CAUCHY – Jurnal Matematika Murni dan Aplikasi
Volume 11 (2) (2026), Pages 2689–2703
p-ISSN: 2086-0382; e-ISSN: 2477-3344

Non-Nilpotent Graph of 2×2 Upper Triangular Matrix Rings over $\mathbb{Z}_{4p}$ for Prime Numbers p

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Abstract

Suppose R represents an arbitrary ring. Let $\mathrm{Nil}(R)$ be the collection of all nilpotent elements in R. The non-nilpotent graph, symbolized by $G_{NN}(R)$, is defined over the vertex set $R \setminus \mathrm{Nil}(R)$, where two distinct vertices form an edge whenever their sum belongs to $\mathrm{Nil}(R)$. Furthermore, let $T_2(\mathbb{Z}_n)$ denote the ring of 2×2 upper triangular matrices with entries in $\mathbb{Z}_n$. This paper investigates the structure of $G_{NN}(T_2(\mathbb{Z}_{4p}))$, where $p \geq 3$ is a prime number. The main findings reveal that $G_{NN}(T_2(\mathbb{Z}_{4p}))$ decomposes into a disjoint union of complete subgraphs and complete bipartite subgraphs. Additionally, key structural invariants of this graph, namely its clique number, girth, diameter, and domination number, are analyzed, along with an examination of its Eulerian and Hamiltonian properties.

Keywords: Nilpotent, Non-Nilpotent Graph, Matrix Ring.

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1. Introduction

The connection between algebraic structures and graphs is an interesting and productive research topic in the field of algebra. This approach was first introduced by [1], who defined the zero-divisor graph of a commutative ring. That initial definition was later modified by [2] to restrict the vertex set only to non-zero zero-divisor elements, which has now become the standard definition and significantly propelled the expansion of the concept.

The importance of research on algebraic graphs lies in its ability to translate complex algebraic properties of a ring, such as zero-divisor, nilpotent, and unit properties, into measurable graph properties, such as diameter, girth, or clique number. This reciprocal relationship allows researchers to gain deeper insights into algebraic structures through graph theory. Since the introduction of the zero-divisor graph, researchers have begun developing various other graph models related to commutative rings to represent different algebraic aspects. Graphs discussed in the literature include zero-divisor graphs [2] [3] [4] [5] [6], nilpotent graph [7] [8], dot total graph [9], prime graph [10], trace graph [11] [12], fibonacci graph [13] and jacobson graph [14]. The exploration of these new graph models demonstrates that this field of research is highly dynamic and open to linking algebraic structures and graphs for conceptual innovation.

In the development of algebraic graphs, the concept of nilpotent elements of a ring has also become a focus of research. The set of nilpotent elements is used to define the non-nilpotent graph. In a ring, an element a is classified as nilpotent provided that $a^n = 0$ for a positive

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Received: September 8, 2026 Revised: November 5, 2026 Accepted: November 11, 2026
DOI: https://doi.org/10.18860/cauchy.v11i2.45999

integer n [15]. The collection consisting of all such nilpotent elements in a ring R is denoted by $Nil(R)$. In this study, we will use the nilpotent elements of an algebraic structure to define a non-nilpotent graph. The notion of non-nilpotent graphs originated in [16]. For a ring R, the non-nilpotent graph $G_{NN}(R)$ features the vertex set $V(G_{NN}(R)) = R \setminus \text{Nil}(R)$, in which two distinct vertices x and y are adjacent by an edge provided that $x + y \in \text{Nil}(R)$ [16]. The non-nilpotent graph $G_{NN}(R)$, originally introduced for commutative rings R with the vertex set $V = R \setminus \text{Nil}(R)$ and the adjacency relation $x + y \in \text{Nil}(R)$, can be extended to non-commutative upper triangular matrix rings. Although upper triangular matrix rings are non-commutative under multiplication, the definition of $G_{NN}(R)$ remains valid as an undirected graph. This is because the adjacency condition is based on matrix addition, which is commutative, thereby preserving the symmetry of adjacency between vertices. Although the adjacency rule relies on the addition $x + y \in \text{Nil}(R)$, the presence of a non-commutative matrix structure significantly influences the characteristics of its nilpotent elements, thereby triggering a much richer and more complex component decomposition compared to commutative rings. Previous research by [8] examined nilpotent graphs of 2×2 upper triangular matrix rings over $\mathbb{Z}_p$ and $\mathbb{Z}_{2p}$, where $p \geq 3$ is prime, and described the general structures and properties of the graphs. In fact, that study employed the definition of non-nilpotent graphs, even though it was referred to as nilpotent graphs. Building upon these works, this research focuses on extending the study of non-nilpotent graphs to 2×2 upper triangular matrix rings over the integers modulo $4p$, denoted by $G_{NN}(T_2(\mathbb{Z}_{4p}))$, where $p \geq 3$ is a prime.

Let G be a graph. The degree of a vertex in graph G is the number of edges incident to that vertex, with each loop counted twice [17]. For a vertex $x \in V(G)$, its degree is represented by $\deg(x)$. A graph G is connected if a path links every pair of vertices; otherwise, it is disconnected [17]. Any disconnected graph can be decomposed into several disjoint connected subgraphs, which are called components [17]. A connected graph is defined as Eulerian if it contains a closed trail traversing every edge, which is known as an Eulerian trail. A connected graph is Hamiltonian if it possesses a cycle passing through every vertex, termed a Hamiltonian cycle [17]. The girth of a graph G, denoted by $gr(G)$, represents the length of its shortest cycle and if G contains no cycles, its girth is taken to be infinite [18]. A clique in G refers to a subset of pairwise adjacent vertices, and the cardinality of the largest clique is defined as the clique number of G, denoted by $\omega(G)$ [18]. The distance between vertices x and y in G, denoted by $d(x, y)$, is the length of a shortest path connecting them; if no such path exists, then $d(x, y) = \infty$ [18]. The diameter of G, denoted by $\text{diam}(G)$, is the maximum distance separating any pair of vertices in G [18]. Let D be a subset of the vertex set of graph G; the set D is called a dominating set of G if every vertex not in D is adjacent to at least one vertex in D and the domination number is the number of vertices in the smallest dominating set of G, denoted by $\gamma(G)$ [19]. For other definitions and terms not explicitly mentioned, this study refers to the book by [17].

The main contribution of this research lies in the complete structural characterization of the non-nilpotent graph $G_{NN}(T_2(\mathbb{Z}_{4p}))$ for prime numbers $p \geq 3$. Specifically, we establish the explicit decomposition of $G_{NN}(T_2(\mathbb{Z}_{4p}))$ into disjoint complete and complete bipartite subgraphs. Based on this structural decomposition, we subsequently determine its key graph invariants, including girth, clique number, diameter, and domination number, as well as analyze its Eulerian and Hamiltonian properties.

2. Methods

This study adopts an analytical mathematical approach to determine the structural properties of non-nilpotent graphs $G_{NN}(T_2(\mathbb{Z}_{4p}))$. The steps taken in this research are as follows:

1. Determining the nilpotent elements of the ring $\mathbb{Z}_{4p}$ and the nilpotent elements of the ring $T_2(\mathbb{Z}_{4p})$
2. Determining the order of the graph $G_{NN}\left(T_2(\mathbb{Z}_{4p})\right)$

3. Determining the condition for adjacent vertices in the graph $G_{NN}(T_2(\mathbb{Z}_{4p}))$
4. Determining the general form of the graph $G_{NN}(T_2(\mathbb{Z}_{4p}))$
5. Analyzing the Eulerian graph and Hamiltonian graph structures of the graph $G_{NN}(T_2(\mathbb{Z}_{4p}))$
6. Determining the girth, clique number, diameter, and domination number of the graph $G_{NN}(T_2(\mathbb{Z}_{4p}))$
7. Formulating theorems
8. Drawing conclusions

Before presenting the result and discussion of the non-nilpotent graph of matrix ring, we first review some definitions, lemmas, theorems, and examples.

Definition 1. [16] Let R be a commutative ring with a nonzero identity. The non-nilpotent graph $G_{NN}(R)$ of R is an undirected simple graph with the set of vertices consisting of all elements $\overline{Nil(R)} = R \setminus Nil(R)$. Two distinct vertices $x, y \in \overline{Nil(R)}$ are adjacent if and only if $x + y \in Nil(R)$.

Example 1. To illustrate the structure of $G_{NN}(T_2(\mathbb{Z}_3))$, the construction begins with the identification of all nilpotent elements in $T_2(\mathbb{Z}_3)$.

$$Nil(T_2(\mathbb{Z}_3)) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \right\}$$

As illustrated in Figure 1, we obtain graph $G_{NN}(T_2(\mathbb{Z}_3))$ as follows.

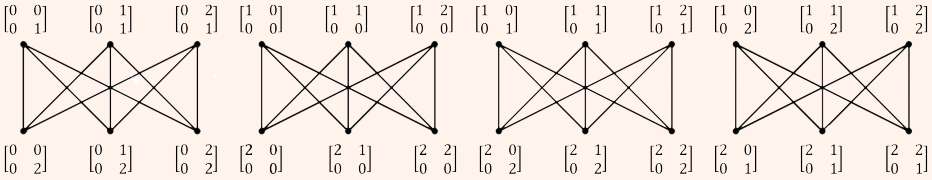


Fig. 1: $G_{NN}(T_2(\mathbb{Z}_3))$

Theorem 1. *[20] Let $x \in \mathbb{Z}_n$, $n = p_1^{k_1} p_2^{k_2} \dots p_m^{k_m}$ where primes p_i and positive integers k_i. It holds that $x \in Nil(\mathbb{Z}_n)$ if and only if $x = qp_1p_2 \dots p_m$ for $q \in \mathbb{N}$.*

Lemma 1. *[21] Let $a, b, c \in \mathbb{Z}_n$. It holds that $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in Nil\left(\begin{bmatrix} \mathbb{Z}_n & \mathbb{Z}_n \\ 0 & \mathbb{Z}_n \end{bmatrix}\right)$ if and only if $a, c \in Nil(\mathbb{Z}_n)$.*

Theorem 2. *[17] A connected graph is Eulerian precisely when all of its vertices possess an even degree.*

Theorem 3. *[18] If G is a graph of order $n \geq 3$ such that $deg(v) \geq \frac{n}{2}$, for each vertex v of G, then G is Hamiltonian*

3. Results and Discussion

In this section, we discuss $G_{NN}(T_2(\mathbb{Z}_{4p}))$ where p is a prime number and $p \geq 3$. Before discussing the non-nilpotent graph further, we redefine the non-nilpotent graph of matrix rings and state

an important theorem to be used in the discussion.

Definition 2. Let $T_2(\mathbb{Z}_{4p})$ be the ring of 2×2 upper triangular matrices over $\mathbb{Z}_{4p}$. The non-nilpotent graph $G_{NN}(T_2(\mathbb{Z}_{4p}))$ of $T_2(\mathbb{Z}_{4p})$ is an undirected simple graph with the vertex set $\overline{Nil(T_2(\mathbb{Z}_{4p}))} = T_2(\mathbb{Z}_{4p}) \setminus Nil(T_2(\mathbb{Z}_{4p}))$. Two distinct vertices $x, y \in \overline{Nil(T_2(\mathbb{Z}_{4p}))}$ are adjacent if and only if $x + y \in Nil(T_2(\mathbb{Z}_{4p}))$.

Theorem 4. $Nil(T_2(\mathbb{Z}_{4p})) = \left\{ \begin{bmatrix} xp & b \\ 0 & yp \end{bmatrix} \mid x, y \in \{0, 2\}, b \in \mathbb{Z}_{4p} \right\}$

Proof. Let $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in T_2(\mathbb{Z}_{4p})$. By Lemma 1, $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in Nil(T_2(\mathbb{Z}_{4p}))$ if and only if $a, c \in Nil(\mathbb{Z}_{4p})$. Thus, to obtain all elements of $Nil(T_2(\mathbb{Z}_{4p}))$, it is necessary to find all elements of $T_2(\mathbb{Z}_{4p})$ of the form $\begin{bmatrix} Nil(\mathbb{Z}_{4p}) & b \\ 0 & Nil(\mathbb{Z}_{4p}) \end{bmatrix}$. By Theorem 1, we obtain $Nil(\mathbb{Z}_{4p}) = \{0, 2p\}$, so there are $16p$ matrices that are nilpotent, namely $\begin{bmatrix} xp & b \\ 0 & yp \end{bmatrix}$ for $x, y \in \{0, 2\}$ and $b \in \mathbb{Z}_{4p}$. Therefore, we obtain $Nil(T_2(\mathbb{Z}_{4p})) = \left\{ \begin{bmatrix} xp & b \\ 0 & yp \end{bmatrix} \mid x, y \in \{0, 2\}, b \in \mathbb{Z}_{4p} \right\}$. □

Based on Theorem 4, two corollaries are obtained, namely the number of vertices in the graph and the condition for adjacent vertices in the graph $G_{NN}(T_2(\mathbb{Z}_{4p}))$.

Corollary 1. *Let $G_{NN}(T_2(\mathbb{Z}_{4p}))$ be the non-nilpotent graph of 2×2 upper triangular matrix rings over $\mathbb{Z}_{4p}$, then $|V(G_{NN}(T_2(\mathbb{Z}_{4p})))| = 64p^3 - 16p$.*

Proof. It is known that $T_2(\mathbb{Z}_{4p}) = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mid a, b, c \in \mathbb{Z}_{4p} \right\}$, so $|T_2(\mathbb{Z}_{4p})| = (4p)^3 = 64p^3$. By Theorem 4, we obtain $Nil(T_2(\mathbb{Z}_{4p})) = \left\{ \begin{bmatrix} xp & b \\ 0 & yp \end{bmatrix}, x, y \in \{0, 2\}, b \in \mathbb{Z}_{4p} \right\}$, so $|Nil(T_2(\mathbb{Z}_{4p}))| = 2^2 \cdot 4p = 16p$. By Definition 2, the vertex set of the graph consists of all non-nilpotent elements of the ring. Therefore, we obtain $|V(G_{NN}(T_2(\mathbb{Z}_{4p})))| = 64p^3 - 16p$. □

Corollary 2. *Let $\begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix}, \begin{bmatrix} a_2 & b_2 \\ 0 & c_2 \end{bmatrix} \in V(G_{NN}(T_2(\mathbb{Z}_{4p})))$. The vertices $\begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix}$ and $\begin{bmatrix} a_2 & b_2 \\ 0 & c_2 \end{bmatrix}$ are adjacent if and only if $a_1 = xp - a_2$ and $c_1 = yp - c_2$, where $x, y \in \{0, 2\}$ but $(a_1, c_1) \notin \{(xp, yp) \mid x, y \in \{0, 2\}\}$.*

Proof. ($\Rightarrow$) Let $\begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix}, \begin{bmatrix} a_2 & b_2 \\ 0 & c_2 \end{bmatrix} \in V(G_{NN}(T_2(\mathbb{Z}_{4p})))$ be adjacent. By Definition 2, the vertices $\begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix}$ and $\begin{bmatrix} a_2 & b_2 \\ 0 & c_2 \end{bmatrix}$ are adjacent when $\begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ 0 & c_2 \end{bmatrix} \in Nil(T_2(\mathbb{Z}_{4p}))$, that

is, $\begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ 0 & c_2 \end{bmatrix} = \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ 0 & c_1+c_2 \end{bmatrix} = \begin{bmatrix} xp & b \\ 0 & yp \end{bmatrix}, x, y \in \{0,2\}, b \in \mathbb{Z}_{4p}$. Based on the equality of these two matrices, it is obtained that if two vertices are adjacent, then $a_1 = xp - a_2$ and $c_1 = yp - c_2$, with $x, y \in \{0,2\}$ but $(a_1, c_1) \notin \{(xp, yp) \mid x, y \in \{0,2\}\}$.
($\Leftarrow$) Let $\begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix}, \begin{bmatrix} a_2 & b_2 \\ 0 & c_2 \end{bmatrix} \in V\left(G_{NN}\left(T_2(\mathbb{Z}_{4p})\right)\right)$ with $a_1 = xp - a_2$ and $c_1 = yp - c_2$, for $x, y \in \{0,2\}$ but $(a_1, c_1) \notin \{(xp, yp) \mid x, y \in \{0,2\}\}$. We obtain $\begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ 0 & c_2 \end{bmatrix} = \begin{bmatrix} xp-a_2 & b_1 \\ 0 & yp-c_2 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ 0 & c_2 \end{bmatrix} = \begin{bmatrix} xp & b_1+b_2 \\ 0 & yp \end{bmatrix}$. Since $\begin{bmatrix} xp & b_1+b_2 \\ 0 & yp \end{bmatrix} \in Nil\left(T_2(\mathbb{Z}_{4p})\right)$, the vertices $\begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix}$ and $\begin{bmatrix} a_2 & b_2 \\ 0 & c_2 \end{bmatrix}$ are adjacent. □

Before discussing the general form of $G_{NN}(T_2(\mathbb{Z}_{4p}))$, we establish the connectivity of the graph in the following lemma.

Lemma 2. *$G_{NN}(T_2(\mathbb{Z}_{4p}))$ is a disconnected graph.*

Proof. Let $A_1 = \begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix}$ and $A_2 = \begin{bmatrix} a_2 & b_2 \\ 0 & c_2 \end{bmatrix}$ be two distinct non-nilpotent vertices in $G_{NN}(T_2(\mathbb{Z}_{4p}))$. By Corollary 2, A_1 and A_2 are adjacent if and only if their sum is nilpotent,

$$A_1 + A_2 = \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ 0 & c_1+c_2 \end{bmatrix} \in \mathrm{Nil}(T_2(\mathbb{Z}_{4p})),$$

which requires the diagonal entries to satisfy

$$a_1 + a_2 \in \{0, 2p\} \quad \text{and} \quad c_1 + c_2 \in \{0, 2p\},$$

regardless of the free upper-right entries b_1 and b_2.
Specifically, for a given vertex $A_1 = \begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix}$, an arbitrary vertex $A_2 = \begin{bmatrix} a_2 & b_2 \\ 0 & c_2 \end{bmatrix}$ is connected to A_1 if and only if its diagonal entries are chosen from the additive inverse classes, namely $a_2 \in -a_1 + \{0, 2p\}$ and $c_2 \in -c_1 + \{0, 2p\}$ modulo $4p$. Conversely, if the diagonal entries of A_2 satisfy $a_2 \notin -a_1 + \{0, 2p\}$ or $c_2 \notin -c_1 + \{0, 2p\}$, their sum yields $a_1 + a_2 \notin \{0, 2p\}$ or $c_1 + c_2 \notin \{0, 2p\}$, which causes the sum matrix $A_1 + A_2$ to be non-nilpotent. To show that $G_{NN}(T_2(\mathbb{Z}_{4p}))$ is a disconnected graph, we take an arbitrary vertex $A_3 = \begin{bmatrix} a_3 & b_3 \\ 0 & c_3 \end{bmatrix} \in V(G_{NN}(T_2(\mathbb{Z}_{4p})))$ such that its diagonal entries do not satisfy the additive inverse conditions with respect to the vertices A_1 or A_2, namely $a_3 \notin -a_1 + \{0, 2p\}$, $a_3 \notin -a_2 + \{0, 2p\}$ or $c_3 \notin -c_1 + \{0, 2p\}$, $c_3 \notin -c_2 + \{0, 2p\}$. Based on the adjacency rules, vertex A_3 is only adjacent to vertices whose sums with A_3 yield nilpotent elements, which are not found in A_1 and A_2. Since no vertex in the component containing A_1 and A_2 satisfies this adjacency relation with A_3, no path can connect vertices outside the component containing A_1 and A_2. Therefore, it is proven that $G_{NN}(T_2(\mathbb{Z}_{4p}))$ is a disconnected graph. □

Building upon Lemma 2, the structural decomposition of the non-nilpotent graph $G_{NN}(T_2(\mathbb{Z}_{4p}))$ into complete and complete bipartite components is established in the following theorem.

Theorem 5. *The non-nilpotent graph $G_{NN}(T_2(\mathbb{Z}_{4p}))$ decomposes into three pairwise disjoint complete subgraphs isomorphic to K_{16p} and $2p^2 - 2$ pairwise disjoint complete bipartite subgraphs isomorphic to $K_{16p,16p}$.*

Proof. The vertex set of the non-nilpotent graph $G_{NN}(T_2(\mathbb{Z}_{4p}))$ consists of all non-nilpotent upper triangular matrices of the form:

$$A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix},$$

where $a, b, c \in \mathbb{Z}_{4p}$, with the condition that at least one of the diagonal entries is a non-nilpotent element in $\mathbb{Z}_{4p}$.

Two distinct vertices $A_1 = \begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix}$ and $A_2 = \begin{bmatrix} a_2 & b_2 \\ 0 & c_2 \end{bmatrix}$ are adjacent if and only if their sum is a nilpotent

$$A_1 + A_2 = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ 0 & c_1 + c_2 \end{bmatrix} \in \mathrm{Nil}(T_2(\mathbb{Z}_{4p})),$$

which occurs if and only if their diagonal entries satisfy the additive conditions:

$$a_1 + a_2 \in \{0, 2p\} \quad \text{and} \quad c_1 + c_2 \in \{0, 2p\},$$

regardless of the upper-right entries b_1 and b_2.

To partition all vertices, we group the diagonal pairs (a, c) using the equivalence classes generated by adding elements from $H = \{0, 2p\} \times \{0, 2p\}$. Each group of diagonal pairs $(a, c) + H$ generates a set of $16p$ matrices of the form

$$V_{(a,c)} = \left\{ \begin{bmatrix} a + h_1 & b \\ 0 & c + h_2 \end{bmatrix} \,\middle|\, h_1, h_2 \in \{0, 2p\},\ b \in \mathbb{Z}_{4p} \right\}.$$

Depending on how a vertex group $V_{(a,c)}$ interacts with its inverse group $V_{(-a,-c)}$, we classify the components into two types

1. **Case 1 : Complete Subgraph K_{16p}:** If a vertex group is equal to its inverse group, that is, $a \equiv -a$ and $c \equiv -c \pmod{2p}$, we obtain a self-inverse group, where $a, c \in \{0, p, 2p, 3p\}$. Excluding the nilpotent group where $a, c \in \{0, 2p\}$, there are exactly 3 non-nilpotent self-inverse groups.
   For any two distinct matrices $A_1 = \begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix}$ and $A_2 = \begin{bmatrix} a_2 & b_2 \\ 0 & c_2 \end{bmatrix}$ inside the same self-inverse group, their diagonal sums satisfy $a_1 + a_2 \in \{0, 2p\}$ and $c_1 + c_2 \in \{0, 2p\}$. Thus, their sum $A_1 + A_2$ is always nilpotent, meaning every pair of distinct matrices in this group is connected by an edge. Since each group contains $16p$ matrices, they form 3 disjoint complete components K_{16p}.
2. **Case 2 : Complete Bipartite Subgraph $K_{16p,16p}$:** The remaining non-nilpotent groups are not self-inverse and naturally form pairs of distinct groups, denoted as $V_{(a,c)}$ and $V_{(-a,-c)}$, where each group contains $16p$ matrices.
   For any two matrices chosen within the same group $V_{(a,c)}$, their diagonal sums do not fall into $\{0, 2p\}$, so no edges exist within $V_{(a,c)}$ nor within $V_{(-a,-c)}$. However, for any matrix $A_1 = \begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix} \in V_{(a,c)}$ and $A_2 = \begin{bmatrix} a_2 & b_2 \\ 0 & c_2 \end{bmatrix} \in V_{(-a,-c)}$, their diagonal sums satisfy $a_1 + a_2 \in \{0, 2p\}$ and $c_1 + c_2 \in \{0, 2p\}$, meaning every vertex in $V_{(a,c)}$ is

connected to every vertex in $V_{(-a,-c)}$ forming mutually disjoint complete bipartite subgraphs $K_{16p,16p}$. The number of vertices satisfying Case 2 is equal to the number of vertices that do not satisfy Case 1, namely $64p^3 - 16p - 3 \cdot 16p = 64p^3 - 64p$. Since each mutually disjoint complete bipartite subgraph $K_{16p,16p}$ of $G_{NN}(T_2(\mathbb{Z}_{4p}))$ has $32p$ vertices, the number of mutually disjoint complete bipartite subgraphs $K_{16p,16p}$ in $G_{NN}(T_2(\mathbb{Z}_{4p}))$ is $\frac{64p^3-64p}{32p} = 2p^2 - 2$.

Thus, the non-nilpotent graph $G_{NN}(T_2(\mathbb{Z}_{4p}))$ decomposes into three pairwise disjoint complete subgraphs isomorphic to K_{16p} and $2p^2 - 2$ pairwise disjoint complete bipartite subgraphs isomorphic to $K_{16p,16p}$. □

To illustrate the general form in Theorem 5, the components for the case $p = 3$, namely $G_{NN}(T_2(\mathbb{Z}_{12}))$, are illustrated in Figure 2 and Figure 3.

Example 2. To illustrate the components for the case $p = 3$, namely $G_{NN}(T_2(\mathbb{Z}_{12}))$, we can explicitly construct the elements of a complete component K_{48} and a complete bipartite component $K_{48,48}$ as follows:

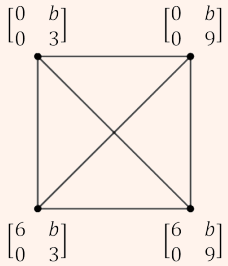


Fig. 2: Schematic representation of the complete subgraph K_{48}

The 48 matrix vertices arise from four subsets, each containing 12 matrices corresponding to $b \in \{0, 1, 2, \ldots, 11\}$:

$$S_1 = \left\{ \begin{bmatrix} 0 & b \\ 0 & 3 \end{bmatrix} \middle| \, b \in \mathbb{Z}_{12} \right\}, \quad S_2 = \left\{ \begin{bmatrix} 0 & b \\ 0 & 9 \end{bmatrix} \middle| \, b \in \mathbb{Z}_{12} \right\},$$
$$S_3 = \left\{ \begin{bmatrix} 6 & b \\ 0 & 3 \end{bmatrix} \middle| \, b \in \mathbb{Z}_{12} \right\}, \quad S_4 = \left\{ \begin{bmatrix} 6 & b \\ 0 & 9 \end{bmatrix} \middle| \, b \in \mathbb{Z}_{12} \right\}.$$

(totaling $4 \times 12 = 48$ matrices)

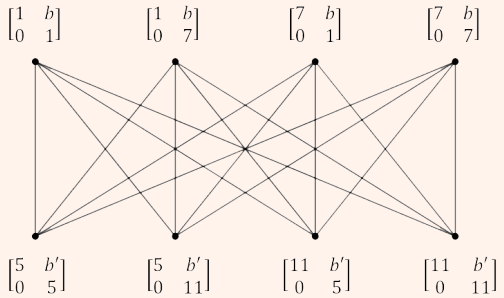


Fig. 3: Schematic representation of the complete bipartite subgraph $K_{48,48}$

The vertex set is partitioned into $V_1 \cup V_2$, where each partition contains 48 vertices divided into two subsets based on diagonal entries:

- Partition V_1

$$V_1 = \left\{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & b \\ 0 & 7 \end{bmatrix}, \begin{bmatrix} 7 & b \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 7 & b \\ 0 & 7 \end{bmatrix} \middle| \, b \in \mathbb{Z}_{12} \right\}$$

  (totaling $4 \times 12 = 48$ matrices).
- Partition V_2

$$V_2 = \left\{ \begin{bmatrix} 5 & b' \\ 0 & 5 \end{bmatrix}, \begin{bmatrix} 5 & b' \\ 0 & 11 \end{bmatrix}, \begin{bmatrix} 11 & b' \\ 0 & 5 \end{bmatrix}, \begin{bmatrix} 11 & b' \\ 0 & 11 \end{bmatrix} \middle| \, b' \in \mathbb{Z}_{12} \right\}$$

  (totaling $4 \times 12 = 48$ matrices).

The graph $G_{NN}(T_2(\mathbb{Z}_{12}))$ consists of two distinct types of pairwise disjoint connected components. The first components are 3 complete subgraphs K_{48}, where every vertex has a degree of $\deg(v) = 47$. The second components are $2p^2 - 2 = 2 \times 3^2 - 2 = 16$ complete bipartite subgraphs $K_{48,48}$, where the vertex set is partitioned into two sets V_1 and V_2 of equal size 48, yielding a uniform vertex degree of $\deg(v) = 48$.

Based on Lemma 2 and Theorem 5, the graph $G_{NN}(T_2(\mathbb{Z}_{4p}))$ is a disconnected graph and decomposes into three pairwise disjoint complete subgraphs isomorphic to K_{16p} and $2p^2 - 2$ pairwise disjoint complete bipartite subgraphs isomorphic to $K_{16p,16p}$, yielding the following results.

Corollary 3. *$G_{NN}(T_2(\mathbb{Z}_{4p}))$ is not an Eulerian graph and the components of $G_{NN}(T_2(\mathbb{Z}_{4p}))$ which are complete subgraphs K_{16p} are also not Eulerian graphs. However, the components of $G_{NN}(T_2(\mathbb{Z}_{4p}))$ which are complete bipartite subgraphs $K_{16p,16p}$ are Eulerian graphs.*

Proof. Since $G_{NN}(T_2(\mathbb{Z}_{4p}))$ is a disconnected graph, $G_{NN}(T_2(\mathbb{Z}_{4p}))$ is not an Eulerian graph. However, each component of $G_{NN}(T_2(\mathbb{Z}_{4p}))$ which is a complete subgraph K_{16p} and a complete bipartite subgraph $K_{16p,16p}$ can be observed. Since the degree of each vertex in the complete subgraph K_{16p} is $16p - 1$, which is an odd number, by Theorem 2, the components of $G_{NN}(T_2(\mathbb{Z}_{4p}))$ which are complete subgraphs K_{16p} are not Eulerian graphs. Meanwhile, the degree of each vertex in the complete bipartite subgraph $K_{16p,16p}$ is $16p$, which is an even number, so the components of $G_{NN}(T_2(\mathbb{Z}_{4p}))$ which are complete bipartite subgraphs $K_{16p,16p}$ are Eulerian graphs. □

Corollary 4. *$G_{NN}(T_2(\mathbb{Z}_{4p}))$ is not a Hamiltonian graph, but every component of $G_{NN}(T_2(\mathbb{Z}_{4p}))$ is a Hamiltonian graph.*

Proof. Since $G_{NN}(T_2(\mathbb{Z}_{4p}))$ is a disconnected graph, $G_{NN}(T_2(\mathbb{Z}_{4p}))$ is not a Hamiltonian graph. However, each component of $G_{NN}(T_2(\mathbb{Z}_{4p}))$ which is a complete subgraph K_{16p} and a complete bipartite subgraph $K_{16p,16p}$ can be observed. Every component of $G_{NN}(T_2(\mathbb{Z}_{4p}))$ is a Hamiltonian graph because it satisfies Theorem 3.
In the complete subgraph K_{16p}, the order of the subgraph is $n = 16p$ and the degree of each vertex is $\deg(v) = 16p - 1$ for every vertex v in the subgraph. This condition satisfies the inequality

$$\deg(v) = 16p - 1 \geq 8p = \frac{16p}{2} = \frac{n}{2}$$

since $16p - 1 \geq 8p$ holds for every prime number $p \geq 3$. By Theorem 3, the complete subgraph K_{16p} is a Hamiltonian graph.

Meanwhile, in the complete bipartite subgraph $K_{16p,16p}$, the order of the subgraph is $n = 32p$ and the degree of each vertex is $\deg(v) = 16p$. This degree satisfies

$$\deg(v) = 16p = \frac{32p}{2} = \frac{n}{2}$$

By Theorem 3, the complete bipartite subgraph $K_{16p,16p}$ is a Hamiltonian graph. □

Furthermore, the girth, clique number, diameter, and domination number of $G_{NN}(T_2(\mathbb{Z}_{4p}))$ will be discussed, which are respectively denoted by $gr(G_{NN}(T_2(\mathbb{Z}_{4p})))$, $\omega(G_{NN}(T_2(\mathbb{Z}_{4p})))$, $diam(G_{NN}(T_2(\mathbb{Z}_{4p})))$, and $\gamma(G_{NN}(T_2(\mathbb{Z}_{4p})))$. In this section, $G_{NNc}(T_2(\mathbb{Z}_{4p}))$ denotes a component of $G_{NN}(T_2(\mathbb{Z}_{4p}))$ in the form of a complete subgraph K_{16p}, and $G_{NNb}(T_2(\mathbb{Z}_{4p}))$ denotes a component of $G_{NN}(T_2(\mathbb{Z}_{4p}))$ in the form of a complete bipartite subgraph $K_{16p,16p}$.

Theorem 6. *Let $T_2(\mathbb{Z}_{4p})$ be the 2×2 upper triangular matrix rings over $\mathbb{Z}_{4p}$, then*

a. *(Girth) $gr(G_{NN}(T_2(\mathbb{Z}_{4p}))) = 3$*
b. *(Clique Number) $\omega(G_{NN}(T_2(\mathbb{Z}_{4p}))) = 16p$*
c. *(Diameter) $diam(G_{NN}(T_2(\mathbb{Z}_{4p}))) = \infty$, $diam(G_{NNc}(T_2(\mathbb{Z}_{4p}))) = 1$, and $diam(G_{NNb}(T_2(\mathbb{Z}_{4p}))) = 2$*
d. *(Domination Number) $\gamma(G_{NN}(T_2(\mathbb{Z}_{4p}))) = 4p^2 - 1$*

Proof. a. (Girth) $gr(G_{NN}(T_2(\mathbb{Z}_{4p}))) = 3$

To prove this Theorem, it is necessary to examine each component of the graph $G_{NN}(T_2(\mathbb{Z}_{4p}))$ to determine the length of the shortest cycle formed. Based on Theorem 5, the graph $G_{NN}(T_2(\mathbb{Z}_{4p}))$ has components in the form of complete subgraphs K_{16p} and complete bipartite subgraphs $K_{16p,16p}$.

(a) **Complete subgraph K_{16p}**

In $G_{NNc}(T_2(\mathbb{Z}_{4p}))$, the existing vertices are of the form $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ with $(a, c) \in \{(xp, yp - p), (xp - p, yp), (xp - p, yp - p)\}$, so that $\forall \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in G_{NNc}(T_2(\mathbb{Z}_{4p}))$, $\begin{bmatrix} a & b_i \\ 0 & c \end{bmatrix}$ is adjacent to $\begin{bmatrix} a & b_j \\ 0 & c \end{bmatrix}$, $i, j \in \{1, 2, \ldots, 4p\}$ and $i \neq j$. Therefore, a cycle of length 3 can be found, namely

$$\begin{bmatrix} a & b_1 \\ 0 & c \end{bmatrix} \rightarrow \begin{bmatrix} a & b_2 \\ 0 & c \end{bmatrix} \rightarrow \begin{bmatrix} a & b_3 \\ 0 & c \end{bmatrix} \rightarrow \begin{bmatrix} a & b_1 \\ 0 & c \end{bmatrix}$$

Thus, the length of the shortest cycle of $G_{NNc}(T_2(\mathbb{Z}_{4p}))$ is 3.

(b) **Complete bipartite subgraph $K_{16p,16p}$**

Since $K_{16p,16p}$ is a complete bipartite graph, the formed cycle must alternate between each partition, namely $\left\{ \begin{bmatrix} xp + a & b_i \\ 0 & yp + c \end{bmatrix} \mid i = 1, 2, \ldots, 4p \text{ and } x, y \in \{0, 2\} \right\}$ and $\left\{ \begin{bmatrix} xp - a & b_j \\ 0 & yp - c \end{bmatrix} \mid j = 1, 2, \ldots, 4p \text{ and } x, y \in \{0, 2\} \right\}$. Therefore, a cycle

of length 4 can be found, namely

$$\begin{bmatrix} xp+a & b_1 \\ 0 & yp+c \end{bmatrix} \rightarrow \begin{bmatrix} xp-a & b_1 \\ 0 & yp-c \end{bmatrix} \rightarrow \begin{bmatrix} xp+a & b_2 \\ 0 & yp+c \end{bmatrix} \rightarrow \begin{bmatrix} xp-a & b_2 \\ 0 & yp-c \end{bmatrix} \rightarrow \begin{bmatrix} xp+a & b_1 \\ 0 & yp+c \end{bmatrix}$$

b. (Clique Number) $\omega\left(G_{NN}\left(T_2(\mathbb{Z}_{4p})\right)\right) = 16p$
To prove this Theorem, we need to examine each component of the graph $G_{NN}\left(T_2(\mathbb{Z}_{4p})\right)$ to find the maximum clique of each component. Based on Theorem 5, the graph $G_{NN}\left(T_2(\mathbb{Z}_{4p})\right)$ has components in the form of complete subgraphs K_{16p} and complete bipartite subgraphs $K_{16p,16p}$.
   (a) **Complete subgraph** K_{16p}
   Since K_{16p} is a complete subgraph with $16p$ vertices, the maximum clique in $G_{NNc}\left(T_2(\mathbb{Z}_{4p})\right)$ is $16p$.
   (b) **Complete bipartite subgraph** $K_{16p,16p}$
   Suppose the clique of $G_{NNb}\left(T_2(\mathbb{Z}_{4p})\right) \geq 3$, then there exists a cycle of length 3, which is the shortest cycle that must be formed. Based on Proof girth on Complete bipartite subgraph $K_{16p,16p}$, the shortest cycle that can be formed has length 4. Since there is a contradiction with the assumption, namely there is no cycle of length 3, the assumption is false. Therefore, the maximum clique of $G_{NNb}\left(T_2(\mathbb{Z}_{4p})\right) < 3$. Consequently, the maximum clique that can be formed in $G_{NNb}\left(T_2(\mathbb{Z}_{4p})\right)$ consists of two vertices, namely the vertices $\begin{bmatrix} xp+a & b \\ 0 & yp+c \end{bmatrix}$ and $\begin{bmatrix} xp-a & b \\ 0 & yp-c \end{bmatrix}$ with $x, y \in \{0, 2\}$.

   Based on the observation of the complete subgraph K_{16p} and the complete bipartite subgraph $K_{16p,16p}$, it is obtained that $16p > 2$ for p a prime number, so $\omega\left(G_{NN}\left(T_2(\mathbb{Z}_{4p})\right)\right) = 16p$.

c. (Diameter) $diam\left(G_{NN}\left(T_2(\mathbb{Z}_{4p})\right)\right) = \infty$, $diam\left(G_{NNc}\left(T_2(\mathbb{Z}_{4p})\right)\right) = 1$, and $diam\left(G_{NNb}\left(T_2(\mathbb{Z}_{4p})\right)\right) = 2$
   Based on Lemma 2, $G_{NN}\left(T_2(\mathbb{Z}_{4p})\right)$ is a disconnected graph, so $diam\left(G_{NN}\left(T_2(\mathbb{Z}_{4p})\right)\right) = \infty$. Furthermore, the diameter of each of its components can be discussed.
   (a) **Complete subgraph** K_{16p}
   Since $G_{NNc}\left(T_2(\mathbb{Z}_{4p})\right)$ is a complete subgraph K_{16p}, the distance from every vertex to any other vertex has the same length, which is 1, so $diam\left(G_{NNc}\left(T_2(\mathbb{Z}_{4p})\right)\right) = 1$.
   (b) **Complete bipartite subgraph** $K_{16p,16p}$
   Notice that each $G_{NNb}\left(T_2(\mathbb{Z}_{4p})\right)$ is a complete bipartite subgraph $K_{16p,16p}$, so $V\left(G_{NNb}\left(T_2(\mathbb{Z}_{4p})\right)\right)$ can be partitioned into 2 vertex sets, namely X and Y. Since in $G_{NNb}\left(T_2(\mathbb{Z}_{4p})\right)$ two adjacent vertices are of the form $\begin{bmatrix} xp+a & b \\ 0 & yp+c \end{bmatrix}$ and $\begin{bmatrix} xp-a & b \\ 0 & yp-c \end{bmatrix}$ with $x, y \in \{0, 2\}$, it can be written that $\begin{bmatrix} xp+a & b_i \\ 0 & yp+c \end{bmatrix} \in X$ and $\begin{bmatrix} xp-a & b_j \\ 0 & yp-c \end{bmatrix} \in Y$ with $i, j \in \{1, 2, \ldots, 4p\}$. From this information, the length of the shortest path from each vertex to another vertex can be observed.

i. *From a vertex in X to a vertex in Y*
The shortest path formed is

$$\begin{bmatrix} xp+a & b_i \\ 0 & yp+c \end{bmatrix} \rightarrow \begin{bmatrix} xp-a & b_j \\ 0 & yp-c \end{bmatrix}$$

It is obtained that the length of the shortest path is 1.

ii. *From a vertex in Y to a vertex in X*
The shortest path formed is

$$\begin{bmatrix} xp-a & b_j \\ 0 & yp-c \end{bmatrix} \rightarrow \begin{bmatrix} xp+a & b_i \\ 0 & yp+c \end{bmatrix}$$

It is obtained that the length of the shortest path is 1.

iii. *From a vertex in X to another vertex in X*
The shortest path formed is

$$\begin{bmatrix} xp+a & b_i \\ 0 & yp+c \end{bmatrix} \rightarrow \begin{bmatrix} xp-a & b_j \\ 0 & yp-c \end{bmatrix} \rightarrow \begin{bmatrix} xp+a & b_k \\ 0 & yp+c \end{bmatrix}, \quad i \neq k$$

It is obtained that the length of the shortest path is 2.

iv. *From a vertex in Y to another vertex in Y*
The shortest path formed is

$$\begin{bmatrix} xp-a & b_i \\ 0 & yp-c \end{bmatrix} \rightarrow \begin{bmatrix} xp+a & b_j \\ 0 & yp+c \end{bmatrix} \rightarrow \begin{bmatrix} xp-a & b_k \\ 0 & yp-c \end{bmatrix}, \quad i \neq k$$

It is obtained that the length of the shortest path is 2.

From those four cases, it can be observed that the maximum length of every shortest path is 2, so $diam\left(G_{NNb}\left(T_2(\mathbb{Z}_{4p})\right)\right) = 2$.

d. (Domination Number) $\gamma\left(G_{NN}\left(T_2(\mathbb{Z}_{4p})\right)\right) = 4p^2 - 1$
To prove this Theorem, it is necessary to examine each component of the graph $G_{NN}\left(T_2(\mathbb{Z}_{4p})\right)$ to determine its domination number. Based on Theorem 5, the graph $G_{NN}\left(T_2(\mathbb{Z}_{4p})\right)$ has components in the form of pairwise disjoint complete subgraphs K_{16p} and complete bipartite subgraphs $K_{16p,16p}$.

(a) **Complete subgraph K_{16p}**

In $G_{NN}\left(T_2(\mathbb{Z}_{4p})\right)$, the existing vertices are of the form $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ with $(a,c) \in \{(xp, yp-p), (xp-p, yp), (xp-p, yp-p)\}$, so that $\forall \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in G_{NNc}\left(T_2(\mathbb{Z}_{4p})\right)$, $\begin{bmatrix} a & b_i \\ 0 & c \end{bmatrix}$ is adjacent to $\begin{bmatrix} a & b_j \\ 0 & c \end{bmatrix}$, $i, j \in \{1, 2, \ldots, 4p\}$ and $i \neq j$. In other words, $\begin{bmatrix} a & b_1 \\ 0 & c \end{bmatrix}$ will be adjacent to $\begin{bmatrix} a & b_2 \\ 0 & c \end{bmatrix}, \begin{bmatrix} a & b_3 \\ 0 & c \end{bmatrix}, \ldots, \begin{bmatrix} a & b_{2^n p} \\ 0 & c \end{bmatrix}$. Therefore, the vertex $\begin{bmatrix} a & b_1 \\ 0 & c \end{bmatrix}$ will dominate $G_{NNc}\left(T_2(\mathbb{Z}_{4p})\right)$. Since based on Theorem 5 there are 3 components of $G_{NNc}\left(T_2(\mathbb{Z}_{4p})\right)$, then $\gamma\left(G_{NNc}\left(T_2(\mathbb{Z}_{4p})\right)\right) = 3$.

(b) **Complete bipartite subgraph $K_{16p,16p}$**
Since $G_{NNb}\left(T_2(\mathbb{Z}_{4p})\right)$ is a complete bipartite subgraph $K_{16p,16p}$,

$V(G_{NNb}(T_2(\mathbb{Z}_{4p})))$ can be partitioned into 2 vertex sets, namely X and Y. Since in $G_{NNb}(T_2(\mathbb{Z}_{4p}))$ two adjacent vertices are of the form $\begin{bmatrix} xp+a & b \\ 0 & yp+c \end{bmatrix}$ and $\begin{bmatrix} xp-a & b \\ 0 & yp-c \end{bmatrix}$ with $x, y \in \{0, 2\}$, it can be written that $\begin{bmatrix} xp+a & b_i \\ 0 & yp+c \end{bmatrix} \in X$ and $\begin{bmatrix} xp-a & b_j \\ 0 & yp-c \end{bmatrix} \in Y$ with $i, j \in \{1, 2, \ldots, 4p\}$. It is obtained that $\begin{bmatrix} xp+a & b_1 \\ 0 & yp+c \end{bmatrix} \in X$ will be adjacent to all vertices in Y, namely $\begin{bmatrix} xp-a & b_1 \\ 0 & yp-c \end{bmatrix}, \begin{bmatrix} xp-a & b_2 \\ 0 & yp-c \end{bmatrix}, \ldots, \begin{bmatrix} xp-a & b_{4p} \\ 0 & yp-c \end{bmatrix}$. Similarly, $\begin{bmatrix} xp-a & b_1 \\ 0 & yp-c \end{bmatrix} \in Y$ will be adjacent to all vertices in X, namely $\begin{bmatrix} xp+a & b_1 \\ 0 & yp+c \end{bmatrix}, \begin{bmatrix} xp+a & b_2 \\ 0 & yp+c \end{bmatrix}, \ldots, \begin{bmatrix} xp+a & b_{4p} \\ 0 & yp+c \end{bmatrix}$. Therefore, the vertices $\begin{bmatrix} xp+a & b_1 \\ 0 & yp+c \end{bmatrix}$ and $\begin{bmatrix} xp-a & b_1 \\ 0 & yp-c \end{bmatrix}$ will dominate $G_{NNb}(T_2(\mathbb{Z}_{4p}))$. Since based on Theorem 5 there are $2p^2-2$ components of $G_{NNb}(T_2(\mathbb{Z}_{4p}))$, then $\gamma(G_{NNb}(T_2(\mathbb{Z}_{4p}))) = 2 \cdot (2p^2-2) = 4p^2-4$.

Based on the domination number of each component of $G_{NN}(T_2(\mathbb{Z}_{4p}))$, it is obtained that $\gamma(G_{NN}(T_2(\mathbb{Z}_{4p}))) = \gamma(G_{NNc}(T_2(\mathbb{Z}_{4p}))) + \gamma(G_{NNb}(T_2(\mathbb{Z}_{4p}))) = 3 + 4p^2 - 4 = 4p^2 - 1$.

□

Example 3. To illustrate and verify the structural results in Theorem 5 and Theorem 6, consider the non-nilpotent graph $G_{NN}(T_2(\mathbb{Z}_{4p}))$, where $p = 3$. The ring $R = T_2(\mathbb{Z}_{12})$ contains $|T_2(\mathbb{Z}_{12})| = 12^3 = 1728$ elements. The set of nilpotent elements $\mathrm{Nil}(T_2(\mathbb{Z}_{12}))$ consists of $16p = 16(3) = 48$ matrices. Thus, the order of the graph is $|V(G_{NN}(T_2(\mathbb{Z}_{12})))| = 1728 - 48 = 1680$ vertices.
The components of $G_{NN}(T_2(\mathbb{Z}_{12}))$ are as follows:

1. **Complete Subgraph Component (K_{48}):** There are 3 pairwise disjoint complete components isomorphic to K_{48}. Consider the vertices of one of the components $G_{NNc}(T_2(\mathbb{Z}_{12}))$

$$V(G_{NNc}(T_2(\mathbb{Z}_{12}))) = \left\{ \begin{bmatrix} x & b \\ 0 & y \end{bmatrix} \;\middle|\; x \in \{0, 6\}, y \in \{3, 9\}, b \in \mathbb{Z}_{12} \right\}.$$

   Since $x \in \{0, 6\}$, $y \in \{3, 9\}$, and $b \in \mathbb{Z}_{12}$, we have $|V(G_{NNc}(T_2(\mathbb{Z}_{12})))| = 2 \times 2 \times 12 = 48$ vertices. Every vertex $v \in V(G_{NNc}(T_2(\mathbb{Z}_{12})))$ is adjacent to all other 47 vertices in $V(G_{NNc}(T_2(\mathbb{Z}_{12})))$, yielding $\deg(v) = 47$.
2. **Complete Bipartite Subgraph Component ($K_{48,48}$):** There are $2(3^2) - 2 = 16$ pairwise disjoint complete bipartite components isomorphic to $K_{48,48}$. Consider the vertices of one of the components $G_{NNb}(T_2(\mathbb{Z}_{12}))$

$$V_1 = \left\{ \begin{bmatrix} x & b \\ 0 & y \end{bmatrix} \;\middle|\; x, y \in \{1, 7\}, b \in \mathbb{Z}_{12} \right\},$$

$$V_2 = \left\{ \begin{bmatrix} x' & b' \\ 0 & y' \end{bmatrix} \;\middle|\; x', y' \in \{5, 11\}, b' \in \mathbb{Z}_{12} \right\}.$$

Each partition contains $2 \times 2 \times 12 = 48$ vertices. No edges exist within V_1 or V_2, but every vertex in V_1 is adjacent to every vertex in V_2. Thus, the degree of every vertex in $G_{NNb}(T_2(\mathbb{Z}_{12}))$ is $\deg(v) = 48$.

3. **Graph Invariants:**
   - **Total Order:** $3(48) + 16(96) = 144 + 1536 = 1680$.
   - **Girth:** $gr(G_{NN}(T_2(\mathbb{Z}_{12}))) = 3$.
   - **Clique Number:** $\omega(G_{NN}(T_2(\mathbb{Z}_{12}))) = 48$.
   - **Diameter:** $diam(G_{NN}(T_2(\mathbb{Z}_{12}))) = \infty$, with $diam(G_{NNc}(T_2(\mathbb{Z}_{12}))) = 1$ and $diam(G_{NNb}(T_2(\mathbb{Z}_{12}))) = 2$.
   - **Domination Number:** $\gamma(G_{NN}(T_2(\mathbb{Z}_{12}))) = 3(1) + (2(3^2) - 2)(2) = 3 + 16(2) = 35 = 4(3^2) - 1$.

4. Conclusion

In this research, the structural characteristics of the non-nilpotent graph $G_{NN}(T_2(\mathbb{Z}_{4p}))$ are investigated. Specifically, $G_{NN}(T_2(\mathbb{Z}_{4p}))$ decomposes into 3 pairwise disjoint complete subgraphs isomorphic to K_{16p} and $2p^2 - 2$ pairwise disjoint complete bipartite subgraphs isomorphic to $K_{16p,16p}$. Furthermore, the properties of $G_{NN}(T_2(\mathbb{Z}_{4p}))$ are summarized as follows:

- $gr\left(G_{NN}\left(T_2(\mathbb{Z}_{4p})\right)\right) = 3$
- $\omega\left(G_{NN}\left(T_2(\mathbb{Z}_{4p})\right)\right) = 16p$
- $diam\left(G_{NN}\left(T_2(\mathbb{Z}_{4p})\right)\right) = \infty$, $diam\left(G_{NNc}\left(T_2(\mathbb{Z}_{4p})\right)\right) = 1$, and $diam\left(G_{NNb}\left(T_2(\mathbb{Z}_{4p})\right)\right) = 2$
- $\gamma\left(G_{NN}\left(T_2(\mathbb{Z}_{4p})\right)\right) = 4p^2 - 1$

CRediT Authorship Contribution Statement

Akmal Bachtiar Ismail Huda: Conceptualization, Methodology, Formal analysis, Writing–Original Draft. **Darmajid:** Supervision, Validation, Writing–Review & Editing. **Noor Hidayat:** Supervision, Validation, Writing–Review & Editing. All authors have read and approved the final version of the manuscript.

Declaration of Generative AI and AI-assisted technologies

No generative AI and AI-assisted technologies were used during the preparation of this manuscript.

Declaration of Competing Interest

The authors declare no competing interests.

Ethics Approval and Consent to Participate

Ethics approval was not required for this study.

Funding and Acknowledgments

This research received no external funding.

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