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Predator Induced Fear in a Modified Rice-Rat-Snake Food Chain Model

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Abstract

Rice-field ecosystems involve trophic interactions among rice, herbivorous rats, and their natural predators, such as snakes. Beyond direct predation, predator presence can influence prey through non-consumptive fear effects that may alter population dynamics. This study develops and analyzes a modified rice-rat-snake food-chain model incorporating predator-induced fear into the effective rice–rat interaction. The model is formulated as a system of three nonlinear ordinary differential equations and investigated through positivity and boundedness analyses, equilibrium analysis, local stability analysis using the Jacobian matrix and the Routh–Hurwitz criterion, and numerical simulations. The analytical results show that the solutions remain nonnegative and ultimately bounded, ensuring the biological feasibility of the model. Four equilibrium states are obtained: total extinction, rice-only persistence, rice–rat coexistence, and full rice–rat–snake coexistence. At the interior equilibrium, the equilibrium rat density is determined by the balance between snake reproduction and mortality, whereas the equilibrium rice and snake densities are coupled through the fear-dependent interaction. The predator-induced fear coefficient therefore modifies the coexistence equilibrium by reducing the effective rice–rat interaction in the presence of snakes. The local stability of the coexistence equilibrium is characterized by the Routh–Hurwitz conditions. Numerical simulations illustrate that increasing predator-induced fear changes the coexistence equilibrium and the transient population dynamics while stable coexistence is maintained for the parameter values considered. These results provide a mathematical framework for understanding how non-consumptive predator effects may influence trophic interactions in rice-field food chains.

Keywords: Predator-induced fear; Rice–rat–snake model; Food chain; Nonlinear ordinary differential equations; Local stability

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1. Introduction

Rice (*Oryza sativa* L.) is one of the world's most important staple crops and plays a fundamental role in global food security. Nevertheless, rice production is continuously threatened by various biotic factors, among which the rice-field rat (*Rattus argentiventer*) is recognized as a major agricultural pest [1]. High rat abundance may cause considerable crop damage and disturb the ecological balance of rice agroecosystems. In rice-field environments, natural predators such as snakes contribute to controlling rat populations through predation, thereby forming a three-level

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trophic interaction involving rice, rats, and snakes [2–4]. Understanding the population dynamics of this agricultural food chain is therefore important for developing ecologically sustainable pest-management strategies.

Mathematical modeling provides a systematic framework for investigating ecological interactions that are difficult to observe or manipulate directly in natural environments. Lotka–Volterra-type models have been extensively used to describe predator–prey interactions and food-chain dynamics because they allow the analytical investigation of population persistence, equilibrium feasibility, stability, and long-term behavior [2, 5]. Various modifications of the classical Lotka–Volterra framework have incorporated logistic growth, prey refuge, harvesting, time delay, and nonlinear functional responses to obtain more biologically realistic representations of ecological systems.

Despite these developments, many predator–prey models represent the influence of predators exclusively through direct consumption. In natural systems, however, predators may also affect prey through non-consumptive mechanisms that alter prey behavior and physiology without causing immediate mortality [6]. This phenomenon, commonly referred to as the predator-induced fear effect, may reduce prey foraging activity, increase vigilance, modify habitat use, and suppress reproductive performance [7]. Consequently, predator presence may influence prey population growth even when the direct predation rate remains unchanged [8, 9].

Mathematical studies have shown that predator-induced fear can substantially modify the qualitative dynamics of ecological systems. The inclusion of fear effects in predator–prey models may alter the existence and stability of equilibria, population persistence, periodic oscillations, and bifurcation behavior [10]. Fear-mediated interactions have also been investigated in multi-trophic food-chain models together with mechanisms such as prey refuge, supplementary food, cooperative hunting, time delay, and generalist predation [11]. These findings indicate that predator-induced fear can influence population dynamics across multiple trophic levels rather than merely decreasing prey abundance [12, 13].

However, most existing fear-mediated food-chain models concern generic ecological communities and do not explicitly represent agricultural food chains [14]. Mathematical models of rice–rat–snake interactions generally describe the influence of snakes on rats only through direct predation [2]. Such formulations do not account for behavioral changes in rats caused by the presence of snakes. In particular, predator-induced fear may reduce rat foraging and reproductive activity and consequently decrease the conversion of consumed rice into rat population growth. Neglecting this indirect mechanism may underestimate the ecological contribution of snakes to rat regulation and rice protection [15].

To address this limitation, the present study develops a modified rice–rat–snake food-chain model in which predator-induced fear reduces the effective conversion of consumed rice into rat population growth. The fear effect is represented by a decreasing factor $\frac{1}{1+fU}$, where U denotes snake abundance and f represents fear intensity. Consequently, the proposed model reduces to the corresponding model without fear when $f = 0$. This formulation distinguishes the non-consumptive influence of snake presence from the direct loss of rats through predation.

The objective of this study is to analyze the dynamical consequences of predator-induced fear in a rice–rat–snake food chain. The research gap addressed in this study is the absence of an explicit fear-mediated reduction in rat growth in the existing rice–rat–snake modeling framework. The novelty of the proposed model lies in incorporating the fear factor into the conversion of consumed rice into rat population growth while retaining direct snake predation as a separate ecological mechanism. The main scientific contribution is the derivation of conditions for solution positivity, boundedness, equilibrium feasibility, and local stability, together with numerical evidence describing how variations in fear intensity influence the long-term coexistence of rice, rats, and snakes. The results are expected to provide additional mathematical insight into the non-consumptive role of natural predators in agricultural pest regulation.

2. Method

This study employs a deterministic mathematical-modeling approach based on a system of nonlinear ordinary differential equations. The model describes the temporal interaction among rice, rats, and snakes in a spatially homogeneous rice-field ecosystem. The methodology consists of establishing the biological assumptions, formulating the mathematical model, defining its variables and parameters, performing qualitative analysis, and conducting numerical simulations.

2.1. Biological Assumptions

The proposed model considers a three-level trophic interaction consisting of rice as the primary producer, rats as herbivorous consumers, and snakes as natural predators of rats. The rice population is assumed to grow naturally under limited environmental resources, so that its growth is constrained by the carrying capacity of the rice-field ecosystem. Rats depend on rice as their primary food resource, and their population growth is therefore associated with the consumption and utilization of rice. As the rat population increases, the consumption pressure on rice also increases. Snakes occupy the highest trophic level in the model and depend on rats as their prey, so that predation reduces the rat population while contributing to the growth of the snake population.

In addition to direct predation, the presence of snakes is assumed to exert a non-consumptive effect on rats through predator-induced fear. This behavioral response represents the tendency of rats to reduce their effective foraging activity and utilization of rice when exposed to greater perceived predation risk. Consequently, the effective rice–rat interaction is reduced as the intensity of predator-induced fear increases. The fear effect does not directly modify the intrinsic logistic growth of rice or the intrinsic growth of snakes; instead, it acts through the interaction between rice and rats.

The model further assumes that all three populations are continuously distributed and interact within a spatially homogeneous environment, with no migration into or out of the system during the observation period. Environmental conditions and model parameters are assumed to remain constant over time, and seasonal variations, disease transmission, age structure, and other external disturbances are not explicitly considered. All population densities are nonnegative, and all biological parameters are assumed to be positive. These assumptions allow the model to focus specifically on the role of predator-induced fear in modifying the trophic dynamics among rice, rats, and snakes.

2.2. Model Formulation

Let $P(t)$, $T(t)$, and $U(t)$ denote the population densities of rice, rats, and snakes, measured in rice plants $ha^{-}1$, rats $ha^{-}1$, and snakes $ha^{-}1$, respectively. Based on the biological assumptions described previously, the dynamics of these populations are formulated as a system of nonlinear ordinary differential equations. In the absence of herbivory, the rice population is assumed to follow logistic growth with intrinsic growth rate r and carrying capacity K. The rice population decreases through interaction with rats, where α denotes the rice consumption coefficient by rats and determines the intensity of the rice–rat interaction. Therefore, the dynamics of the rice population are formulated as

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right) - \frac{\alpha PT}{1 + fU} \tag{1}$$

The growth of the rat population depends on the utilization of consumed rice. The parameter β represents the conversion efficiency of consumed rice into rat population growth. In the absence of predator-induced fear, this contribution is represented by the interaction between rice and rats. However, the presence of snakes generates a non-consumptive effect that reduces the effective contribution of rice consumption to rat population growth.

To incorporate this mechanism, the predator-induced fear effect is represented by the decreasing function in Eq. (2)

$$F(U) = \frac{1}{1 + fU}, \tag{2}$$

where $f \geq 0$ denotes the predator-induced fear coefficient. The function satisfies $F(0) = 1$ and decreases as either snake abundance or the fear coefficient increases. Hence, using the fear function defined in Eq. (2), the effective contribution of rice consumption to rat population growth is given by Eq. (3):

$$\beta\alpha PTF(U) = \frac{\beta\alpha PT}{1 + fU}. \tag{3}$$

In addition to the fear-mediated growth mechanism, rats experience natural mortality at rate c and direct predation by snakes at rate m. Consequently, the rat population dynamics are formulated as

$$\frac{dT}{dt} = \frac{\beta\alpha PT}{1 + fU} - cT - mTU. \tag{4}$$

The snake population increases through the consumption of rats. The parameter γ represents the conversion efficiency of consumed rats into snake population growth, while m represents the predation coefficient between rats and snakes. Snakes are also subject to natural mortality at rate s. Therefore, the snake population dynamics are expressed as

$$\frac{dU}{dt} = \gamma mTU - sU. \tag{5}$$

Combining Eq. (1), Eq. (4), and Eq. (5), the proposed modified rice–rat–snake food chain model with predator-induced fear is given by

$$\begin{aligned} \frac{dP}{dt} &= rP\left(1 - \frac{P}{K}\right) - \frac{\alpha PT}{1 + fU}, \\ \frac{dT}{dt} &= \frac{\beta\alpha PT}{1 + fU} - cT - mTU, \\ \frac{dU}{dt} &= \gamma mTU - sU. \end{aligned} \tag{6}$$

In Eq. (6), the term $(1 + fU)^{-1}$ distinguishes the proposed model from the conventional rice–rat–snake formulation by incorporating the non-consumptive influence of snake presence on rat population growth. When $f = 0$, Eq. (2) reduces to

$$F(U) = 1, \tag{7}$$

Thus, Eq. (7) confirms that the fear effect disappears when $f = 0$.

In the present formulation, the same fear function is applied to the effective rice–rat interaction in both the rice and rat equations. Thus, the term

$$\frac{\alpha PT}{1 + fU}$$

represents the effective rate of rice consumption by rats under predator-induced fear, while

$$\frac{\beta\alpha PT}{1 + fU}$$

represents the corresponding contribution of consumed rice to rat population growth. Therefore, increasing either the fear coefficient f or snake abundance U reduces the effective rice–rat

interaction. The intrinsic logistic growth of rice and the direct predation of rats by snakes remain represented separately in the model.

Accordingly, when $f = 0$, Eq. (4) reduces to

$$\frac{dT}{dt} = \beta\alpha PT - cT - mTU. \tag{8}$$

Thus, the model without predator-induced fear is recovered as a special case of the proposed system. For $f > 0$, increasing either f or U decreases the effective contribution of consumed rice to rat population growth. The proposed formulation therefore incorporates both the direct consumptive effect of snake predation and the indirect non-consumptive effect of predator-induced fear within the same food-chain framework.

2.3. Parameter Description

The proposed model involves parameters representing the demographic characteristics and trophic interactions among rice, rats, and snakes. The parameter r denotes the intrinsic growth rate of rice, while K represents the environmental carrying capacity that limits rice population growth. The interaction between rice and rats is characterized by the consumption coefficient α, which represents the intensity of rice consumption by rats, and the conversion efficiency β, which determines the proportion of consumed rice that contributes effectively to rat population growth. The rat population is additionally regulated by its natural mortality rate c.

The interaction between rats and snakes is characterized by the predation coefficient m, which represents the intensity of direct predation of rats by snakes. The parameter γ denotes the conversion efficiency through which consumed rats contribute to snake population growth, whereas s represents the natural mortality rate of snakes. Together, these parameters describe the direct trophic transfer from rice to rats and subsequently from rats to snakes.

The principal modification introduced in this study is the predator-induced fear coefficient f. This parameter quantifies the strength of the non-consumptive influence of snake presence on the effective rice–rat interaction. Increasing f represents a stronger behavioral response of rats to perceived predation risk and therefore reduces the effective rate of rice consumption and its corresponding contribution to rat population growth. A value of $f = 0$ corresponds to the absence of predator-induced fear, whereas $f > 0$ represents the presence of a fear response. Increasing f indicates a stronger response of rats to the presence of snakes and consequently a greater reduction in the effective contribution of consumed rice to rat population growth. Unlike the remaining demographic and interaction parameters, f is treated as a variable parameter in the numerical investigation to evaluate how different levels of predator-induced fear affect the dynamics and stability of the food-chain system. From a modeling perspective, the factor $1/(1 + fU)$ preserves positivity and introduces a saturating fear response: its marginal influence is strongest at low perceived risk and weakens as the product fU increases. This formulation is biologically plausible because avoidance behavior cannot reduce feeding activity indefinitely, while it provides a direct mechanism through which predator abundance modifies crop pressure without changing the direct predation term.

The parameter values used for numerical simulations are summarized in Table 1. The baseline values of the demographic and trophic parameters are adopted from the previously established rice–rat–snake model to maintain comparability between the original and modified systems. The fear coefficient is varied across simulation scenarios rather than assigned a single fixed value, allowing the model behavior to be examined from the no-fear condition to progressively stronger predator-induced fear. All parameters are assumed to be positive, except that $f = 0$ is permitted as the limiting case representing the absence of fear.

The baseline parameter values were selected primarily from the previous rice–rat–snake model [2], while parameters without direct empirical estimates were treated as assumed values for qualitative numerical investigation. In particular, the fear coefficient f was varied over the

Table 1: Parameter Values, Units, Biological Interpretations, and Sources

Parameter	Value	Unit	Biological Interpretation	Source
r	1.0	month^{-1}	Intrinsic growth rate of rice	Assumed
K	2000	rice plants ha^{-1}	Carrying capacity of rice	Based on [2]
α	0.0005	ha $(\text{rat})^{-1}$ month^{-1}	Consumption coefficient of rice by rats	Modified from [2]
β	0.30	rats $(\text{rice plant})^{-1}$	Conversion efficiency of consumed rice into rat population growth	[2]
c	0.20	month^{-1}	Natural mortality rate of rats	[2]
m	0.001	ha $(\text{snake})^{-1}$ month^{-1}	Predation coefficient of rats by snakes	[2]
γ	0.40	snakes $(\text{rat})^{-1}$	Conversion efficiency of consumed rats into snake population growth	[2]
s	0.10	month^{-1}	Natural mortality rate of snakes	[2]
f	0–0.01	ha $(\text{snake})^{-1}$	Predator-induced fear coefficient affecting effective rat population growth	Hypothetical range

interval $[0, 0.01]$ to examine the response of the system from the absence of predator-induced fear to progressively stronger fear effects. Therefore, the numerical parameter set is intended to illustrate the qualitative dynamical behavior of the model rather than to represent a calibrated field population. The robustness of the resulting dynamics with respect to parameter variation is further examined through sensitivity analysis.

2.4. Research Procedure

The analysis of the proposed rice–rat–snake food chain model is conducted through a systematic analytical and numerical procedure. First, the qualitative properties of the model are examined by establishing the positivity and boundedness of its solutions. Positivity is investigated to ensure that the population variables remain nonnegative for nonnegative initial conditions, while boundedness is examined to verify that the population trajectories remain within biologically meaningful limits over time. These properties provide the mathematical foundation for the subsequent dynamical analysis.

The equilibrium points of the system are then determined by imposing the steady-state conditions on all population equations. Each equilibrium is examined with respect to its biological feasibility to identify the conditions under which the corresponding population state can exist. The local stability of the feasible equilibria is subsequently investigated by linearizing the nonlinear system around each equilibrium using the Jacobian matrix. The resulting eigenvalue conditions are used to characterize local dynamical behavior, while the Routh–Hurwitz criterion is applied when the characteristic polynomial cannot be evaluated conveniently through explicit eigenvalues.

Finally, numerical simulations are performed using Python to complement and verify the analytical results. The system of ordinary differential equations is numerically solved using the parameter values and initial conditions specified for the rice–rat–snake system. Particular attention is given to the predator-induced fear coefficient, which is varied across several scenarios to investigate its influence on the temporal behavior and long-term coexistence of the three populations. The numerical trajectories are interpreted together with the analytical stability conditions to assess whether the simulated dynamics are consistent with the theoretical results. This procedure provides an integrated framework for evaluating both the mathematical properties and ecological implications of predator-induced fear in the proposed food-chain model.

3. Results and Discussion

This section presents the qualitative and numerical analysis of the proposed rice–rat–snake food chain model. The mathematical properties of the system are first investigated through positivity and boundedness analysis, followed by the determination of equilibrium points and their local stability. Numerical simulations are subsequently used to verify the analytical results and to examine the influence of predator-induced fear on the long-term dynamics of the three interacting populations.

3.1. Positivity of Solutions

Since the variables of the proposed model represent biological populations, their values must remain nonnegative for all $t \geq 0$. Therefore, it is necessary to establish that solutions originating from nonnegative initial conditions remain in the nonnegative region. Consider the proposed system

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right) - \frac{\alpha PT}{1 + fU}, \tag{9}$$

$$\frac{dT}{dt} = \frac{\beta\alpha PT}{1 + fU} - cT - mTU, \tag{10}$$

$$\frac{dU}{dt} = \gamma mTU - sU, \tag{11}$$

with initial conditions

$$P(0) = P_0 > 0, \qquad T(0) = T_0 > 0, \qquad U(0) = U_0 > 0. \tag{12}$$

Under the initial conditions in Eq. (12), Eq. (9) can be rewritten as

$$\frac{1}{P}\frac{dP}{dt} = r\left(1 - \frac{P}{K}\right) - \frac{\alpha T}{1 + fU}. \tag{13}$$

Integrating Eq. (13) from 0 to t gives

$$P(t) = P_0 \exp\left[\int_0^t \left(r\left(1 - \frac{P(\tau)}{K}\right) - \frac{\alpha T(\tau)}{1 + fU(\tau)}\right) d\tau\right]. \tag{14}$$

Since the exponential function in Eq. (14) is strictly positive and $P_0 > 0$, Eq. (14) implies

$$P(t) > 0 \tag{15}$$

for every $t \geq 0$ for which the solution exists. Similarly, Eq. (10) can be expressed as

$$\frac{1}{T}\frac{dT}{dt} = \frac{\beta\alpha P}{1 + fU} - c - mU. \tag{16}$$

Integrating Eq. (16) gives

$$T(t) = T_0 \exp\left[\int_0^t \left(\frac{\beta\alpha P(\tau)}{1 + fU(\tau)} - c - mU(\tau)\right) d\tau\right]. \tag{17}$$

For $f \geq 0$ and $U(t) \geq 0$, the denominator $1 + fU(t)$ remains positive. Therefore, since $T_0 > 0$, Eq. (17) implies

$$T(t) > 0 \tag{18}$$

for all $t \geq 0$. For the snake population, Eq. (11) gives

$$\frac{1}{U}\frac{dU}{dt} = \gamma mT - s, \tag{19}$$

Integrating Eq. (19) gives

$$U(t) = U_0 \exp\left[\int_0^t (\gamma m T(\tau) - s)\, d\tau\right]. \tag{20}$$

Because $U_0 > 0$ and the exponential term in Eq. (20) is positive, Eq. (20) implies

$$U(t) > 0 \tag{21}$$

for all $t \geq 0$.

Therefore, for any strictly positive initial condition

$$(P_0, T_0, U_0) \in \mathbb{R}^3_+, \tag{22}$$

the corresponding solution satisfies

$$P(t) > 0, \qquad T(t) > 0, \qquad U(t) > 0 \tag{23}$$

for all $t \geq 0$ within its interval of existence. Moreover, if one of the initial populations is zero, the corresponding coordinate plane is invariant because each population equation contains its respective population variable as a multiplicative factor. Hence, the nonnegative orthant

$$\mathbb{R}^3_+ = \left\{(P, T, U) \in \mathbb{R}^3 : P \geq 0,\ T \geq 0,\ U \geq 0\right\} \tag{24}$$

is positively invariant under the proposed system.

This result confirms the biological consistency of the model because populations that are initially nonnegative cannot become negative during their temporal evolution. The positivity property therefore provides the necessary foundation for the subsequent boundedness, equilibrium, and stability analyses.

3.2. Boundedness of Solutions

In addition to positivity, boundedness is required to ensure that the population trajectories remain within biologically meaningful limits. Consider the auxiliary function

$$W(t) = P(t) + \frac{1}{\beta} T(t) + \frac{1}{\beta\gamma} U(t). \tag{25}$$

Differentiating Eq. (25) along the solutions of the proposed system gives

$$\frac{dW}{dt} = \left[rP\left(1 - \frac{P}{K}\right) - \frac{\alpha P T}{1 + fU}\right] + \frac{1}{\beta}\left[\frac{\beta\alpha P T}{1 + fU} - cT - mTU\right] + \frac{1}{\beta\gamma}\left[\gamma m T U - sU\right].$$

The fear-dependent rice–rat interaction terms cancel exactly,

$$-\frac{\alpha P T}{1 + fU} + \frac{\alpha P T}{1 + fU} = 0,$$

while the direct predation terms also cancel,

$$-\frac{mTU}{\beta} + \frac{mTU}{\beta} = 0.$$

Therefore,

$$\frac{dW}{dt} = rP\left(1 - \frac{P}{K}\right) - \frac{c}{\beta} T - \frac{s}{\beta\gamma} U. \tag{26}$$

From Eq. (26), let $0 < \delta \leq \min\{c, s\}$. Adding δW to both sides gives

$$\frac{dW}{dt} + \delta W \leq (r + \delta)P - \frac{r}{K} P^2. \tag{27}$$

The right-hand side is a concave quadratic function of P whose maximum is

$$\frac{K(r+\delta)^2}{4r}.$$

Using the maximum of the right-hand side of the inequality in Eq. (27), we obtain the bound in Eq. (28):

$$\frac{dW}{dt} + \delta W \leq \frac{K(r+\delta)^2}{4r}. \tag{28}$$

Applying the differential inequality theorem to Eq. (28) yields Eq. (29):

$$W(t) \leq W(0)e^{-\delta t} + \frac{K(r+\delta)^2}{4r\delta}\left(1 - e^{-\delta t}\right). \tag{29}$$

Taking the limit superior of Eq. (29) as $t \to \infty$ gives Eq. (30):

$$\limsup_{t\to\infty} W(t) \leq \frac{K(r+\delta)^2}{4r\delta}. \tag{30}$$

Thus, all solutions of the proposed system initiating in the nonnegative region are ultimately bounded. Together with the positivity result, this establishes that the model evolves within a biologically feasible region and that none of the three populations can grow without bound.

3.3. Equilibrium Analysis

The equilibrium points are obtained by setting all population growth rates equal to zero. Thus, the equilibrium conditions are given by Eq. (31)–Eq. (33):

$$P\left[r\left(1 - \frac{P}{K}\right) - \frac{\alpha T}{1+fU}\right] = 0, \tag{31}$$

$$T\left[\frac{\beta\alpha P}{1+fU} - c - mU\right] = 0, \tag{32}$$

$$U\left[\gamma m T - s\right] = 0. \tag{33}$$

Several biologically distinct equilibrium states can be identified from these conditions.

The first equilibrium is the extinction equilibrium

$$E_0 = (0, 0, 0),$$

which represents the absence of all three populations. The second equilibrium is the rice-only equilibrium

$$E_1 = (K, 0, 0),$$

where rice reaches its carrying capacity in the absence of rats and snakes. In the absence of snakes, when rice and rats coexist, the third equilibrium is $E_2 = (P_2, T_2, 0)$, where

$$P_2 = \frac{c}{\beta\alpha} \quad \text{and} \quad T_2 = \frac{r}{\alpha}\left(1 - \frac{c}{\beta\alpha K}\right).$$

The equilibrium E_2 is biologically feasible provided that $\beta\alpha K > c$. Furthermore, the interior equilibrium is given by $E^* = (P^*, T^*, U^*)$.

This equilibrium represents coexistence among rice, rats, and snakes. From Eq. (33),

$$T^* = \frac{s}{\gamma m}. \tag{34}$$

Substitution into Eq. (31) yields

$$P^* = K\left(1 - \frac{\alpha s}{r\gamma m(1+fU^*)}\right). \tag{35}$$

Therefore, a necessary condition for a positive rice population at coexistence is

$$r\gamma m > \alpha s. \tag{36}$$

The remaining equilibrium condition is obtained from Eq. (32) and gives

$$\frac{\beta\alpha P^*}{1+fU^*} = c + mU^*. \tag{37}$$

For $f > 0$, substituting Eq. (35) into Eq. (37) gives

$$\beta\alpha K\left(1 + fU^* - \frac{\alpha s}{r\gamma m}\right) = (c + mU^*)(1 + fU^*)^2. \tag{38}$$

Rearranging Eq. (38) yields the cubic Eq. (39), whose biologically relevant positive root determines U^*.

$$\begin{aligned} &mf^2(U^*)^3 + (2mf + cf^2)(U^*)^2 \\ &\quad + (m + 2cf - \beta\alpha K f)U^* + c - \beta\alpha K + \frac{\beta\alpha^2 Ks}{r\gamma m} = 0. \end{aligned} \tag{39}$$

Unlike a quadratic equation, this cubic equation does not in general admit a simple biologically convenient expression for its positive root. Therefore, the biologically relevant positive root of the cubic equation is determined numerically for the parameter values considered. When $f = 0$, Eq. (37) reduces to

$$U^* = \frac{\beta\alpha P^* - c}{m}. \tag{40}$$

Furthermore, the dependence of the equilibrium snake population U^* on the fear coefficient f can be examined by explicitly accounting for the dependence of the equilibrium rice population P^* on both f and U^*. Substituting Eq. (35) into Eq. (37) gives

$$\frac{\beta\alpha K}{1+fU^*}\left(1 - \frac{\alpha s}{r\gamma m(1+fU^*)}\right) - c - mU^* = 0. \tag{41}$$

For convenience, define

$$A = \frac{\alpha s}{r\gamma m}, \qquad B = \beta\alpha K, \qquad x = 1 + fU^*. \tag{42}$$

Using the definitions in Eq. (42), the equilibrium condition in Eq. (41) can be rewritten as Eq. (43):

$$B\left(\frac{1}{x} - \frac{A}{x^2}\right) - c - mU^* = 0. \tag{43}$$

Implicitly differentiating the equilibrium condition in Eq. (43) with respect to f, using $x = 1 + fU^*(f)$ as defined in Eq. (42), yields Eq. (44):

$$\frac{dU^*}{df} = -\frac{BU^*(x - 2A)}{mx^3 + Bf(x - 2A)}. \tag{44}$$

Eq. (44) implies that

$$\frac{dU^*}{df} < 0 \tag{45}$$

provided that

$$x > 2A, \tag{46}$$

or equivalently,

$$1 + fU^* > \frac{2\alpha s}{r\gamma m}. \tag{47}$$

For the parameter values considered in this study,

$$\frac{\alpha s}{r\gamma m} = \frac{(0.0005)(0.10)}{(1)(0.40)(0.001)} = 0.125, \tag{48}$$

and therefore

$$\frac{2\alpha s}{r\gamma m} = 0.25.$$

Since $1 + fU^* \geq 1 > 0.25$, the above condition is satisfied for all fear levels considered in the numerical investigation. Consequently,

$$\frac{dU^*}{df} < 0 \tag{49}$$

for the parameter regime investigated. Thus, the equilibrium snake population decreases monotonically with increasing predator-induced fear for the parameter configuration considered in this study. This analytical result is consistent with the numerical equilibrium values reported in Section 3.5.

3.4. Local Stability Analysis

The local stability of each equilibrium point is investigated using the Jacobian matrix. The Jacobian matrix of the proposed system is

$$J(P,T,U) = \begin{pmatrix} r - \frac{2rP}{K} - \frac{\alpha T}{1+fU} & -\frac{\alpha P}{1+fU} & \frac{f\alpha PT}{(1+fU)^2} \\ \frac{\beta\alpha T}{1+fU} & \frac{\beta\alpha P}{1+fU} - c - mU & -\frac{\beta\alpha PTf}{(1+fU)^2} - mT \\ 0 & \gamma mU & \gamma mT - s \end{pmatrix}. \tag{50}$$

At the extinction equilibrium E_0, the eigenvalues are

$$\lambda_1 = r, \qquad \lambda_2 = -c, \qquad \lambda_3 = -s. \tag{51}$$

Since $\lambda_1 = r > 0$ in Eq. (51), the extinction equilibrium is unstable.

At the rice-only equilibrium $E_1 = (K, 0, 0)$, the eigenvalues are

$$\lambda_1 = -r, \qquad \lambda_2 = \beta\alpha K - c, \qquad \lambda_3 = -s. \tag{52}$$

Therefore, from the eigenvalues in Eq. (52), E_1 is locally asymptotically stable if

$$\beta\alpha K < c. \tag{53}$$

At the rice–rat equilibrium E_2, the eigenvalue associated with invasion by the snake population is

$$\lambda_3 = \gamma m T_2 - s. \tag{54}$$

The rice–rat subsystem has negative trace and positive determinant whenever E_2 is biologically feasible. Consequently, E_2 is locally asymptotically stable if

$$\gamma m T_2 < s, \tag{55}$$

whereas it becomes unstable to snake invasion when

$$\gamma m T_2 > s. \tag{56}$$

Evaluating the Jacobian matrix in Eq. (50) at the coexistence equilibrium E^* gives the following compact form:

$$J(E^*) = \begin{pmatrix} -A & -B & C_1 \\ D & 0 & -E \\ 0 & C_2 & 0 \end{pmatrix}, \tag{57}$$

where

$$A = \frac{rP^*}{K}, \tag{58}$$

$$B = \frac{\alpha P^*}{1 + fU^*}, \tag{59}$$

$$C_1 = \frac{f\alpha P^* T^*}{(1 + fU^*)^2}, \tag{60}$$

$$D = \frac{\beta\alpha T^*}{1 + fU^*}, \tag{61}$$

$$E = mT^* + \frac{f\beta\alpha P^* T^*}{(1 + fU^*)^2}, \tag{62}$$

$$C_2 = \gamma m U^*. \tag{63}$$

For a biologically feasible coexistence equilibrium, A, B, D, E and C_2 are positive, while $C_1 \geq 0$, with $C_1 > 0$ when $f > 0$ and $C_1 = 0$ when $f = 0$. In particular, when $f = 0$, the fear-dependent coupling vanishes and $C_1 = 0$. The quantities defined in Eq. (58)–Eq. (63) represent the entries of the compact Jacobian matrix in Eq. (57).

Substituting the Jacobian matrix in Eq. (57), with its entries defined in Eq. (58)–Eq. (63), gives the characteristic polynomial:

$$\lambda^3 + A\lambda^2 + (BD + EC_2)\lambda + C_2(AE - DC_1) = 0. \tag{64}$$

For the Routh–Hurwitz analysis, Eq. (64) is written in the standard form:

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0,$$

The coefficients are

$$a_1 = A, \tag{65}$$

$$a_2 = BD + EC_2, \tag{66}$$

$$a_3 = C_2(AE - DC_1). \tag{67}$$

Using the coefficients defined in Eq. (65)–Eq. (67), the Routh–Hurwitz criterion states that the coexistence equilibrium is locally asymptotically stable if

$$a_1 > 0, \qquad a_2 > 0, \qquad a_3 > 0, \qquad a_1 a_2 > a_3.$$

Since A, B, D, E, and C_2 are positive, the first two conditions are automatically satisfied. Furthermore,

$$a_1 a_2 - a_3 = A(BD + EC_2) - C_2(AE - DC_1) = ABD + C_2 DC_1 > 0.$$

Therefore, the remaining nontrivial condition is

$$AE > DC_1.$$

Hence, the coexistence equilibrium E^* is locally asymptotically stable whenever it is biologically feasible and satisfies

$$AE > DC_1.$$

Thus, local asymptotic stability is determined not only by the biological feasibility of the coexistence equilibrium but also by the Routh–Hurwitz conditions evaluated at that equilibrium. In particular, the nontrivial stability condition is

$$AE > DC_1.$$

Therefore, the coexistence equilibrium E^* is locally asymptotically stable when it is biologically feasible and the above Routh–Hurwitz condition is satisfied. For the parameter values considered in the numerical simulations, this condition is satisfied, and the numerical trajectories consequently converge toward the corresponding coexistence equilibrium. This result indicates that, under the proposed model structure, predator-induced fear changes the location of the coexistence equilibrium while local asymptotic stability is maintained for the parameter sets investigated.

3.5. Numerical Simulations

Numerical simulations were conducted using Python to complement the analytical results and to investigate the influence of predator-induced fear on the temporal dynamics of the rice–rat–snake system. The baseline parameter values were

$$r = 1, \quad K = 2000, \quad \alpha = 0.0005, \quad \beta = 0.30,$$

and

$$c = 0.20, \quad m = 0.001, \quad \gamma = 0.40, \quad s = 0.10.$$

The initial conditions were

$$P(0) = 1500, \qquad T(0) = 100, \qquad U(0) = 10.$$

The equilibrium rat density is independent of the fear coefficient and is given by

$$T^* = \frac{s}{\gamma m} = \frac{0.10}{(0.40)(0.001)} = 250.$$

For $f = 0$, the coexistence equilibrium is

$$E^* = (1750,\ 250,\ 62.5).$$

For $f > 0$, the positive equilibrium U^* was obtained from the fear-dependent cubic equilibrium equation, after which P^* was calculated from

$$P^* = K\left(1 - \frac{\alpha s}{r\gamma m(1 + fU^*)}\right).$$

For the four fear levels considered, the resulting coexistence equilibria are summarized in Table 2.

Table 2: Coexistence equilibrium for different values of the predator-induced fear coefficient.

f	P^*	T^*	U^*
0	1750.0000	250	62.5000
0.001	1762.2238	250	51.4089
0.005	1783.8633	250	31.3350
0.010	1794.2458	250	21.5042

Figure 1 shows the temporal dynamics of the three populations for different values of the predator-induced fear coefficient. The trajectories exhibit damped oscillatory behavior before approaching their corresponding coexistence equilibria. For $f = 0$, the trajectories approach

$$E^* = (1750,\ 250,\ 62.5).$$

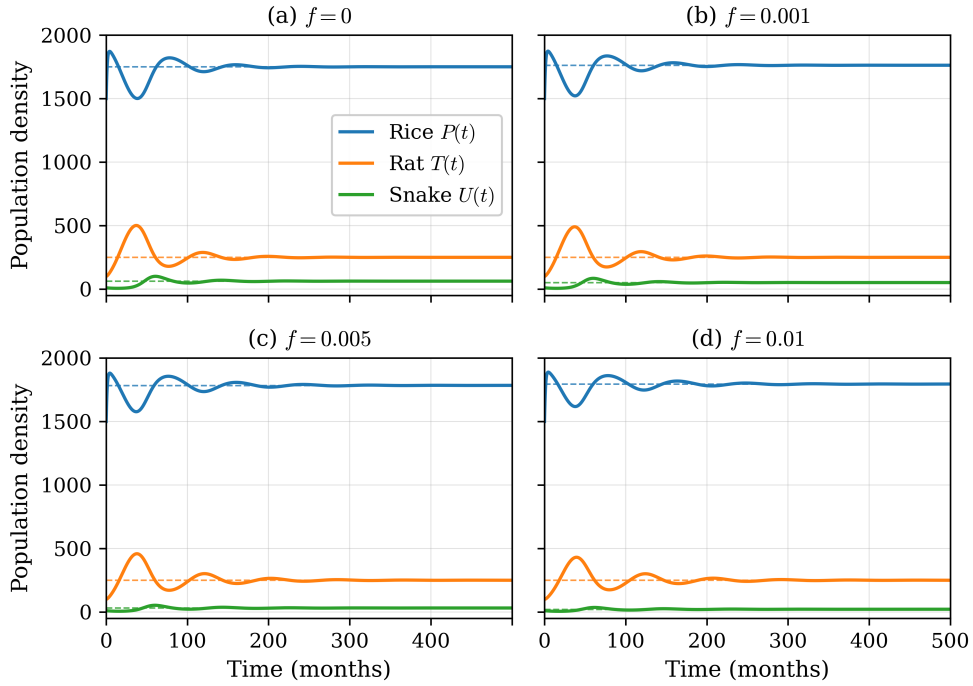


Fig. 1: Time-series dynamics of the rice, rat, and snake populations for different values of the predator-induced fear coefficient: (a) $f = 0$, (b) $f = 0.001$, (c) $f = 0.005$, and (d) $f = 0.01$. Dashed lines indicate the corresponding coexistence equilibrium.

As f increases, both the transient trajectories and the location of the coexistence equilibrium change. In particular, the equilibrium rice population increases, whereas the equilibrium snake population decreases. The rat equilibrium remains fixed at $T^* = 250$.

Taken together, these trajectories show that predator-induced fear redistributes the long-term population levels across the food chain without eliminating convergence to coexistence. Ecologically, the increase in rice density and the decrease in snake density demonstrate how a behavioral response at the intermediate trophic level can propagate indirectly to both the basal resource and the top predator. The phase portraits in Figure 2 provide a geometric representation of the convergence toward the corresponding coexistence equilibria. For all parameter values considered, the trajectories converge toward the interior equilibrium, consistent with the local stability conditions obtained from the Routh–Hurwitz analysis.

The phase portraits in Figure 2 provide a geometric representation of the convergence toward the coexistence equilibrium. For each value of f, the trajectory forms a shrinking spiral toward an interior equilibrium. The change in the position of the attracting equilibrium becomes increasingly visible along the snake population axis as the fear coefficient increases. These numerical observations further support the local stability results obtained from the Routh–Hurwitz criterion.

Figure 3 illustrates the relationship between the predator-induced fear coefficient and the equilibrium snake population. The numerical results show that U^* decreases as f increases. For the selected fear levels,

$$U^* = 62.5000, \quad 51.4089, \quad 31.3350, \quad 21.5042$$

for $f = 0$, 0.001, 0.005, and 0.01, respectively.

At the same time, the corresponding equilibrium rice population increases from 1750.0000 to 1794.2458, while the equilibrium rat population remains equal to 250. Therefore, predator-induced fear changes the coexistence equilibrium through the coupled dependence of P^* and U^*. Overall, the numerical results are consistent with the analytical findings. The simulated solutions remain positive and bounded, the trajectories converge toward the predicted coexistence equilibria, and

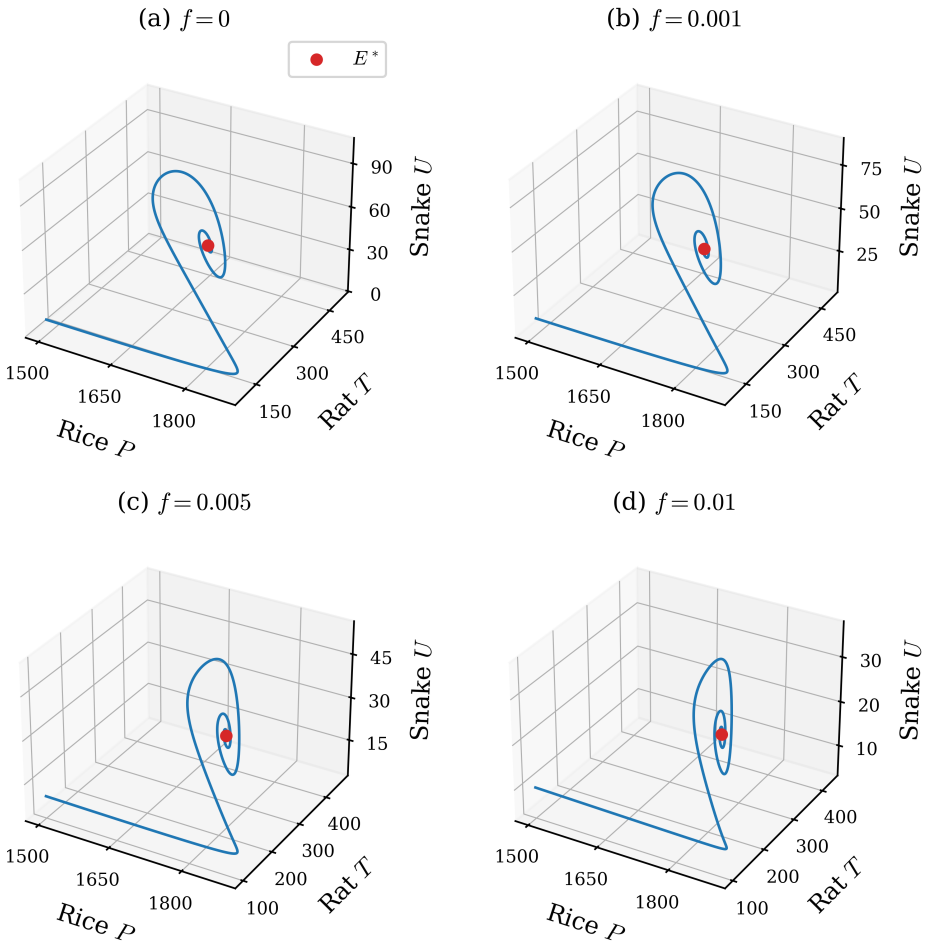


Fig. 2: Three-dimensional phase portraits of the rice–rat–snake system for different values of the predator-induced fear coefficient: (a) $f = 0$, (b) $f = 0.001$, (c) $f = 0.005$, and (d) $f = 0.01$.

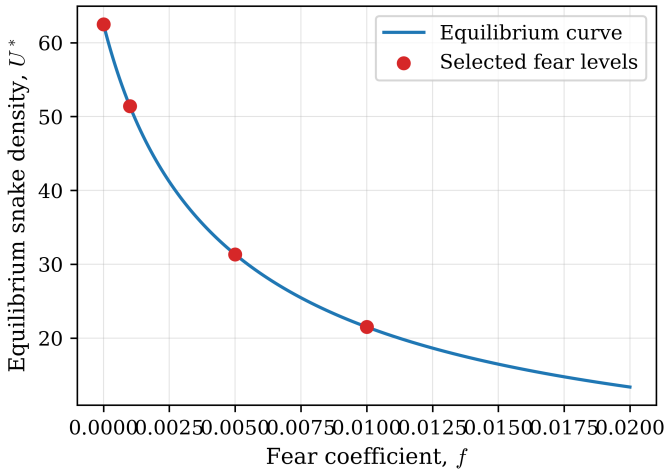


Fig. 3: Effect of the predator-induced fear coefficient f on the equilibrium abundance of the snake population U^*. The marked points indicate the fear levels considered in the numerical simulations.

increasing predator-induced fear modifies the equilibrium configuration without destroying local stability.

3.6. Sensitivity Analysis

To assess the robustness of the predicted coexistence equilibrium with respect to parameter variation, a local one-at-a-time (OAT) sensitivity analysis was performed. The baseline fear coefficient was set to $f = 0.005$, while the remaining parameters were fixed at their baseline values. Each parameter was independently perturbed by -20% and $+20\%$, while all other parameters were kept unchanged.

The equilibrium snake population U^* was selected as the primary response variable because the study focuses on the influence of the fear-dependent trophic interaction on the top predator. The sensitivity index for each parameter q_i was defined as

$$S_i = \max\left\{\frac{|U^*(0.8q_i) - U_0^*|}{U_0^*}, \frac{|U^*(1.2q_i) - U_0^*|}{U_0^*}\right\} \times 100\%, \tag{68}$$

where U_0^* denotes the equilibrium snake population obtained from the baseline parameter set. The sensitivity indices calculated using Eq. (68) are summarized in Figure 4.

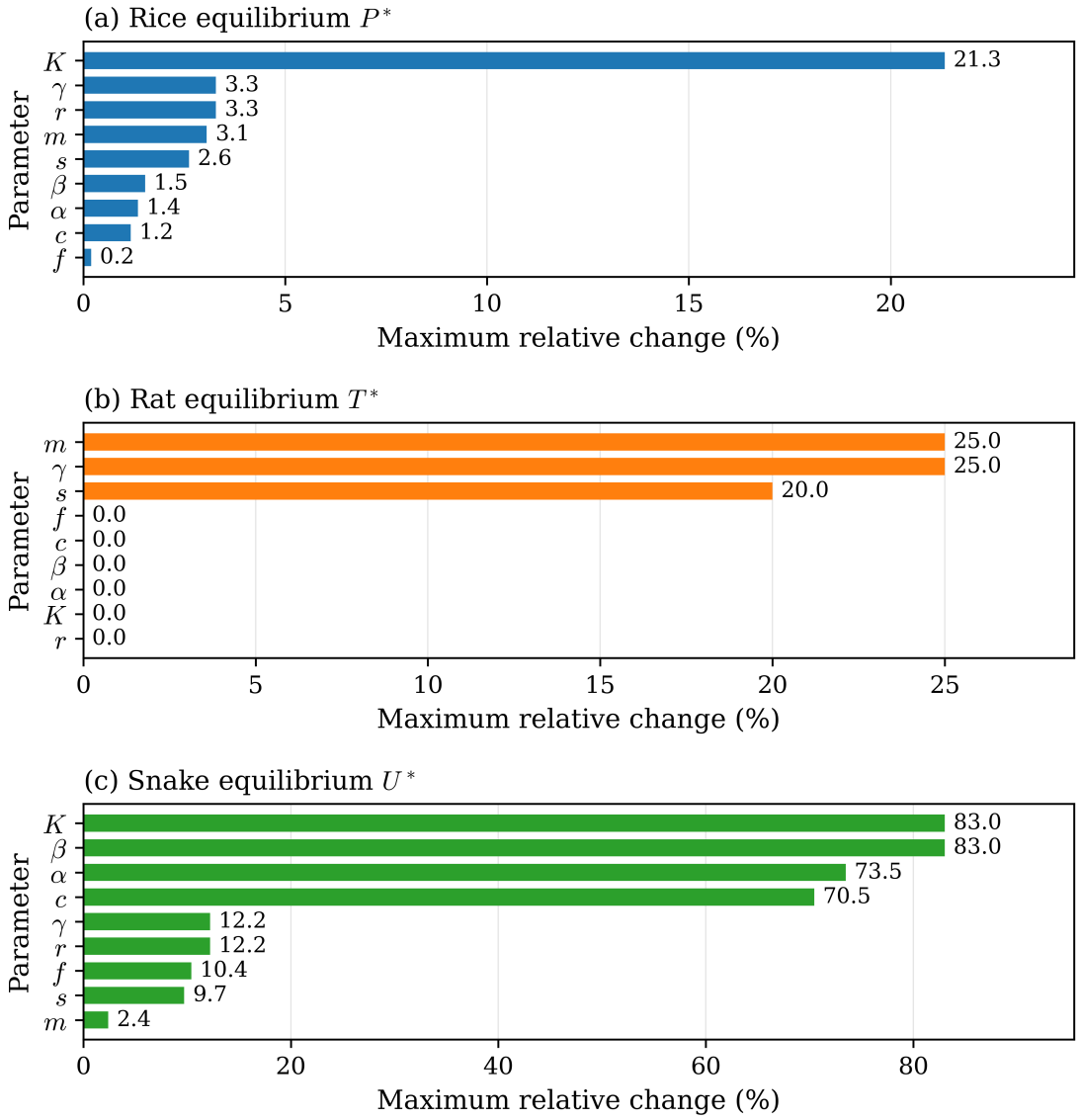


Fig. 4: Sensitivity analysis of the coexistence equilibrium with respect to the model parameters: (a) rice equilibrium P^*, (b) rat equilibrium T^*, and (c) snake equilibrium U^*.

Figure 4 presents the sensitivity analysis of the coexistence equilibrium for the rice, rat, and snake populations, represented by P^*, T^*, and U^*, respectively. The results indicate that the

equilibrium populations respond differently to variations in the model parameters. For the snake equilibrium U^*, the carrying capacity K and conversion efficiency β produce the largest relative changes, followed by the rice consumption coefficient α and the rat mortality parameter c. The fear coefficient f also affects U^*, with a relative change of approximately 10.40%.

The fear coefficient f produces a maximum relative change of approximately 10.40% in U^* around the baseline value $f = 0.005$. The parameters r and γ each produce changes of approximately 12.22%, while s produces approximately 9.73%. The predation coefficient m produces the smallest local change, approximately 2.40%. These results should be interpreted as local sensitivity measures around the selected baseline parameter set. They do not constitute a global sensitivity analysis and should not be interpreted as evidence that a parameter is biologically unimportant outside the investigated range.

3.7. Discussion

The analytical and numerical results demonstrate that incorporating predator-induced fear modifies the rice–rat–snake food chain while preserving the fundamental biological properties of the system. Positivity guarantees that nonnegative population densities remain biologically meaningful, whereas boundedness prevents unbounded population growth. Four biologically distinct equilibrium states are identified: total extinction, rice-only persistence, rice–rat coexistence without snakes, and full coexistence among rice, rats, and snakes.

The biological interpretation of the fear mechanism is central to the proposed formulation. The function

$$F(U) = \frac{1}{1 + fU}$$

decreases as either the fear coefficient f or snake abundance U increases. In biological terms, this factor represents a reduction in the effective foraging activity of rats under greater perceived predation risk. Consequently, the effective rate of rice consumption by rats decreases, and the contribution of consumed rice to rat population growth decreases simultaneously. The fear mechanism therefore represents a non-consumptive predator effect rather than an additional mortality term.

The equilibrium analysis reveals that

$$T^* = \frac{s}{\gamma m},$$

so the equilibrium rat density is determined by the balance between snake reproduction and snake mortality. In contrast,

$$P^* = K\left(1 - \frac{\alpha s}{r\gamma m(1 + fU^*)}\right),$$

which shows that the equilibrium rice density depends on the fear coefficient indirectly through the equilibrium snake density. Thus, predator-induced fear can modify both the equilibrium rice and snake populations even though it does not directly alter the intrinsic logistic growth rate of rice.

The numerical simulations support this analytical interpretation. As the fear coefficient increases from 0 to 0.01, the equilibrium snake population decreases from 62.5000 to approximately 21.5042, whereas the equilibrium rice population increases from 1750.0000 to approximately 1794.2458. The equilibrium rat population remains at 250 because it is determined by the snake demographic balance. Therefore, stronger predator-induced fear reduces the effective rice–rat interaction and changes the equilibrium configuration of the food chain.

The local stability analysis further shows that the coexistence equilibrium is locally asymptotically stable when the Routh–Hurwitz conditions are satisfied. In particular, the nontrivial condition is

$$AE > DC_1.$$

For the parameter values used in the numerical simulations, this condition is satisfied and the numerical trajectories converge toward the corresponding coexistence equilibria. Hence, the observed convergence in the simulations is consistent with the local stability analysis.

The sensitivity analysis indicates that the predicted snake equilibrium is particularly responsive to parameters associated with resource availability and trophic conversion, especially K, β, α, and c. The fear coefficient produces a more moderate local change in U^* around the selected baseline value. However, these results represent local sensitivity around a particular parameter set and should not be interpreted as a global ranking of ecological importance.

From a modeling perspective, the proposed formulation distinguishes direct predation from non-consumptive predator effects within the same food-chain framework. Direct predation reduces the rat population through the term mTU, whereas predator-induced fear modifies the effective rice–rat interaction through the factor $(1 + fU)^{-1}$. This distinction provides a mathematical representation of how predator presence may influence prey behavior in addition to causing direct mortality.

Nevertheless, the model is based on a spatially homogeneous deterministic framework and uses several assumed or previously adopted parameter values. The numerical results should therefore be interpreted as qualitative model predictions rather than field-calibrated population estimates. Empirical estimation of the fear response and field-based parameter calibration would be required for quantitative ecological prediction.

4. Conclusion

This study developed a modified rice–rat–snake food-chain model by incorporating predator-induced fear into the effective rice–rat interaction. The proposed formulation represents the non-consumptive effect of snake presence through the decreasing factor

$$F(U) = \frac{1}{1 + fU},$$

while direct snake predation remains represented separately by the term mTU. When $f = 0$, the model reduces to the corresponding no-fear system.

The qualitative analysis established the positivity and ultimate boundedness of the solutions, ensuring the biological feasibility of the model. Four equilibrium states were identified, including total extinction, rice-only persistence, rice–rat coexistence, and full rice–rat–snake coexistence. For the interior equilibrium, the rat density satisfies

$$T^* = \frac{s}{\gamma m},$$

while the rice and snake densities are coupled through the fear-dependent equilibrium conditions.

The local stability analysis based on the Jacobian matrix and the Routh–Hurwitz criterion showed that the coexistence equilibrium is locally asymptotically stable when the corresponding stability conditions are satisfied. Numerical simulations using the selected parameter values confirmed convergence toward the coexistence equilibria for the investigated fear levels. Increasing the predator-induced fear coefficient changes the coexistence equilibrium by reducing the equilibrium snake abundance and increasing the equilibrium rice abundance, while the equilibrium rat density remains fixed by the snake demographic balance. These results illustrate that predator-induced fear can alter trophic interactions through a non-consumptive behavioral mechanism without necessarily eliminating local coexistence.

The sensitivity analysis further showed that the predicted equilibrium snake abundance is particularly responsive to parameters associated with resource availability, conversion efficiency, and rat mortality. Because the model uses assumed and previously adopted parameter values, the numerical results should be interpreted as qualitative predictions. Future studies may incorporate empirical estimation of predator-induced fear, spatial movement, seasonal variation, nonlinear

functional responses, and field-based calibration to obtain a more detailed representation of rice-field ecosystems.

CRediT Authorship Contribution Statement

Zakaria: Conceptualization, Methodology, Software, Formal Analysis, Investigation, Visualization, writing–original Draft. **Dian Savitri:** Supervision, Validation, and review.

Declaration of Generative AI and AI-assisted technologies

ChatGPT (OpenAI, GPT-6) was used during the preparation of this manuscript to assist with English-language proofreading, language editing, improving the clarity and consistency of selected passages, preparing responses to editorial comments, and providing technical assistance with Python code for figure preparation and export. The tool was not used to independently generate research data or determine the scientific conclusions. All AI-assisted suggestions were reviewed and verified by the authors, who take full responsibility for the content of the manuscript.

Declaration of Competing Interest

The authors declare no competing interests.

Ethics Approval and Consent to Participate

Ethics approval was not required for this study, as the research involved mathematical modeling and numerical simulations only and did not involve experiments on human participants or animals.

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Data and Code Availability

No external datasets were generated or analyzed during this study. The parameter values used in the numerical simulations are provided in Table 1, and the numerical results are presented in the manuscript. The Python code used for numerical simulations and figure generation is available from the corresponding author upon reasonable request.

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