



Generalized Analytic Function Classes Using an Extended Two-Factor q -Differential Operator: Fekete–Szegő Problem

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Abstract

In this paper, we introduce an extended two-factor q -differential multiplier operator denoted by $\mathcal{M}_{q,\lambda,\beta}^{n,m}f(z)$. Using this operator, we define the associated subclasses $\mathcal{X}_{q,\lambda,\beta}^{n,m}(\phi)$ and $\mathcal{Y}_{q,\lambda,\beta}^{n,m}(\phi, \delta)$ of analytic functions. We obtain upper bounds for the initial coefficients a_2 , a_3 and Fekete–Szegő inequalities for the functional $|a_3 - \mu a_2^2|$ for functions belonging to these subclasses. The bounds are derived by reducing the coefficient problem to standard Carathéodory estimates. The estimate for $\mathcal{X}$ is sharp, whereas for $\mathcal{Y}$ sharpness is established at the Carathéodory level and actual extremality is conditional on solvability of the defining differential equation. Additional consequences include third coefficient estimates, a logarithmic coefficient inequality, an upper bound for the second Hankel determinant, and a classical limiting theorem.

Keywords: analytic functions; coefficient bounds; Fekete–Szegő functional; q -differential operator; subordination

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1. Introduction

The theory of analytic and univalent functions constitutes one of the central areas of geometric function theory. Let $\mathcal{A}$ denote the class of functions analytic in the open unit disk

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\},$$

normalized by $f(0) = 0$ and $f'(0) = 1$. Thus, every $f \in \mathcal{A}$ has the Taylor expansion

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j. \quad (1)$$

Coefficient estimates for the Taylor coefficients of functions in geometrically defined subclasses of $\mathcal{A}$ have been a fundamental topic in geometric function theory. In particular, the Fekete–Szegő functional

$$|a_3 - \mu a_2^2|$$

provides an important measure of the interaction between the second and third Taylor coefficients and has been extensively studied since the classical work of Fekete and Szegő [1], as discussed in the standard literature [2].

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Subordination theory provides a natural framework for constructing and investigating generalized subclasses of analytic functions. For analytic functions f and univalent g in $\mathbb{U}$, with $f(0) = g(0)$, we write $f \prec g$ if there exists an analytic Schwarz function ω in $\mathbb{U}$, satisfying $\omega(0) = 0$ and $|\omega(z)| < 1$, such that $f(z) = g(\omega(z))$ [3]. The classical theory of differential subordinations developed by Miller and Mocanu [3] has been particularly influential in the study of coefficient problems, while more recent investigations have continued to exploit subordination in conjunction with generalized differential and convolution operators [3, 4].

Quantum calculus, or q -calculus, provides a natural deformation of classical differential calculus. The Jackson q -derivative

$$D_q f(z) = \frac{f(z) - f(qz)}{(1-q)z}, \quad z \neq 0,$$

with $0 < q < 1$, satisfies

$$D_q(z^j) = [j]_q z^{j-1}, \quad [j]_q = \frac{1-q^j}{1-q}.$$

This is equivalent to the convention $D_q f(z) = [f(qz) - f(z)]/[(q-1)z]$ used in some references. The systematic development of this operator goes back to Jackson [5].

Since

$$\lim_{q \rightarrow 1^-} [j]_q = j,$$

the classical derivative is recovered in the limit $q \rightarrow 1^-$. The use of q -difference and q -differential operators in geometric function theory has developed rapidly. Ramachandran et al. [6] investigated subclasses associated with the q -difference operator, while Seoudy and Aouf [7] obtained coefficient estimates for q -starlike and q -convex functions.

Further developments include the construction of subclasses associated with q -Sălăgean operators and Janowski functions [8, 9], applications of q -Sălăgean operators to multivalent functions [9], and classes associated with generalized conic domains [10]. These studies demonstrate that q -differential operators provide a flexible mechanism for producing new analytic function classes while preserving meaningful limiting relations with classical operators. The Fekete–Szegő problem has also received considerable attention in the q -calculus setting. Alsoboh and Darus [11] studied a Fekete–Szegő problem associated with a q -derivative operator, while Janteng, Liew and Omar [4] studied Fekete–Szegő functionals for classes of analytic functions involving the q -derivative.

Selvaraj, Varadharajan and Lakshmi [12] considered Fekete–Szegő type coefficient inequalities involving q -derivatives with respect to (j, k) -symmetric points. More recently, Gul et al. [13] investigated Fekete–Szegő functionals for subclasses associated with the q -Sălăgean operator. These developments demonstrate the continuing importance of coefficient functionals in q -geometric function theory.

Related coefficient estimates have also been obtained for meromorphic classes using the q -derivative operator by Aouf and Orhan [14]. Earlier work by Janteng and Halim [15] considered coefficient estimates for close-to-convex functions with respect to symmetric points, providing related coefficient-estimate background outside the q -calculus setting. Bulut [16] studied a subclass associated with a q -Sălăgean operator and an operator on a Hilbert space. A closely related recent study by Yie, Janteng, and Abbas [17] considered coefficient problems for subclasses of holomorphic functions associated with the Sălăgean q -differential operator in complex order. Their formulation is based on the Sălăgean q -differential structure, whereas the present construction introduces a product of two multiplier factors with independently adjustable parameters λ and β and corresponding orders n and m . Thus, the present operator extends the one-factor setting by allowing asymmetric parameter choices when $\lambda \neq \beta$, while the existing Sălăgean formulation is recovered as a special case under suitable parameter restrictions. No additional theoretical development is required for this comparison; the distinction lies in the multiplier structure used to define the associated subclasses.

Recent investigations have also moved beyond the Fekete–Szegő functional to higher-order coefficient functionals. In particular, the second Hankel determinant has been studied together with the Fekete–Szegő functional for classes defined through generalized q -Sălăgean derivatives [18]. Other recent studies have investigated coefficient problems for classes defined through q -derivatives [19]. Thus, the development of more flexible multiplier operators capable of simultaneously incorporating several independent parameters is a natural direction for further research.

Kanas and Răducanu [20] studied a class of analytic functions related to conic domains using a q -difference operator and an extended Ruscheweyh differential operator. These results provide relevant background for the interaction between q -difference operators and conic domains. These results indicate that the interaction between q -differential operators, convolution multipliers and subordination continues to generate new coefficient problems.

Despite these developments, many operators considered in the cited literature are governed by a single multiplier sequence or by parameters entering a common factor. This motivates the introduction of a two-factor multiplier structure in which the two factors possess independently adjustable parameters. In this paper we introduce the operator

$$\mathcal{M}_{q,\lambda,\beta}^{n,m}f(z) = z + \sum_{j=2}^{\infty} A_j a_j z^j,$$

where

$$A_j = \left(\frac{[j]_q + \lambda}{1 + \lambda} \right)^n \left(\frac{[j]_q + \beta}{1 + \beta} \right)^m.$$

The parameters λ and β control two distinct multiplier factors, while n and m determine their respective orders. This produces a more flexible coefficient multiplier than the one-factor constructions generally considered in the preceding literature.

Our operator contains several important special cases. When $m = 0$, $\lambda = \beta = 0$, we obtain the multiplier $A_j = [j]_q^n$ associated with the q -Sălăgean-type setting considered in earlier studies. When $\lambda = \beta$, the two factors merge and

$$A_j = \left(\frac{[j]_q + \lambda}{1 + \lambda} \right)^{n+m},$$

which gives a one-factor generalized Al-Oboudi/Cătaş-type multiplier after an appropriate reparameterization. Thus, the proposed operator simultaneously contains a one-factor family and a genuinely asymmetric two-factor family (the case $\lambda \neq \beta$). This asymmetric case is the principal source of the additional flexibility of the present construction. Using $\mathcal{M}_{q,\lambda,\beta}^{n,m}$, we introduce the subclasses

$$\mathcal{X}_{q,\lambda,\beta}^{n,m}(\phi)$$

defined through

$$D_q(\mathcal{M}_{q,\lambda,\beta}^{n,m}f(z)) \prec \phi(z),$$

and

$$\mathcal{Y}_{q,\lambda,\beta}^{n,m}(\phi, \delta),$$

defined through a convex combination of q -starlike- and q -convex-type expressions. The parameter δ therefore provides an interpolation between two geometrically distinct endpoint conditions.

Within the literature examined for this study, we did not identify a previous study employing the simultaneous two-factor multiplier structure of the form

$$\left(\frac{[j]_q + \lambda}{1 + \lambda} \right)^n \left(\frac{[j]_q + \beta}{1 + \beta} \right)^m$$

within the present q -subordination framework. This distinguishes the proposed construction from the one-factor multiplier sequences commonly used in q -geometric function theory, permitting

asymmetric deformation of the Taylor coefficients through two independent multiplier mechanisms. The resulting framework contains the classical q -Sălăgean-type multiplier as a special case and collapses to a one-factor generalized Al-Oboudi/Cătaş-type multiplier when $\lambda = \beta$.

Beyond the principal Fekete–Szegő estimates, the present work develops additional consequences of the two-factor structure, including third coefficient bounds, a logarithmic coefficient inequality, an upper bound for the second Hankel determinant, and a rigorous classical limit as $q \rightarrow 1^-$. These results demonstrate that the proposed operator provides a unified parameterized framework for coefficient problems involving q -differential operators. The objective of this study is to establish coefficient bounds and Fekete–Szegő inequalities for subclasses of analytic functions defined through the extended two-factor q -differential multiplier operator. The research gap addressed is the limited treatment, within the literature examined for this study, of coefficient problems based on two independently parameterized multiplier factors in the present q -subordination setting.

The principal contribution of the work is the introduction of the asymmetric two-factor multiplier with independently adjustable parameters and orders, together with the associated subclasses. The scientific contribution is a collection of coefficient estimates, Fekete–Szegő inequalities, logarithmic coefficient and Hankel determinant bounds, and a classical limiting result that extends the corresponding one-factor settings. The remainder of the paper is organized as follows. Section 2 introduces the extended q -differential operator and the associated subclasses. Section 3 establishes the principal coefficient and Fekete–Szegő inequalities. Additional results concerning logarithmic coefficients, Hankel determinants and limiting cases are also obtained. The final section summarizes the principal contributions and indicates possible directions for future investigations.

2. Methodology

This section introduces the extended two-factor q -differential operator and the associated subclasses of analytic functions. We begin by defining the operator and describing its basic properties, including its special cases and limiting behavior. We then introduce the subclasses $\mathcal{X}_{q,\lambda,\beta}^{n,m}(\phi)$ and $\mathcal{Y}_{q,\lambda,\beta}^{n,m}(\phi, \delta)$ via subordination, and finally collect the auxiliary lemmas that will be used throughout the proofs.

2.1. The Extended q -Differential Operator

The methodology in this paper is theoretical and proof-based, employing techniques from geometric function theory and q -calculus. We begin by defining the central operator that will be used throughout the study. Throughout this paper, we assume $0 < q < 1$, $n, m \in \mathbb{N}_0$, $\lambda \geq 0$, and $\beta \in \mathbb{C}$ with $|\beta| < 1$. Let

$$\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$$

with $\phi(0) = 1$ be analytic and univalent in $\mathbb{U}$ with $B_1 > 0$.

For $j \geq 2$, since $[j]_q > 1$, $\lambda \geq 0$, and $|\beta| < 1$, we have

$$[j]_q + \lambda > 0,$$

$$|[j]_q + \beta| \geq [j]_q - |\beta| > 0,$$

and

$$1 + \beta \neq 0.$$

Hence $A_j \neq 0$ for every $j \geq 2$.

For $f \in \mathcal{A}$ given by Eq. (1), define the extended q -differential operator $\mathcal{M}_{q,\lambda,\beta}^{n,m} : \mathcal{A} \rightarrow \mathcal{A}$ by

$$\mathcal{M}_{q,\lambda,\beta}^{n,m} f(z) = z + \sum_{j=2}^{\infty} A_j a_j z^j, \quad (2)$$

where

$$A_j = \left(\frac{[j]_q + \lambda}{1 + \lambda} \right)^n \left(\frac{[j]_q + \beta}{1 + \beta} \right)^m, \quad j \geq 2. \quad (3)$$

Since $[j]_q \rightarrow (1 - q)^{-1}$ as $j \rightarrow \infty$, the sequence $\{A_j\}_{j \geq 2}$ is bounded for fixed q, n, m, λ, β . Hence, if $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ is analytic in $\mathbb{U}$, then $\sum_{j=2}^{\infty} A_j a_j z^j$ converges in $\mathbb{U}$. Thus $\mathcal{M}_{q,\lambda,\beta}^{n,m} f \in \mathcal{A}$, so the operator in Eq. (2) is well defined.

The key novelty of this operator is the independent two-factor multiplier sequence in Eq. (3), which allows for independent parameterization through λ and β . This contrasts with the single-factor formulation recovered when $(\lambda, \beta, m) = (0, 0, 0)$. This operator generalizes several known operators in the literature. Specifically:

- When $\lambda = 0$, $\beta = 0$, and $m = 0$, we obtain the operator $M_q^n f(z)$ as studied by Tseu and Janteng [21].
- When $n = 0$, $m = 1$, and $\beta = 0$, we obtain

$$\mathcal{M}_{q,\lambda,0}^{0,1} f(z) = z + \sum_{j=2}^{\infty} [j]_q a_j z^j = z D_q f(z).$$

- When $\lambda = \beta$, the operator reduces to

$$\mathcal{M}_{q,\lambda,\lambda}^{n,m} f(z) = z + \sum_{j=2}^{\infty} \left(\frac{[j]_q + \lambda}{1 + \lambda} \right)^{n+m} a_j z^j. \quad (4)$$

The multiplier in Eq. (4) has the form of a one-factor generalized Al-Oboudi/Cătaş-type multiplier after an appropriate reparameterization.

As $q \rightarrow 1^-$, we have $[j]_q \rightarrow j$, and consequently the multiplier coefficients satisfy

$$\left(\frac{[j]_q + \lambda}{1 + \lambda} \right)^n \left(\frac{[j]_q + \beta}{1 + \beta} \right)^m \rightarrow \left(\frac{j + \lambda}{1 + \lambda} \right)^n \left(\frac{j + \beta}{1 + \beta} \right)^m.$$

Thus, the multiplier sequence converges to the corresponding classical multiplier sequence. The q -derivative of the extended operator is given by

$$D_q \left(\mathcal{M}_{q,\lambda,\beta}^{n,m} f(z) \right) = 1 + \sum_{j=2}^{\infty} [j]_q A_j a_j z^{j-1}. \quad (5)$$

For $j = 2, 3$, Eq. (5) becomes

$$D_q \left(\mathcal{M}_{q,\lambda,\beta}^{n,m} f(z) \right) = 1 + [2]_q A_2 a_2 z + [3]_q A_3 a_3 z^2 + \cdots. \quad (6)$$

2.2. New Subclasses

The concept of subordination is fundamental to defining our new subclasses. For analytic functions f and univalent g in $\mathbb{U}$, with $f(0) = g(0)$, we write $f \prec g$ if there exists an analytic Schwarz function $\omega(z)$ satisfying $\omega(0) = 0$ and $|\omega(z)| < 1$ in $\mathbb{U}$ such that $f(z) = g(\omega(z))$ [3]. Let

$$\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$$

with $B_1 > 0$ be analytic and univalent in $\mathbb{U}$. Using the extended multiplier operator, we define the following associated subclasses.

Definition 1. A function $f \in \mathcal{A}$ belongs to the class $\mathcal{X}_{q,\lambda,\beta}^{n,m}(\phi)$ if

$$D_q \left(\mathcal{M}_{q,\lambda,\beta}^{n,m} f(z) \right) \prec \phi(z). \quad (7)$$

Definition 2. For $0 \leq \delta \leq 1$, a function $f \in \mathcal{A}$ belongs to the class $\mathcal{Y}_{q,\lambda,\beta}^{n,m}(\phi, \delta)$ if

$$(1 - \delta) \frac{z D_q \left(\mathcal{M}_{q,\lambda,\beta}^{n,m} f(z) \right)}{\mathcal{M}_{q,\lambda,\beta}^{n,m} f(z)} + \delta \left(1 + \frac{q z D_q^2 \left(\mathcal{M}_{q,\lambda,\beta}^{n,m} f(z) \right)}{D_q \left(\mathcal{M}_{q,\lambda,\beta}^{n,m} f(z) \right)} \right) \prec \phi(z), \quad (8)$$

provided that $F(z) := \mathcal{M}_{q,\lambda,\beta}^{n,m} f(z) \neq 0$ for $z \in \mathbb{U} \setminus \{0\}$ and $D_q F(z) \neq 0$ for $z \in \mathbb{U}$, so that the quotient in Eq. (8) has a removable singularity at $z = 0$.

These classes generalize several known subclasses. When $\delta = 0$, the class $\mathcal{Y}_{q,\lambda,\beta}^{n,m}(\phi, 0)$ reduces to a class related to q -starlike-type functions. When $\delta = 1$, it reduces to a class related to q -convex-type functions. The class $\mathcal{Y}_{q,\lambda,\beta}^{n,m}(\phi, \delta)$ provides an interpolating family in which the parameter δ connects the endpoint subclasses corresponding to the q -starlike-type and q -convex-type conditions.

2.3. Auxiliary Lemmas

The following lemmas are essential for our proofs. The Carathéodory lemma provides bounds on the coefficients of functions with positive real part.

Lemma 1 (Carathéodory Lemma). *Let*

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots$$

be analytic in $\mathbb{U}$ with $\operatorname{Re}(p(z)) > 0$. Then for all $k \geq 1$, $|p_k| \leq 2$. Moreover, for any complex number ν ,

$$|p_2 - \nu p_1^2| \leq 2 \max\{1, |2\nu - 1|\}. \quad (9)$$

The bounds are sharp.

Lemma 2. *If $\omega(z)$ is a Schwarz function with $\omega(0) = 0$ and $|\omega(z)| < 1$ in $\mathbb{U}$, and we define*

$$p(z) = \frac{1 + \omega(z)}{1 - \omega(z)} = 1 + p_1 z + p_2 z^2 + \cdots,$$

then $\operatorname{Re}(p(z)) > 0$ and

$$\omega(z) = \frac{1}{2} \left(p_1 z + \left(p_2 - \frac{p_1^2}{2} \right) z^2 + \left(p_3 + \frac{p_1^3}{4} - p_1 p_2 \right) z^3 + \cdots \right). \quad (10)$$

These lemmas provide the analytical tools needed to derive coefficient bounds for functions in our new subclasses.

3. Results and Discussion

This section presents the main results of the paper. We first establish Fekete–Szegő inequalities and initial coefficient estimates for the class $\mathcal{X}_{q,\lambda,\beta}^{n,m}(\phi)$, and then obtain the corresponding results for the class $\mathcal{Y}_{q,\lambda,\beta}^{n,m}(\phi, \delta)$. We subsequently derive third-coefficient estimates, a logarithmic coefficient inequality, an upper bound for the second Hankel determinant, and the classical limiting theorem. The section concludes with special cases, reductions, and a discussion of the results.

3.1. Results for the Class $\mathcal{X}_{q,\lambda,\beta}^{n,m}(\phi)$

We now present the main results for the class $\mathcal{X}_{q,\lambda,\beta}^{n,m}(\phi)$. The following theorem provides Fekete–Szegő inequalities.

Theorem 1. *Let*

$$\phi(z) = 1 + B_1 z + B_2 z^2 + \dots$$

be analytic and univalent in $\mathbb{U}$, with $B_1 > 0$, and let $f \in \mathcal{A}$ given by Eq. (1) be in the class $\mathcal{X}_{q,\lambda,\beta}^{n,m}(\phi)$ as defined in Definition 1. For any complex number μ , we have

$$|a_3 - \mu a_2^2| \leq \frac{|B_1|}{[3]_q |A_3|} \max \left\{ 1, \left| \frac{B_2}{B_1} - \frac{[3]_q A_3 \mu B_1}{[2]_q^2 A_2^2} \right| \right\}. \quad (11)$$

The estimate Eq. (11) follows from the sharp Carathéodory coefficient inequality. The estimate is sharp under the coefficientwise realization of the corresponding extremal data.

Proof. If $f \in \mathcal{X}_{q,\lambda,\beta}^{n,m}(\phi)$, then by Definition 1, that is, by Eq. (7), there exists a Schwarz function $\omega(z)$ such that

$$D_q \left(\mathcal{M}_{q,\lambda,\beta}^{n,m} f(z) \right) = \phi(\omega(z)).$$

Define the Carathéodory function $p(z)$ as in Eq. (10). From Eq. (10) and the expansion of ϕ , we obtain

$$\phi(\omega(z)) = 1 + \frac{B_1 p_1}{2} z + \left(\frac{B_1}{2} \left(p_2 - \frac{p_1^2}{2} \right) + \frac{B_2 p_1^2}{4} \right) z^2 + \dots \quad (12)$$

From Eq. (6), we have

$$D_q \left(\mathcal{M}_{q,\lambda,\beta}^{n,m} f(z) \right) = 1 + [2]_q A_2 a_2 z + [3]_q A_3 a_3 z^2 + \dots \quad (13)$$

Comparing coefficients in Eq. (12) and Eq. (13) yields

$$a_2 = \frac{B_1 p_1}{2[2]_q A_2}, \quad (14)$$

and

$$[3]_q A_3 a_3 = \frac{B_1}{2} \left(p_2 - \frac{p_1^2}{2} \right) + \frac{B_2 p_1^2}{4}. \quad (15)$$

From Eq. (15),

$$a_3 = \frac{B_1}{2[3]_q A_3} p_2 - \frac{B_1}{4[3]_q A_3} p_1^2 + \frac{B_2}{4[3]_q A_3} p_1^2.$$

Therefore,

$$a_3 - \mu a_2^2 = \frac{B_1}{2[3]_q A_3} \left(p_2 - \gamma p_1^2 \right), \quad (16)$$

where

$$\gamma = \frac{1}{2} \left(1 - \frac{B_2}{B_1} + \frac{[3]_q A_3 \mu B_1}{[2]_q^2 A_2^2} \right).$$

Applying Eq. (9) of Lemma 1 to Eq. (16), we have

$$|p_2 - \gamma p_1^2| \leq 2 \max\{1, |2\gamma - 1|\}.$$

Noting that

$$2\gamma - 1 = -\frac{B_2}{B_1} + \frac{[3]_q A_3 \mu B_1}{[2]_q^2 A_2^2},$$

the desired bound follows. $\square$

When $\lambda = 0$, $\beta = 0$, $m = 0$, Theorem 1 reduces to

$$|a_3 - \mu a_2^2| \leq \frac{|B_1|}{[3]_q^{n+1}} \max \left\{ 1, \left| \frac{B_2}{B_1} - \frac{[3]_q^{n+1} \mu B_1}{[2]_q^{2n+2}} \right| \right\}, \quad (17)$$

which corresponds to the analogous result of Tseu and Janteng [21] under the same parameter restrictions.

3.2. Results for the Class $\mathcal{Y}_{q,\lambda,\beta}^{n,m}(\phi, \delta)$

For the class $\mathcal{Y}_{q,\lambda,\beta}^{n,m}(\phi, \delta)$, we obtain the following theorem. Define

$$\Gamma_1 = [(1 - \delta)([2]_q - 1) + \delta q[2]_q] A_2, \quad (18)$$

$$\Gamma_2 = [(1 - \delta)([3]_q - 1) + \delta q[2]_q[3]_q] A_3, \quad (19)$$

and

$$\Gamma_3 = [(1 - \delta)([2]_q - 1) + \delta q[2]_q^2] A_2^2. \quad (20)$$

Since $[2]_q - 1 = q > 0$ and $q[2]_q[3]_q > 0$, we have $\Gamma_1 \neq 0$ and $\Gamma_2 \neq 0$.

Theorem 2. *Let*

$$\phi(z) = 1 + B_1 z + B_2 z^2 + \dots$$

be analytic and univalent in $\mathbb{U}$, with $B_1 > 0$, and let $f \in \mathcal{A}$ be in the class $\mathcal{Y}_{q,\lambda,\beta}^{n,m}(\phi, \delta)$ as defined in Definition 2. For any complex number μ ,

$$|a_3 - \mu a_2^2| \leq \frac{|B_1|}{|\Gamma_2|} \max \left\{ 1, \left| \frac{B_2}{B_1} + \frac{\Gamma_3 B_1}{\Gamma_1^2} - \frac{\mu \Gamma_2 B_1}{\Gamma_1^2} \right| \right\}. \quad (21)$$

The estimate Eq. (21) is obtained from the sharp Carathéodory coefficient inequality. The coefficient estimate is sharp for the induced Carathéodory problem; equality for the original class requires existence of the corresponding analytic solution of the defining subordination equation Eq. (8).

Proof. If $f \in \mathcal{Y}_{q,\lambda,\beta}^{n,m}(\phi, \delta)$, then by Definition 2 there exists a Schwarz function $\omega(z)$ such that the left-hand side of Eq. (8) equals $\phi(\omega(z))$.

Let

$$F(z) = \mathcal{M}_{q,\lambda,\beta}^{n,m} f(z) = z + A_2 a_2 z^2 + A_3 a_3 z^3 + \dots$$

Then, using the q -derivative definitions, direct expansion gives

$$\frac{z D_q F(z)}{F(z)} = 1 + ([2]_q - 1) A_2 a_2 z + \left\{ ([3]_q - 1) A_3 a_3 - ([2]_q - 1) A_2^2 a_2^2 \right\} z^2 + \dots \quad (22)$$

Similarly,

$$D_q F(z) = 1 + [2]_q A_2 a_2 z + [3]_q A_3 a_3 z^2 + \dots$$

and

$$D_q^2 F(z) = [2]_q A_2 a_2 + [2]_q [3]_q A_3 a_3 z + \dots$$

Using

$$\frac{1}{1 + \alpha z + \beta z^2} = 1 - \alpha z + (\alpha^2 - \beta) z^2 + \dots,$$

we obtain

$$1 + \frac{q z D_q^2 F(z)}{D_q F(z)} = 1 + q[2]_q A_2 a_2 z + \left[q[2]_q [3]_q A_3 a_3 - q[2]_q^2 A_2^2 a_2^2 \right] z^2 + \dots \quad (23)$$

Let $g(z)$ denote the left-hand side of Eq. (8). Combining Eq. (22) and Eq. (23), and using the notation in Eq. (18), Eq. (19), and Eq. (20), we have

$$g(z) = 1 + \Gamma_1 a_2 z + (\Gamma_2 a_3 - \Gamma_3 a_2^2) z^2 + \cdots. \quad (24)$$

Comparing coefficients of Eq. (24) with Eq. (12) gives

$$a_2 = \frac{B_1 p_1}{2\Gamma_1}, \quad (25)$$

and

$$\Gamma_2 a_3 - \Gamma_3 a_2^2 = \frac{B_1}{2} \left(p_2 - \frac{p_1^2}{2} \right) + \frac{B_2 p_1^2}{4}. \quad (26)$$

Solving Eq. (26) for a_3 and substituting a_2 from Eq. (25), we have

$$a_3 = \frac{\Gamma_3}{\Gamma_2} a_2^2 + \frac{B_1}{2\Gamma_2} \left(p_2 - \frac{p_1^2}{2} \right) + \frac{B_2}{4\Gamma_2} p_1^2.$$

Then

$$a_3 - \mu a_2^2 = \frac{B_1}{2\Gamma_2} p_2 - \frac{B_1}{4\Gamma_2} p_1^2 + \frac{B_2}{4\Gamma_2} p_1^2 + \frac{\Gamma_3}{\Gamma_2} a_2^2 - \mu a_2^2.$$

Substituting

$$a_2^2 = \frac{B_1^2 p_1^2}{4\Gamma_1^2},$$

we get

$$a_3 - \mu a_2^2 = \frac{B_1}{2\Gamma_2} (p_2 - \eta p_1^2), \quad (27)$$

where

$$\eta = \frac{1}{2} \left(1 - \frac{B_2}{B_1} - \frac{\Gamma_3 B_1}{\Gamma_1^2} + \frac{\mu \Gamma_2 B_1}{\Gamma_1^2} \right). \quad (28)$$

Therefore,

$$2\eta - 1 = -\frac{B_2}{B_1} - \frac{\Gamma_3 B_1}{\Gamma_1^2} + \frac{\mu \Gamma_2 B_1}{\Gamma_1^2}. \quad (29)$$

Applying Eq. (9) of Lemma 1 to Eq. (27), with η given by Eq. (28), we obtain

$$|p_2 - \eta p_1^2| \leq 2 \max\{1, |2\eta - 1|\}.$$

Substituting Eq. (29) gives the desired bound. $\square$

3.3. Initial Coefficient Estimates

For the class $\mathcal{X}_{q,\lambda,\beta}^{n,m}(\phi)$, the initial coefficient estimates are

$$|a_2| \leq \frac{B_1}{[2]_q |A_2|}, \quad (30)$$

and

$$|a_3| \leq \frac{B_1}{[3]_q |A_3|} \max \left\{ 1, \left| \frac{B_2}{B_1} \right| \right\}. \quad (31)$$

For the class $\mathcal{Y}_{q,\lambda,\beta}^{n,m}(\phi, \delta)$, the initial coefficient estimates are

$$|a_2| \leq \frac{B_1}{|\Gamma_1|}, \quad (32)$$

and

$$|a_3| \leq \frac{B_1}{|\Gamma_2|} \max \left\{ 1, \left| \frac{B_2}{B_1} + \frac{\Gamma_3 B_1}{\Gamma_1^2} \right| \right\}. \quad (33)$$

The estimates Eq. (30) and Eq. (32) for a_2 follow directly from the first coefficient comparison and $|p_1| \leq 2$, while the estimates Eq. (31) and Eq. (33) for a_3 follow by setting $\mu = 0$ in Theorems 1 and 2.

3.4. Third-Coefficient Estimates and Sharpness

The following theorem records the sharp form of the third-coefficient estimate obtained from the Fekete–Szegő inequality.

Theorem 3. Let $f \in \mathcal{X}_{q,\lambda,\beta}^{n,m}(\phi)$, where

$$\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots, \quad B_1 > 0.$$

Then

$$|a_3| \leq \frac{B_1}{[3]_q |A_3|} \max \left\{ 1, \left| \frac{B_2}{B_1} \right| \right\}. \quad (34)$$

The estimate Eq. (34) is sharp at the coefficient level.

Proof. Set $\mu = 0$ in Theorem 1. We immediately obtain

$$|a_3| \leq \frac{B_1}{[3]_q |A_3|} \max \left\{ 1, \left| \frac{B_2}{B_1} \right| \right\}.$$

The two extremal Carathéodory choices

$$p(z) = \frac{1 + e^{i\theta} z}{1 - e^{i\theta} z}$$

and

$$p(z) = \frac{1 + e^{i\theta} z^2}{1 - e^{i\theta} z^2}$$

give the two branches of the maximum at the coefficient level. $\square$

Similarly, for the second class we obtain:

Theorem 4. Let $f \in \mathcal{Y}_{q,\lambda,\beta}^{n,m}(\phi, \delta)$. Then

$$|a_3| \leq \frac{B_1}{|\Gamma_2|} \max \left\{ 1, \left| \frac{B_2}{B_1} + \frac{\Gamma_3 B_1}{\Gamma_1^2} \right| \right\}. \quad (35)$$

The coefficient estimate Eq. (35) is sharp for the induced Carathéodory problem. Equality for the original class $\mathcal{Y}_{q,\lambda,\beta}^{n,m}(\phi, \delta)$ is attained whenever the corresponding extremal solution of the defining subordination equation exists.

Proof. Put $\mu = 0$ in Theorem 2. We obtain

$$|a_3| \leq \frac{B_1}{|\Gamma_2|} \max \left\{ 1, \left| \frac{B_2}{B_1} + \frac{\Gamma_3 B_1}{\Gamma_1^2} \right| \right\}.$$

The two branches are realized at the Carathéodory level by

$$p_1 = 0, \quad |p_2| = 2$$

and

$$p_1 = 2e^{i\theta}, \quad p_2 = 2e^{2i\theta},$$

respectively. The qualification concerning the original class follows from the defining differential-subordination equation. $\square$

3.5. A Logarithmic Coefficient Inequality

Let

$$F(z) = \mathcal{M}_{q,\lambda,\beta}^{n,m} f(z) = z + A_2 a_2 z^2 + A_3 a_3 z^3 + \dots$$

and assume that $F(z) \neq 0$ for $z \in \mathbb{U} \setminus \{0\}$. Define the logarithmic coefficients $\gamma_1, \gamma_2, \dots$ by

$$\log \frac{F(z)}{z} = 2 \sum_{k=1}^{\infty} \gamma_k z^k. \quad (36)$$

Then, by Eq. (36),

$$2\gamma_1 = A_2 a_2,$$

and

$$2\gamma_2 = A_3 a_3 - \frac{1}{2} A_2^2 a_2^2.$$

After substituting the coefficient relations induced by the defining subordination, the multiplier factors A_2 and A_3 cancel from the resulting logarithmic coefficient estimate, so the bound depends only on the q -differential structure and the coefficients of the subordinating function ϕ .

Theorem 5. *If $f \in \mathcal{X}_{q,\lambda,\beta}^{n,m}(\phi)$ and $F(z) := \mathcal{M}_{q,\lambda,\beta}^{n,m} f(z) \neq 0$ for $z \in \mathbb{U} \setminus \{0\}$, then, for every $\nu \in \mathbb{C}$,*

$$|\gamma_2 - \nu \gamma_1^2| \leq \frac{B_1}{2[3]_q} \max \left\{ 1, \left| -\frac{B_2}{B_1} + \frac{B_1[3]_q}{2[2]_q^2} (1 + \nu) \right| \right\}. \quad (37)$$

Proof. From the coefficient relations Eq. (14) and Eq. (15) in the proof of Theorem 1,

$$A_2 a_2 = \frac{B_1 p_1}{2[2]_q}$$

and

$$A_3 a_3 = \frac{B_1}{2[3]_q} p_2 + \frac{B_2 - B_1}{4[3]_q} p_1^2.$$

Therefore,

$$2\gamma_2 = \frac{B_1}{2[3]_q} p_2 + \left[\frac{B_2 - B_1}{4[3]_q} - \frac{B_1^2}{8[2]_q^2} \right] p_1^2.$$

Thus,

$$\gamma_2 = \frac{B_1}{4[3]_q} p_2 + \left[\frac{B_2 - B_1}{8[3]_q} - \frac{B_1^2}{16[2]_q^2} \right] p_1^2.$$

Moreover,

$$\gamma_1^2 = \frac{B_1^2 p_1^2}{16[2]_q^2}.$$

Consequently,

$$\gamma_2 - \nu\gamma_1^2 = \frac{B_1}{4[3]_q} (p_2 - \Lambda_\nu p_1^2),$$

where

$$\Lambda_\nu = \frac{1}{2} \left(1 - \frac{B_2}{B_1} \right) + \frac{B_1[3]_q}{4[2]_q^2} (1 + \nu).$$

By the Carathéodory inequality,

$$|p_2 - \Lambda_\nu p_1^2| \leq 2 \max\{1, |2\Lambda_\nu - 1|\}.$$

Since

$$2\Lambda_\nu - 1 = -\frac{B_2}{B_1} + \frac{B_1[3]_q}{2[2]_q^2} (1 + \nu),$$

we obtain

$$|\gamma_2 - \nu\gamma_1^2| \leq \frac{B_1}{2[3]_q} \max \left\{ 1, \left| -\frac{B_2}{B_1} + \frac{B_1[3]_q}{2[2]_q^2} (1 + \nu) \right| \right\}.$$

This proves Eq. (37) and hence the theorem. $\square$

3.6. A Second Hankel Determinant Estimate

We obtain an upper bound for the second Hankel determinant and leave the determination of sharp bounds for future work.

Theorem 6. *Let*

$$f \in \mathcal{X}_{q,\lambda,\beta}^{n,m}(\phi),$$

where

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots, \quad B_1 > 0.$$

Then

$$|H_2(2)| \leq \frac{B_1(4|B_1| + 4|B_2| + |B_3|)}{[2]_q[4]_q|A_2A_4|} + \frac{B_1^2}{[3]_q^2|A_3|^2} \max \left\{ 1, \left| \frac{B_2}{B_1} \right| \right\}^2. \quad (38)$$

Proof. Let

$$\omega(z) = \frac{1}{2} \left[p_1 z + \left(p_2 - \frac{p_1^2}{2} \right) z^2 + \left(p_3 + \frac{p_1^3}{4} - p_1 p_2 \right) z^3 + \cdots \right].$$

Thus,

$$\omega_1 = \frac{p_1}{2},$$

$$\omega_2 = \frac{p_2}{2} - \frac{p_1^2}{4},$$

and

$$\omega_3 = \frac{p_3}{2} + \frac{p_1^3}{8} - \frac{p_1 p_2}{2}.$$

Expanding

$$\phi(\omega(z)) = 1 + B_1 \omega(z) + B_2 \omega(z)^2 + B_3 \omega(z)^3 + \cdots,$$

the coefficient of z^3 is

$$B_1 \omega_3 + 2B_2 \omega_1 \omega_2 + B_3 \omega_1^3.$$

Substituting the expressions for $\omega_1, \omega_2, \omega_3$, we obtain

$$\frac{B_1}{2}p_3 + \frac{B_2 - B_1}{2}p_1p_2 + \left(\frac{B_1}{8} - \frac{B_2}{4} + \frac{B_3}{8}\right)p_1^3.$$

Comparing with

$$D_q F(z) = 1 + [2]_q A_2 a_2 z + [3]_q A_3 a_3 z^2 + [4]_q A_4 a_4 z^3 + \dots$$

gives

$$[4]_q A_4 a_4 = \frac{B_1}{2}p_3 + \frac{B_2 - B_1}{2}p_1p_2 + \left(\frac{B_1}{8} - \frac{B_2}{4} + \frac{B_3}{8}\right)p_1^3. \quad (39)$$

Hence, from Eq. (39) and using

$$|p_1| \leq 2, \quad |p_2| \leq 2, \quad |p_3| \leq 2,$$

we obtain

$$|[4]_q A_4 a_4| \leq |B_1| + 2|B_2 - B_1| + 8 \left| \frac{B_1}{8} - \frac{B_2}{4} + \frac{B_3}{8} \right|.$$

Furthermore,

$$|a_2| \leq \frac{B_1}{[2]_q |A_2|}$$

and, by Theorem 3,

$$|a_3| \leq \frac{B_1}{[3]_q |A_3|} \max \left\{ 1, \left| \frac{B_2}{B_1} \right| \right\}.$$

A direct estimate of

$$|H_2(2)| = |a_2 a_4 - a_3^2|$$

then yields

$$|H_2(2)| \leq |a_2| |a_4| + |a_3|^2.$$

Substitution of the preceding estimates gives Eq. (38). $\square$

3.7. The Classical Limit of the Extended Operator

It is important to distinguish between two limiting processes: the q -derivative operator D_q converges to the classical derivative D as $q \rightarrow 1^-$, while the multiplier operator $\mathcal{M}_{q,\lambda,\beta}^{n,m}$ converges to the classical multiplier operator $\mathcal{M}_{1,\lambda,\beta}^{n,m}$. The limiting multiplier is not necessarily the ordinary derivative unless n, m and the parameters are selected appropriately.

Theorem 7. Let $n, m \in \mathbb{N}_0$, $\lambda \geq 0$, and $\beta \in \mathbb{C}$ with $|\beta| < 1$. Define

$$A_j(q) = \left(\frac{[j]_q + \lambda}{1 + \lambda} \right)^n \left(\frac{[j]_q + \beta}{1 + \beta} \right)^m.$$

Then, for every fixed $j \geq 2$,

$$\lim_{q \rightarrow 1^-} A_j(q) = \left(\frac{j + \lambda}{1 + \lambda} \right)^n \left(\frac{j + \beta}{1 + \beta} \right)^m. \quad (40)$$

Moreover, if

$$\mathcal{M}_{q,\lambda,\beta}^{n,m} f(z) = z + \sum_{j=2}^{\infty} A_j(q) a_j z^j,$$

then

$$\lim_{q \rightarrow 1^-} \mathcal{M}_{q,\lambda,\beta}^{n,m} f(z) = z + \sum_{j=2}^{\infty} \left(\frac{j+\lambda}{1+\lambda} \right)^n \left(\frac{j+\beta}{1+\beta} \right)^m a_j z^j$$

locally uniformly in $\mathbb{U}$.

Consequently, the Fekete–Szegő estimate in Theorem 1 converges to

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{3|C_3|} \max \left\{ 1, \left| \frac{B_2}{B_1} - \frac{C_3 \mu B_1}{C_2^2} \right| \right\}, \quad (41)$$

where

$$C_j = \left(\frac{j+\lambda}{1+\lambda} \right)^n \left(\frac{j+\beta}{1+\beta} \right)^m.$$

Proof. For every fixed integer $j \geq 2$,

$$[j]_q = \frac{1-q^j}{1-q} \longrightarrow j \quad (q \rightarrow 1^-).$$

Since the maps

$$x \mapsto \left(\frac{x+\lambda}{1+\lambda} \right)^n$$

and

$$x \mapsto \left(\frac{x+\beta}{1+\beta} \right)^m$$

are continuous at $x = j$, we obtain Eq. (40), that is,

$$A_j(q) \longrightarrow C_j.$$

Moreover, there exists a constant $C > 0$, independent of $q \in (1/2, 1)$, such that

$$|A_j(q)| \leq C(j+1)^{n+m}$$

for all $j \geq 2$. If $|z| \leq r < 1$, analyticity of f gives exponential decay of $|a_j| r^j$, so

$$\sum_{j=2}^{\infty} C(j+1)^{n+m} |a_j| r^j$$

converges. The Weierstrass M -test therefore establishes the local uniform convergence. Finally,

$$[2]_q \rightarrow 2, \quad [3]_q \rightarrow 3,$$

and

$$A_2(q) \rightarrow C_2, \quad A_3(q) \rightarrow C_3.$$

Taking the limit in Theorem 1 gives Eq. (41). $\square$

3.8. Special Cases and Reductions

The following special cases demonstrate the generality of our results and show precisely how the operator reduces to known operators under specific parameter choices.

Remark 1. When $\lambda = \beta$, the operator coefficients reduce to

$$A_j = \left(\frac{[j]_q + \lambda}{1 + \lambda} \right)^{n+m}, \quad j \geq 2. \quad (42)$$

The multiplier in Eq. (42) gives a one-factor generalized Al-Oboudi/Cătaş-type multiplier after an appropriate reparameterization.

Remark 2. When $m = 0$, $\lambda = 0$, and $\beta = 0$, the operator coefficients reduce to

$$A_j = [j]_q^n, \quad j \geq 2. \quad (43)$$

Consequently, with the coefficients in Eq. (43), Theorem 1 reduces to the corresponding result of Tseu and Janteng [21] given in Eq. (17).

These reductions make the mathematical contribution of the new parameters λ , β , and m explicit: the two-factor multiplier sequence with independent parameters λ and β provides additional flexibility that is not available in the earlier single-factor formulation. The symmetric case $\lambda = \beta$ collapses the two factors into a one-factor generalized Al-Oboudi/Cătaş-type multiplier after an appropriate reparameterization.

3.9. Discussion of Results

The extended operator $\mathcal{M}_{q,\lambda,\beta}^{n,m}f(z)$ with independent parameters λ , β , and m recovers several existing q -differential multiplier settings as special cases, as demonstrated in the preceding special cases. The Fekete–Szegő inequalities obtained explicitly exhibit the dependence on the parameters $q, \lambda, \beta, n, m, \delta$, thereby providing a flexible framework for further coefficient problems and related subclasses. The class $\mathcal{X}_{q,\lambda,\beta}^{n,m}(\phi)$ generalizes classes defined by the q -derivative operator, while $\mathcal{Y}_{q,\lambda,\beta}^{n,m}(\phi, \delta)$ provides an interpolating family in which the parameter δ connects the endpoint subclasses corresponding to the q -starlike-type and q -convex-type conditions.

The methodology employed in this paper reduces the coefficient problem to standard subordination and Carathéodory coefficient estimates. The sharpness results for $\mathcal{X}$ are established at the coefficient level, while for $\mathcal{Y}$ the sharpness at the Carathéodory level is conditional on the existence of analytic solutions to the defining subordination equation. The coefficient bounds obtained in this work may serve as a foundation for investigating higher-order coefficient problems, including sharp Hankel determinant estimates and logarithmic coefficient inequalities, as well as extensions to meromorphic and multivalent settings.

4. Conclusion

In this paper, we have introduced an extended q -differential operator $\mathcal{M}_{q,\lambda,\beta}^{n,m}f(z)$ with independent parameters λ , β , and m , thereby extending the single-factor operator previously studied [21]. Using this operator, we defined the associated subclasses $\mathcal{X}_{q,\lambda,\beta}^{n,m}(\phi)$ and $\mathcal{Y}_{q,\lambda,\beta}^{n,m}(\phi, \delta)$ of analytic functions. We obtained coefficient estimates for the initial coefficients a_2 and a_3 , and Fekete–Szegő inequalities for $|a_3 - \mu a_2^2|$ for functions in these classes. The results recover several existing q -differential multiplier settings as special cases and provide a foundation for further research in q -analytic function theory. The two-factor multiplier structure with independent parameters λ and β represents the main novelty. We obtained an upper bound for the second Hankel determinant and leave the determination of sharp bounds for future work. Future research directions include investigating higher-order coefficient bounds (a_4 , a_5 , etc.) for these classes, studying the Hankel determinant problems more deeply, extending the results to meromorphic functions or multivalent functions, and considering q -integral operator analogues.

CRediT Authorship Contribution Statement

Victor Wanjala: Conceptualization, Methodology, Formal Analysis, Investigation, Writing–Original Draft Preparation, Writing–Review & Editing.

Beatrice Obiero Adhiambo: Conceptualization, Validation, Supervision, Writing–Review & Editing.

Declaration of Generative AI and AI-assisted Technologies

No generative AI or AI-assisted technologies were used during the preparation of this manuscript.

Declaration of Competing Interest

The authors declare no competing interests.

Ethical Considerations

Ethical approval was not required for this study.

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No datasets were generated or analyzed during the current study.

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