfloor \frac{m}{2} \rfloor$. The graph has $2(m + 1)$ vertices. Hence at least $2(m + 1) - 2 \lfloor \frac{m}{2} \rfloor = 2 \lfloor \frac{m}{2} \rfloor + 2 = 2a + 2$ vertices must receive colors greater than 1. Each fan copy has diameter 2, so every color $\ell \geq 2$ can occur at most once within a given fan. In particular, $|C_2| \leq 2$. Furthermore, $\text{diam}(B_{u_i}(F_m)) = 3$. Indeed, vertices in the same fan are at distance at most 2, whereas vertices in different fans are at distance at most 3 through $u_1 u_2$; moreover, $d(v_{1,1}, v_{2,1}) = 3$. Consequently every color $\ell \geq 3$ can occur at most once in the entire graph. If a packing coloring uses k colors, the number of vertices that can receive colors greater than 1 is therefore at most $2 + (k - 2) = k$. Since at least $2a + 2$ vertices require such colors, $k \geq 2a + 2$. Combining the lower and upper bounds gives $\chi_\rho(B_{u_i}(F_m)) = 2a + 2$. $\square$

Figure 3 illustrates the center-to-center coupling and the controlled reuse of color 2 across the two fan copies. The second barbell-type construction connects two fan copies through end-vertices of their path parts rather than through their centers. This change produces larger cross-fan distances and consequently different possibilities for color reuse.

Definition 5. Let $r \geq 2$. Take two vertex-disjoint copies $F_r^{(1)}$ and $F_r^{(2)}$ of F_r . For $i \in \{1, 2\}$, let u_i be the center and $v_{i,1}, \dots, v_{i,r}$ the path vertices of $F_r^{(i)}$. The barbell fan graph type II, denoted by $B_{v_{i,1}}(F_r)$, is obtained by adding the edge $v_{1,1} v_{2,1}$. No edge is added between u_1 and u_2 . Thus $v_{1,1}$ and $v_{2,1}$ are precisely the endpoints of the bridge between the two fan copies.

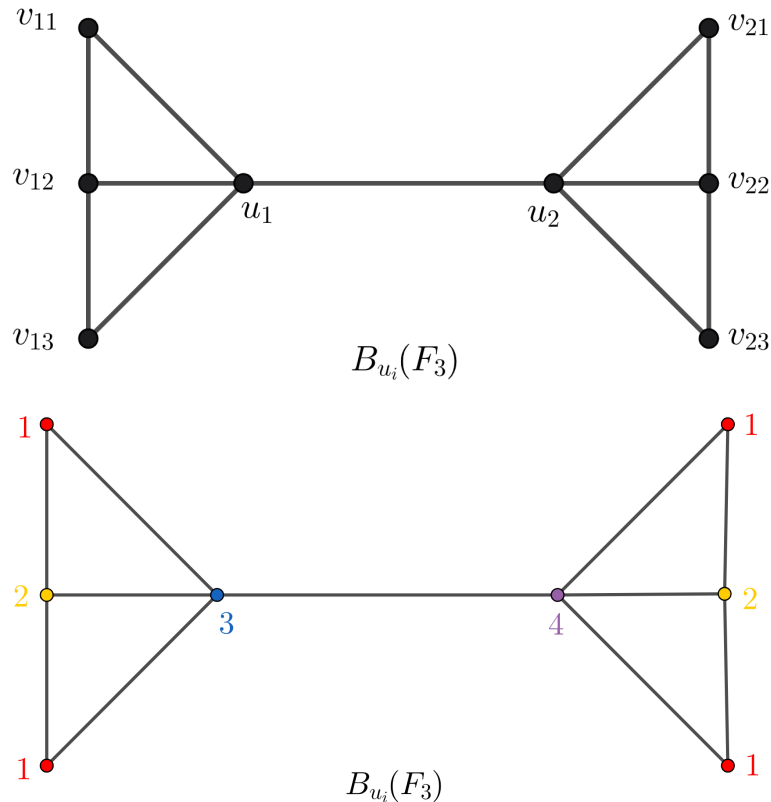


Fig. 3: The barbell fan graph type I $B_{u_i}(F_3)$ and a packing coloring attaining $\chi_\rho(B_{u_i}(F_3)) = 4$. The upper panel shows the graph construction, while the lower panel displays a valid packing coloring.

Theorem 4. For every $r \geq 2$,

$$\chi_\rho(B_{v_{i,1}}(F_r)) = \begin{cases} 4, & r = 2 \text{ or } r = 3, \\ 5, & r = 4, \\ r, & r \geq 5. \end{cases}$$

Proof. Let $G = B_{v_{i,1}}(F_r)$. We first establish two structural properties that will be used throughout the proof. Since the two fan copies are connected only through the bridge $v_{1,1}v_{2,1}$, $d_G(u_1, u_2) = d_G(u_1, v_{1,1}) + 1 + d_G(v_{2,1}, u_2) = 3$. Every vertex of a fan is at distance at most 1 from its center. Consequently, any two vertices lying in different fan copies are at distance at most $1 + 3 + 1 = 5$, and hence $\text{diam}(G) \leq 5$. We next determine the independence number of G . The path vertices induce the path $v_{1,r}v_{1,r-1} \cdots v_{1,1}v_{2,1}v_{2,2} \cdots v_{2,r}$, which has order $2r$. Selecting alternate vertices gives an independent set of cardinality r , so $\alpha(G) \geq r$. Conversely, let I be an independent set of G . If neither center belongs to I , then I is contained in the induced path P_{2r} , and therefore $|I| \leq r$. If exactly one center belongs to I , say u_1 , then no path vertex of the first fan can belong to I , while at most $\lceil r/2 \rceil$ path vertices can be selected from the second fan. Hence $|I| \leq 1 + \lceil r/2 \rceil \leq r$, $r \geq 2$. If both centers belong to I , then no path vertex can be selected, and therefore $|I| = 2 \leq r$. Thus $\alpha(G) = r$. Since $|V(G)| = 2r + 2$, every packing coloring of G assigns colors greater than 1 to at least $(2r + 2) - \alpha(G) = r + 2$ vertices.

Case $r = 2$. Define $c(u_1) = c(u_2) = 2$, $c(v_{1,1}) = c(v_{2,2}) = 1$, and $c(v_{1,2}) = 3$, $c(v_{2,1}) = 4$. The vertices receiving color 1 are nonadjacent, and $d_G(u_1, u_2) = 3$, so the repeated color 2

satisfies the packing condition. Colors 3 and 4 occur only once. Hence this is a packing 4-coloring, and $\chi_\rho(B_{v_{i,1}}(F_2)) \leq 4$. For the reverse inequality, suppose that a packing 3-coloring exists. Each copy of F_2 is a triangle. Since the three vertices of a triangle are pairwise adjacent, no color can occur twice within the same triangle. Hence any packing coloring using only the colors 1, 2, 3 must assign all three colors exactly once to each triangle. In particular, color 3 occurs once in each fan. Every vertex of the first triangle is at distance at most 1 from $v_{1,1}$, and every vertex of the second triangle is at distance at most 1 from $v_{2,1}$. Since $v_{1,1}v_{2,1} \in E(G)$, the two vertices receiving color 3 are at distance at most $1 + 1 + 1 = 3$. This contradicts the required separation of at least 4 for color 3. Therefore $\chi_\rho(B_{v_{i,1}}(F_2)) = 4$.

Case $r = 3$. Consider the coloring $c(u_1) = c(u_2) = 2$, $c(v_{1,1}) = c(v_{1,3}) = c(v_{2,2}) = 1$, and $c(v_{1,2}) = c(v_{2,3}) = 3$, $c(v_{2,1}) = 4$. The color 1 vertices form an independent set, while $d_G(u_1, u_2) = 3$. Moreover, $d_G(v_{1,2}, v_{2,3}) = d_G(v_{1,2}, v_{1,1}) + 1 + d_G(v_{2,1}, v_{2,3}) = 1 + 1 + 2 = 4$. Thus the repeated colors 2 and 3 satisfy their required distance conditions, while color 4 occurs only once. Hence $\chi_\rho(B_{v_{i,1}}(F_3)) \leq 4$. Deleting $v_{1,3}$ and $v_{2,3}$ from $B_{v_{i,1}}(F_3)$ leaves a subgraph isomorphic to $B_{v_{i,1}}(F_2)$. By Lemma 1, $\chi_\rho(B_{v_{i,1}}(F_3)) \geq \chi_\rho(B_{v_{i,1}}(F_2)) = 4$. Therefore $\chi_\rho(B_{v_{i,1}}(F_3)) = 4$.

Case $r = 4$. For the upper bound, define $c(u_1) = c(u_2) = 2$, $c(v_{1,1}) = c(v_{1,3}) = c(v_{2,2}) = c(v_{2,4}) = 1$, and $c(v_{1,2}) = c(v_{2,3}) = 3$, $c(v_{2,1}) = 4$, $c(v_{1,4}) = 5$. The vertices receiving color 1 form an independent set. Furthermore, $d_G(u_1, u_2) = 3$ and $d_G(v_{1,2}, v_{2,3}) = 4$. Colors 4 and 5 occur only once. Hence this is a packing 5-coloring, so $\chi_\rho(B_{v_{i,1}}(F_4)) \leq 5$. It remains to prove that four colors are insufficient. Suppose, to the contrary, that $G = B_{v_{i,1}}(F_4)$ admits a packing 4-coloring. Since $\alpha(G) = 4$, $|C_1| \leq 4$. Each fan has diameter 2, so every color $\ell \geq 2$ can occur at most once in each fan. Therefore $|C_2|, |C_3|, |C_4| \leq 2$. Since $|V(G)| = 10$, $10 = |C_1| + |C_2| + |C_3| + |C_4| \leq 4 + 2 + 2 + 2 = 10$. Equality must therefore hold throughout, giving $|C_1| = 4$ and $|C_2| = |C_3| = |C_4| = 2$. The class C_1 cannot contain a center. Indeed, an independent set containing exactly one center has size at most $1 + \alpha(P_4) = 3$, while an independent set containing both centers has size 2. Hence all four vertices of C_1 are path vertices. Each path P_4 contains at most two independent vertices. Since $|C_1| = 4$, exactly two vertices of C_1 lie in each fan. Every independent set of cardinality 2 in $v_{i,1}v_{i,2}v_{i,3}v_{i,4}$ contains one vertex from $\{v_{i,1}, v_{i,2}\}$ and one vertex from $\{v_{i,3}, v_{i,4}\}$. Thus, in each fan, C_1 contains exactly one vertex from each of these two sets. Now consider C_4 . Since $|C_4| = 2$ and color 4 can occur at most once in each fan, its two vertices lie in different fan copies. They must be at distance at least 5. Because the bridge $v_{1,1}v_{2,1}$ is the unique connection between the two fans, for $x \in V(F_4^{(1)})$ and $y \in V(F_4^{(2)})$, $d_G(x, y) = d_G(x, v_{1,1}) + 1 + d_G(v_{2,1}, y)$. Within either fan, $d_G(u_i, v_{i,1}) = 1$, and

$$d_G(v_{i,j}, v_{i,1}) = \begin{cases} 0, & j = 1, \\ 1, & j = 2, \\ 2, & j = 3, 4. \end{cases}$$

Therefore a cross-fan distance of 5 is possible only when both vertices are at distance 2 from their respective bridge endpoints. Consequently, the two vertices of C_4 must lie respectively in $\{v_{1,3}, v_{1,4}\}$ and $\{v_{2,3}, v_{2,4}\}$. The class C_1 has already occupied one vertex from each of these two sets. Since C_4 must occupy the other one, the union $C_1 \cup C_4$ exhausts the vertices $v_{i,3}$ and $v_{i,4}$ in each fan. Thus the only vertices remaining in each fan are its center and one vertex from $\{v_{i,1}, v_{i,2}\}$. Any pair consisting of one such remaining vertex from each fan is at distance at most 3. However, $|C_3| = 2$, and color 3 can occur at most once in each fan. Its two occurrences must therefore lie in different fan copies, where their distance is at

most 3. This contradicts the required separation of at least 4 for color 3. Hence no packing 4-coloring exists, and therefore $\chi_\rho(B_{v_{i,1}}(F_4)) = 5$.

Case $r \geq 5$. We first construct a packing r -coloring. Set $c(u_1) = c(u_2) = 2$. Since $d_G(u_1, u_2) = 3$, the repeated use of color 2 is valid. Assign color 1 alternately to the path vertices by $c(v_{1,2j-1}) = 1$ ($2j - 1 \leq r$), and $c(v_{2,2j}) = 1$ ($2j \leq r$). These vertices form an independent set. In particular, the two bridge endpoints do not both receive color 1. Next, set $c(v_{1,2}) = c(v_{2,3}) = 3$. Since the bridge is the only edge joining the two fan copies, $d_G(v_{1,2}, v_{2,3}) = d_G(v_{1,2}, v_{1,1}) + 1 + d_G(v_{2,1}, v_{2,3}) = 1 + 1 + 2 = 4$, so color 3 satisfies the packing condition. Similarly, assign $c(v_{1,4}) = c(v_{2,5}) = 4$. Since $d_G(v_{1,4}, v_{1,1}) = 2$ and $d_G(v_{2,1}, v_{2,5}) = 2$, we obtain $d_G(v_{1,4}, v_{2,5}) = 2 + 1 + 2 = 5$. Hence color 4 is also admissible. After these assignments, exactly $r - 4$ path vertices remain uncolored. List them as $z_1, z_2, \dots, z_{r-4}$ and define $c(z_h) = h + 4$, $1 \leq h \leq r - 4$. Thus the remaining vertices receive the pairwise distinct colors $5, 6, \dots, r$. Each of these colors occurs only once. We therefore obtain a packing r -coloring, and hence $\chi_\rho(G) \leq r$. For the lower bound, recall that at least $r + 2$ vertices must receive colors greater than 1. Since each fan has diameter 2, every color $\ell \geq 2$ can occur at most once in each fan. In particular, $|C_2|, |C_3|, |C_4| \leq 2$. Moreover, since $\text{diam}(G) \leq 5$, every color $\ell \geq 5$ can occur at most once in the entire graph. Suppose first that a packing coloring uses $k \leq 4$ colors. Then at most $2(k - 1) \leq 6$ vertices can receive colors greater than 1. This is impossible because $r \geq 5$ implies $r + 2 \geq 7$. Therefore $k \geq 5$. For $k \geq 5$, colors 2, 3, and 4 can occupy at most six vertices in total, while each of the $k - 4$ colors $5, 6, \dots, k$ can occur at most once. Hence the number of vertices receiving colors greater than 1 is at most $2 + 2 + 2 + (k - 4) = k + 2$. Since at least $r + 2$ vertices must receive such colors, $r + 2 \leq k + 2$, and therefore $k \geq r$. Thus $\chi_\rho(G) \geq r$. Combining this lower bound with the explicit packing r -coloring gives $\chi_\rho(B_{v_{i,1}}(F_r)) = r$, $r \geq 5$. Combining the four cases completes the proof. $\square$

The different distance structure of the endpoint-bridge construction is illustrated in Fig. 4. We next consider a chain of q fan graphs obtained by joining the centers of consecutive copies of F_m . This construction is formalized in the following definition.

Definition 6. Let $m, q \geq 2$. The q -fan chain graph, denoted by $CF_m(q)$, is obtained from q vertex-disjoint copies $F_m^{(1)}, F_m^{(2)}, \dots, F_m^{(q)}$ of F_m . For each $i \in \{1, \dots, q\}$, let u_i be the center of $F_m^{(i)}$ and let $v_{i,1}, v_{i,2}, \dots, v_{i,m}$ be its path vertices. Consecutive centers are joined by $u_i u_{i+1} \in E(CF_m(q))$, $1 \leq i \leq q - 1$. Hence the vertices $u_1, u_2, \dots, u_q$ induce the path P_q .

Here m denotes the number of path vertices in each fan, whereas q denotes the number of fan copies. The index i identifies a fan copy and j identifies a path vertex inside that copy. In the arguments below, ℓ denotes a color, while k denotes the total number of colors in a packing coloring.

The next lemma records the two structural restrictions that drive the general counting argument.

Lemma 5. For $m, q \geq 2$, $\alpha(CF_m(q)) = q \lceil \frac{m}{2} \rceil$. Moreover, if c is a packing coloring of $CF_m(q)$, then for every $\ell \geq 2$, $|C_\ell| \leq \lceil \frac{q}{\ell-1} \rceil$.

Proof. Let I be an independent set of $CF_m(q)$. Consider a fixed fan copy $F_m^{(i)}$. If $u_i \in I$, then none of its m path vertices can belong to I , so this fan contributes only one vertex. If $u_i \notin I$, at most $\alpha(P_m) = \lceil \frac{m}{2} \rceil$ of its path vertices may be selected. Since $m \geq 2$, either case contributes at most $\lceil m/2 \rceil$ vertices. Summing over the q fan copies gives $|I| \leq q \lceil \frac{m}{2} \rceil$.

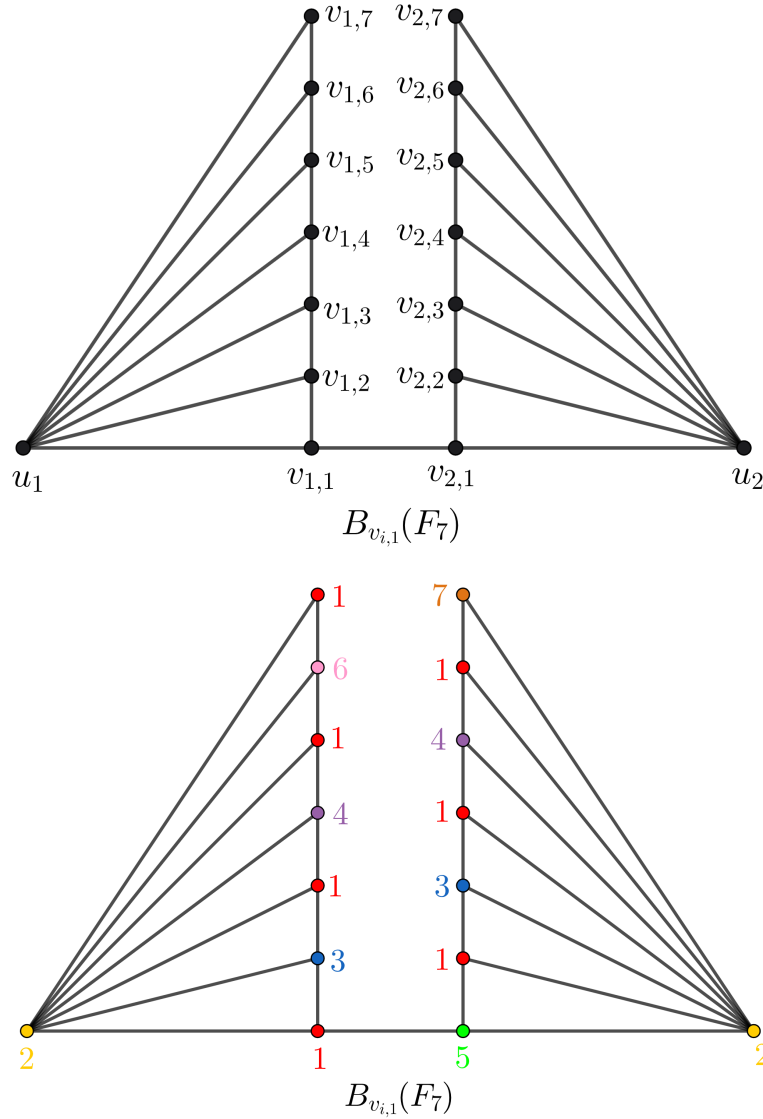


Fig. 4: The barbell fan graph type II $B_{v_{i,1}}(F_7)$ and a packing coloring attaining $\chi_\rho(B_{v_{i,1}}(F_7)) = 7$. The upper panel shows the endpoint-bridge construction, while the lower panel displays a valid packing coloring.

Equality is attained by choosing $v_{i,1}, v_{i,3}, v_{i,5}, \dots$ in every fan. No two of these vertices are adjacent, including vertices belonging to different fan copies. Hence $\alpha(CF_m(q)) = q \lceil \frac{m}{2} \rceil$. Now fix a color $\ell \geq 2$. Every fan $F_m^{(i)}$ has diameter 2, so color ℓ occurs at most once in each fan. Suppose that its occurrences lie in fan copies with indices $i_1 < i_2 < \dots < i_h$. For two consecutive occurrence indices $i_t < i_{t+1}$, choose $x \in V(F_m^{(i_t)})$ and $y \in V(F_m^{(i_{t+1})})$ with $c(x) = c(y) = \ell$. Every vertex of a fan is at distance at most 1 from its center. Hence a route through the centers gives $d(x, y) \leq 1 + (i_{t+1} - i_t) + 1 = (i_{t+1} - i_t) + 2$. On the other hand, the packing condition gives $d(x, y) \geq \ell + 1$. Therefore $i_{t+1} - i_t \geq \ell - 1$. Applying this inequality successively, $i_h \geq i_1 + (h - 1)(\ell - 1) \geq 1 + (h - 1)(\ell - 1)$. Since $i_h \leq q$, $1 + (h - 1)(\ell - 1) \leq q$. Thus $h \leq \left\lfloor \frac{q-1}{\ell-1} \right\rfloor + 1 = \left\lceil \frac{q}{\ell-1} \right\rceil$. Hence $|C_\ell| \leq \left\lceil \frac{q}{\ell-1} \right\rceil$. $\square$

We now derive bounds valid for all $m, q \geq 2$.

Theorem 5. Let $a = \lfloor \frac{m}{2} \rfloor$ and $s = \min\{a, q - 1\}$. Define

$$LB_{m,q} = \min \left\{ k \geq 2 : \sum_{\ell=2}^k \left\lceil \frac{q}{\ell-1} \right\rceil \geq q(a+1) \right\},$$

and

$$UB_{m,q} = 1 + q(a+1) - \sum_{t=1}^s \left(\left\lceil \frac{q}{t} \right\rceil - 1 \right).$$

Then $LB_{m,q} \leq \chi_\rho(CF_m(q)) \leq UB_{m,q}$.

Proof. The graph $CF_m(q)$ has $q(m+1)$ vertices. By Lemma 5, $|C_1| \leq q \lceil \frac{m}{2} \rceil$. Consequently, at least $q(m+1) - q \lceil \frac{m}{2} \rceil = q(m+1 - \lceil \frac{m}{2} \rceil) = q(\lfloor \frac{m}{2} \rfloor + 1) = q(a+1)$ vertices must receive colors greater than 1. If a packing coloring uses k colors, then Lemma 5 yields $\sum_{\ell=2}^k |C_\ell| \leq \sum_{\ell=2}^k \left\lceil \frac{q}{\ell-1} \right\rceil$. Therefore a necessary condition for a packing k -coloring is $\sum_{\ell=2}^k \left\lceil \frac{q}{\ell-1} \right\rceil \geq q(a+1)$. By the definition of $LB_{m,q}$, $\chi_\rho(CF_m(q)) \geq LB_{m,q}$. We next construct a coloring that gives the upper bound. Assign color 1 to every odd-indexed path vertex: $c(v_{i,2j-1}) = 1$, for $1 \leq i \leq q$, $1 \leq j \leq \lceil \frac{m}{2} \rceil$. These vertices form an independent set. In every fan, the a even-indexed path vertices and the center remain uncolored. Hence the number of uncolored vertices is exactly $q(a+1)$. For every $t \in \{1, \dots, s\}$, define $I_t = \{1 + ht : h = 0, 1, \dots, \lfloor \frac{q-1}{t} \rfloor\}$. Its cardinality is $|I_t| = \lfloor \frac{q-1}{t} \rfloor + 1 = \lceil \frac{q}{t} \rceil$. Since $t \leq s \leq a$, $2t \leq 2a \leq m$, so $v_{i,2t}$ exists for every $i \in I_t$. Define $c(v_{i,2t}) = t+1$, $i \in I_t$. To verify this repeated color, let $i, j \in I_t$ with $i < j$. By the definition of I_t , $j - i \geq t$. The fan copies communicate only through their centers. Hence

$$\begin{aligned} d(v_{i,2t}, v_{j,2t}) &= d(v_{i,2t}, u_i) + d(u_i, u_j) + d(u_j, v_{j,2t}) \\ &= 1 + (j - i) + 1 \\ &= (j - i) + 2 \\ &\geq t + 2. \end{aligned}$$

The common color is $t+1$, for which the required distance is $(t+1) + 1 = t+2$. Thus every repeated color introduced in this construction satisfies the packing condition. The preceding repeated classes color $\sum_{t=1}^s \lceil \frac{q}{t} \rceil$ vertices. Let $N = q(a+1) - \sum_{t=1}^s \lceil \frac{q}{t} \rceil$ be the number of vertices still uncolored. Choose any fixed ordering $z_1, z_2, \dots, z_N$ of these vertices and define $c(z_h) = s+1+h$, $1 \leq h \leq N$. If $N = 0$, this final assignment is empty. Every one of these colors is used only once. The largest color appearing in this construction is therefore

$$\begin{aligned} s+1+N &= s+1+q(a+1) - \sum_{t=1}^s \left\lceil \frac{q}{t} \right\rceil \\ &= 1+q(a+1) - \sum_{t=1}^s \left(\left\lceil \frac{q}{t} \right\rceil - 1 \right) \\ &= UB_{m,q}. \end{aligned}$$

Consequently, $\chi_\rho(CF_m(q)) \leq UB_{m,q}$. Together with the lower bound, this proves the theorem. $\square$

A representative instance of the generalized fan-chain family is shown in Fig. 5. The displayed graph consists of three fan copies whose centers are joined sequentially, while the accompanying coloring illustrates how colors may be reused across different fan copies when the required distance constraints are satisfied. This behavior provides a concrete illustration of the spacing principle

used in Theorem 5.

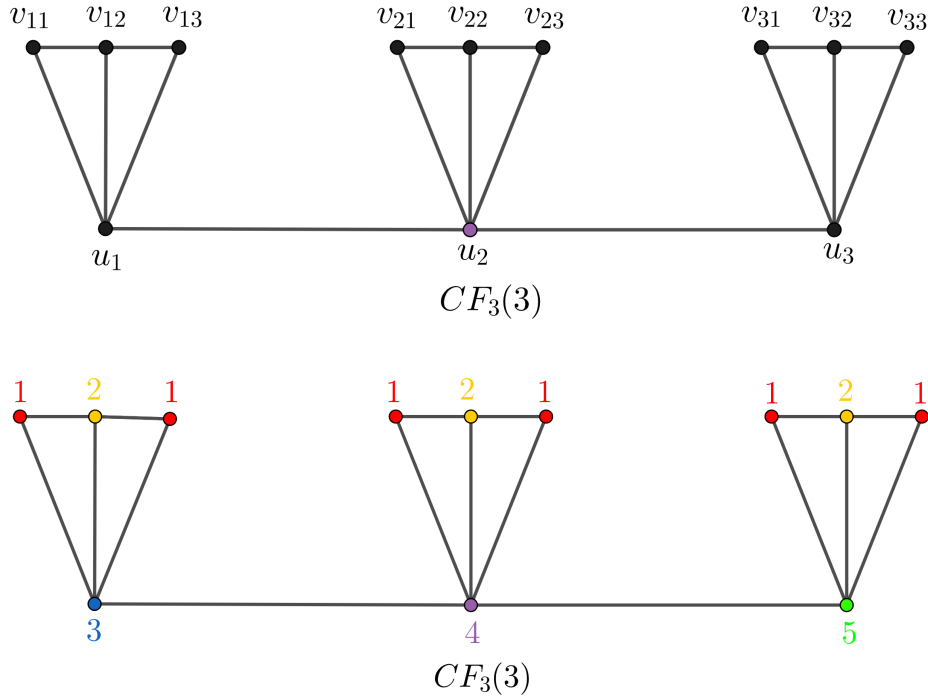


Fig. 5: A representative three-fan instance $CF_3(3)$ of the generalized fan-chain family $CF_m(q)$ and a valid packing coloring. The upper panel illustrates the sequential connection of the fan centers, while the lower panel illustrates the reuse of colors across different fan copies. The figure is included to visualize the general construction; the bounds in Theorem 5 apply to arbitrary $m, q \geq 2$.

For the generalized family $CF_m(q)$, the bounds in Theorem 5 coincide when the fan size is sufficiently large relative to the number of copies, yielding the following exact result.

Theorem 6. Let $m, q \geq 2$ and suppose that $m \geq 2q - 2$. Define $A_q = \sum_{t=1}^{q-1} \left\lceil \frac{q}{t} \right\rceil$. Then $\chi_\rho(CF_m(q)) = q \left(\left\lfloor \frac{m}{2} \right\rfloor + 2 \right) - A_q$.

Proof. Set $a = \left\lfloor \frac{m}{2} \right\rfloor$. The assumption $m \geq 2q - 2$ implies $a \geq q - 1$. Hence the auxiliary parameter s in Theorem 5 satisfies $s = \min\{a, q - 1\} = q - 1$. The upper bound from that theorem becomes

$$\begin{aligned} UB_{m,q} &= 1 + q(a + 1) - \sum_{t=1}^{q-1} \left(\left\lceil \frac{q}{t} \right\rceil - 1 \right) \\ &= 1 + q(a + 1) - A_q + (q - 1) \\ &= q(a + 2) - A_q. \end{aligned}$$

We now show that the lower bound has the same value. The quantity A_q is precisely the total capacity supplied by colors $2, 3, \dots, q$ in the lower bound sum. Since A_q contains $q - 1$ terms and each term is at most q , $A_q \leq q(q - 1) < q^2$. On the other hand, $a \geq q - 1$ implies $q(a + 1) \geq q^2$. Therefore $A_q < q(a + 1)$. Thus the capacity supplied by colors $2, \dots, q$ is insufficient to accommodate all vertices that must receive colors greater than 1. Consequently, $LB_{m,q} > q$. For every $k \geq q$, setting $t = \ell - 1$ gives $\sum_{\ell=2}^k \left\lceil \frac{q}{\ell-1} \right\rceil = \sum_{t=1}^{k-1} \left\lceil \frac{q}{t} \right\rceil$. The first $q - 1$ terms sum to A_q , while for every $t \geq q$, $\left\lceil \frac{q}{t} \right\rceil = 1$. Hence $\sum_{\ell=2}^k \left\lceil \frac{q}{\ell-1} \right\rceil = A_q + (k - q)$. This quantity reaches the required value $q(a + 1)$ exactly when $A_q + (k - q) \geq q(a + 1)$, or equivalently, $k \geq q(a + 2) - A_q$. Since $LB_{m,q} > q$, this calculation applies at the minimizing

value in the definition of $LB_{m,q}$. Therefore $LB_{m,q} = q(a+2) - A_q$. Hence $LB_{m,q} = UB_{m,q} = q(a+2) - A_q$, and consequently $\chi_\rho(CF_m(q)) = q(a+2) - A_q = q(\lfloor \frac{m}{2} \rfloor + 2) - A_q$. $\square$

Taken together, the preceding results show that the packing chromatic behavior of fan-derived graphs is governed not simply by graph order, but by the distance structure created by the corresponding graph operation. Subdivision provides a particularly clear example. Replacing every fan edge by a path of length two increases the separation among the original vertices sufficiently to produce a packing chromatic number that remains constant as the underlying fan grows. The umbrella construction behaves differently: once $r \geq 2$, the attached path can be accommodated through a periodic arrangement of low colors without increasing the number of colors required by the fan itself.

The two barbell constructions make the role of cross-component distance even more apparent. In $B_{u_i}(F_m)$, joining the centers gives diameter 3. Color 2 may be reused across the two fan copies at distance 3, whereas every color at least 3 is globally unique. Thus the color requirements of the individual fans cannot simply be added. In $B_{v_{i,1}}(F_r)$, moving the bridge from the centers to path end-vertices creates larger cross-fan distances. These distances permit controlled reuse of colors 2, 3, and 4. The separate arguments for $r = 2, 3, 4$ also demonstrate why conclusions obtained from the general counting pattern should not be extrapolated to small parameter values without direct verification.

For $CF_m(q)$, the decisive feature is the spacing between the indices of fan copies carrying the same color. Lemma 5 shows that a color $\ell \geq 2$ can occur on at most $\lceil \frac{q}{\ell-1} \rceil$ vertices. Thus low colors may be reused across several copies, while the permitted multiplicity decreases as the color index increases. Theorem 5 converts these local distance restrictions into bounds that hold for arbitrary m and q , whereas Theorem 6 identifies a parameter range in which the capacity argument and the explicit coloring construction coincide.

The parameterized construction is also consistent with the two-fan case. When $q = 2$, $CF_m(2) \cong B_{u_i}(F_m)$, and $A_2 = 2$. Moreover, the condition $m \geq 2q - 2$ reduces to $m \geq 2$. Hence Theorem 6 gives $\chi_\rho(CF_m(2)) = 2\lfloor \frac{m}{2} \rfloor + 2$, in agreement with Theorem 3. The case $q = 3$ is likewise contained naturally in the same parameterized family and therefore does not require a separate corollary. Treating q as a graph parameter makes the effect of repeated-color spacing visible within one general framework rather than through isolated fixed-size constructions.

4. Conclusion

This study determines the packing chromatic behavior of several fan-derived graph constructions by emphasizing the distance relationships that govern color reuse. We obtain $\chi_\rho(S(F_n)) = 4$ for $n \geq 2$, while for the umbrella graph the packing chromatic number coincides with that of the underlying fan whenever $r \geq 2$. For the two barbell-type constructions, we establish $\chi_\rho(B_{u_i}(F_m)) = 2\lfloor m/2 \rfloor + 2$ and $\chi_\rho(B_{v_{i,1}}(F_r)) = 4$ for $r = 2, 3$, 5 for $r = 4$, and r for $r \geq 5$, showing that the position of the connecting edge substantially affects the admissible reuse of low colors. The fixed three-fan configuration is extended to the generalized fan chain $CF_m(q)$, for which color class capacities yield lower and upper bounds for all $m, q \geq 2$ and the exact value $\chi_\rho(CF_m(q)) = q(\lfloor \frac{m}{2} \rfloor + 2) - \sum_{t=1}^{q-1} \lceil \frac{q}{t} \rceil$ whenever $m \geq 2q - 2$. These results show that the packing chromatic number of fan-derived graphs is governed not simply by the number of constituent fans, but by the cross-component distances created by the underlying graph operation; determining the exact value of $\chi_\rho(CF_m(q))$ outside the above parameter range remains a natural direction for further study.

CRedit Authorship Contribution Statement

Raventino: Methodology, Formal analysis, Investigation, Writing–Original Draft, Visualization.
Fransiskus Fran: Conceptualization, Formal analysis, Supervision, Validation, Writing–Review

& Editing. **Yudhi:** Formal analysis, Supervision, Validation, Writing–Review & Editing.

Declaration of Generative AI and AI-assisted Technologies

Generative AI tools, including ChatGPT, were used as auxiliary tools for translation, language refinement, grammar correction, and improvement of the clarity and readability of the manuscript. Any AI-assisted suggestions were evaluated by the authors and were not used as a substitute for mathematical reasoning or proof. All mathematical content, including the definitions, constructions, calculations, theorems, proofs, and conclusions, was independently verified by the authors, who take full responsibility for the accuracy and integrity of the work.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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References

- [1] Wayne Goddard, Sandra M. Hedetniemi, Stephen T. Hedetniemi, John M. Harris, and Douglas F. Rall. “Broadcast Chromatic Numbers of Graphs”. In: *Ars Combinatoria* 86 (2008), pp. 33–49. <https://combinatorialpress.com/ars-articles/volume-086-ars-articles/broadcast-chromatic-numbers-of-graphs/>.
- [2] Boštjan Brešar, Jasmina Ferme, Sandi Klavžar, and Douglas F. Rall. “A Survey on Packing Colorings”. In: *Discussiones Mathematicae Graph Theory* 40.4 (2020), pp. 923–970. DOI: [10.7151/dmgt.2320](https://doi.org/10.7151/dmgt.2320).
- [3] Jiří Fiala and Petr A. Golovach. “Complexity of the Packing Coloring Problem for Trees”. In: *Discrete Applied Mathematics* 158.7 (2010), pp. 771–778. DOI: [10.1016/j.dam.2008.09.001](https://doi.org/10.1016/j.dam.2008.09.001).
- [4] Minki Kim, Bernard Lidický, Tomáš Masařík, and Florian Pfender. “Notes on Complexity of Packing Coloring”. In: *Information Processing Letters* 137 (2018), pp. 6–10. DOI: [10.1016/j.ipl.2018.04.012](https://doi.org/10.1016/j.ipl.2018.04.012).
- [5] Albert William, S. Roy, and Indra Rajasingh. “Packing Chromatic Number of Circular Fans and Mesh of Trees”. In: *International Journal of Mathematics and Soft Computing* 4.2 (2014), pp. 145–152. DOI: [10.26708/IJMSC.2014.2.4.15](https://doi.org/10.26708/IJMSC.2014.2.4.15).
- [6] S. Roy. “Packing Chromatic Number of Certain Fan and Wheel Related Graphs”. In: *AKCE International Journal of Graphs and Combinatorics* 14.1 (2017), pp. 63–69. DOI: [10.1016/j.akcej.2016.11.001](https://doi.org/10.1016/j.akcej.2016.11.001).
- [7] K. Rajalakshmi, M. Venkatachalam, M. Barani, and Dafik. “A Study on Packing Coloring of Fan and Jump Graph Families”. In: *Palestine Journal of Mathematics* 10 (2021). Special Issue II, pp. 1–11. https://pjm.ppu.edu/sites/default/files/papers/PJM_Special_Issue_July_2021_1to_11.pdf.

- [8] Alje Marie M. Urenda, Jacel Angeline V. Lingcong, Normina A. Batucan, Joel G. Adanza, and Michael P. Baldado Jr. “Packing Chromatic Number of the Join of Some Classes of Graphs”. In: *International Mathematical Forum* 17.2 (2022), pp. 75–87. DOI: [10.12988/imf.2022.912315](https://doi.org/10.12988/imf.2022.912315).
- [9] Boštjan Brešar, Sandi Klavžar, and Douglas F. Rall. “On the Packing Chromatic Number of Cartesian Products, Hexagonal Lattice, and Trees”. In: *Discrete Applied Mathematics* 155.17 (2007), pp. 2303–2311. DOI: [10.1016/j.dam.2007.06.008](https://doi.org/10.1016/j.dam.2007.06.008).
- [10] Boštjan Brešar, Sandi Klavžar, Douglas F. Rall, and Kirsti Wash. “Packing Chromatic Number Under Local Changes in a Graph”. In: *Discrete Mathematics* 340.5 (2017), pp. 1110–1115. DOI: [10.1016/j.disc.2016.09.030](https://doi.org/10.1016/j.disc.2016.09.030).
- [11] József Balogh, Alexandr Kostochka, and Xujun Liu. “Packing Chromatic Number of Subdivisions of Cubic Graphs”. In: *Graphs and Combinatorics* 35.2 (2019), pp. 513–537. DOI: [10.1007/s00373-019-02016-3](https://doi.org/10.1007/s00373-019-02016-3).
- [12] Rachid Lemdani, Moncef Abbas, and Jasmina Ferme. “Packing Chromatic Numbers of Finite Super Subdivisions of Graphs”. In: *Filomat* 34.10 (2020), pp. 3275–3286. DOI: [10.2298/FIL2010275L](https://doi.org/10.2298/FIL2010275L).
- [13] S. Morteza Mirafzal. “Some Algebraic Properties of the Subdivision Graph of a Graph”. In: *Communications in Combinatorics and Optimization* 9.2 (2024), pp. 297–307. DOI: [10.22049/cco.2023.28270.1494](https://doi.org/10.22049/cco.2023.28270.1494).
- [14] Rafiantika Megahnia Prihandini, Ridho Alfarisi, Ika Hesti Agustin, and Dafik. “Super (a, d) -Cycles-Antimagic Labeling of Subdivision of a Fan Graph”. In: *Far East Journal of Mathematical Sciences* 103.4 (2018), pp. 819–830. DOI: [10.17654/MS103040819](https://doi.org/10.17654/MS103040819).
- [15] Yuli Anti Mitselina Rumahorbo, Saib Suwilo, Mardiningsih, and Putri Khairiah Nasution. “Dimensi Partisi pada Graf Payung”. In: *MES: Journal of Mathematics Education and Science* 9.2 (2024), pp. 146–155. DOI: [10.30743/mes.v9i2.8613](https://doi.org/10.30743/mes.v9i2.8613).
- [16] S. M. Citra, A. I. Kristiana, R. Adawiyah, Dafik, and R. M. Prihandini. “On the Packing Chromatic Number of Vertex Amalgamation of Some Related Tree Graph”. In: *Journal of Physics: Conference Series* 1836.1 (2021), p. 012025. DOI: [10.1088/1742-6596/1836/1/012025](https://doi.org/10.1088/1742-6596/1836/1/012025).
- [17] Akmal, Asmiati, and Aang Nuryaman. “On the Upperbound of the Locating Chromatic Numbers of the Amalgamation of Paths and Its Barbel Graph”. In: *Jurnal Matematika UNAND* 14.4 (2025), pp. 304–310. DOI: [10.25077/jmua.14.4.304-310.2025](https://doi.org/10.25077/jmua.14.4.304-310.2025).