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Rainbow Antimagic Coloring of Closed Sunflower Graphs and Their Multilayer Extensions

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Abstract

Rainbow antimagic coloring combines a bijective vertex labeling with rainbow connectivity, so repeated induced edge weights must be controlled without obstructing rainbow paths. This paper studies the rainbow antimagic connection number of closed sunflower graphs and introduces a multilayer closed sunflower family whose two-layer case recovers the original graph. Lower bounds are obtained from vertex degrees, distances, and regular induced subgraphs, while constructive upper bounds are established through explicit bijective labelings and rainbow-path arguments. For the closed sunflower graph CSF_n, we obtain $\max\{n, 7\} \leq \mathrm{rac}(\mathrm{CSF}_n) \leq \left\lceil \frac{5n+4}{2} \right\rceil$, and determine the smallest case exactly as $\mathrm{rac}(\mathrm{CSF}_3) = 7$. For $n \geq 10$ with $n \equiv 2 \pmod 4$, a refined construction improves the upper bound to $n+2$, leaving a gap of at most two from the lower bound. For the multilayer family $\mathrm{MCSF}_{n,k}$, an alternating labeling across successive layers yields $\max\{n, k, 7\} \leq \mathrm{rac}(\mathrm{MCSF}_{n,k}) \leq k(n+2) + \left\lfloor \frac{n-3}{2} \right\rfloor$. The multilayer construction extends the problem beyond the two-layer case by introducing additional cyclic layers and consecutive interlayer interfaces. Except for CSF_3 and the isomorphic case $\mathrm{MCSF}_{3,2}$, the results provide lower and constructive upper bounds rather than exact characterizations.

Keywords: Rainbow antimagic coloring; rainbow antimagic connection number; closed sunflower graph; multilayer graph; vertex labeling.

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1. Introduction

Graph coloring problems are determined not only by the number of colors used, but also by the restrictions imposed on where colors may repeat. Proper colorings require adjacent elements to receive distinct colors, whereas path-based colorings impose distinctness conditions along selected connecting paths [1, 2]. Related coloring parameters on directed graphs, such as the diachromatic and digrundy numbers, further illustrate how different structural restrictions lead to different coloring problems [3, 4]. Studies on particular graph families provide useful settings in which these structural effects can be examined explicitly [5, 6].

Rainbow connection is one such path-based coloring problem. Introduced by Chartrand et al. [2], it requires an edge coloring in which every pair of vertices is joined by a path whose edges have pairwise distinct colors. Unlike proper edge coloring, adjacent edges are allowed to share a color as long as suitable rainbow paths remain available. Subsequent studies have

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considered rainbow k-connection on special graph families, rainbow connection on planter and octopus graphs, and computational aspects of rainbow coloring [7–9]. These results show that the structure of the available paths plays a central role in determining how colors can be reused.

A different source of coloring restrictions arises from graph labeling. Antimagic labeling was introduced by Hartsfield and Ringel [10], and several related labeling-based colorings have subsequently been developed. In local antimagic vertex coloring, induced vertex weights are required to distinguish adjacent vertices [11]. In local edge antimagic coloring, a bijection $f : V(G) \longrightarrow \{1, 2, \ldots, |V(G)|\}$ assigns to each edge $uv \in E(G)$ the induced weight $w_f(uv) = f(u) + f(v)$, and adjacent edges are required to receive distinct induced weights [12]. Unlike an ordinary edge coloring, the edge colors in this setting are determined by the vertex labels and therefore cannot be chosen independently.

Rainbow antimagic coloring combines this endpoint-sum construction with rainbow connectivity. Early studies were given by Kusumawardani et al., Sulistiyono et al., and Dafik et al. [13–15]. A bijective vertex labeling is called rainbow antimagic when the induced edge coloring is rainbow connected, and the minimum number of distinct induced edge colors over all such labelings is the rainbow antimagic connection number, denoted by $\mathrm{rac}(G)$ [15]. Further results have been obtained for lollipop, stacked book, Dutch windmill, comb-product, and corona-product graph families [16–19]. Rainbow vertex antimagic coloring has also been studied as a related labeling problem, although its induced weights and rainbow condition are defined differently [20].

An earlier study on the same underlying graph family considers a different notion of rainbow connectivity. Hasan et al. studied the total rainbow connection number of closed sunflower graphs as part of their investigation of wheel-related graph families [21]. Their work concerns the same underlying closed sunflower structure, but the coloring mechanism is fundamentally different. In a total rainbow coloring, colors are assigned directly to both vertices and edges, and the edges together with the internal vertices of a required connecting path must receive mutually distinct colors. In rainbow antimagic coloring, by contrast, the vertex values are prescribed by a bijection and every edge color is forced by the endpoint sum $f(u) + f(v)$. Consequently, a total rainbow coloring of CSF_n need not be induced by any rainbow antimagic labeling, and results for the total rainbow connection number do not determine $\mathrm{rac}(\mathrm{CSF}_n)$.

Related rainbow antimagic studies have also considered sunflower and other wheel-related graph families. Dafik et al. investigated sunflower graphs together with several special graphs [22], while later work considered sunlet, helm, closed helm, and web graphs [23]. The sunflower graph occurring in [22] contains pendant vertices and is structurally different from the closed sunflower graph considered here. Hence the existing rainbow antimagic results for sunflower-type graphs cannot be transferred directly to CSF_n. The closed sunflower structure contains two cyclic layers linked by vertical and diagonal edges, together with a central vertex adjacent to the inner cycle, and these additional connections substantially change both the available rainbow paths and the endpoint-sum structure.

This paper studies the rainbow antimagic connection number of the closed sunflower graph CSF_n and considers a multilayer extension denoted by $\mathrm{MCSF}_{n,k}$. The case $k = 2$ recovers CSF_n, whereas $k > 2$ introduces additional cyclic layers and $k - 1$ consecutive vertical–diagonal interfaces. The multilayer case therefore introduces features that do not occur in the known two-layer graph. In particular, the distance from the central vertex to the outermost layer increases with k, and the induced edge weights must be controlled simultaneously across several consecutive interfaces. Thus the contribution for $\mathrm{MCSF}_{n,k}$ is not merely a reformulation of the case $k = 2$, but a construction and analysis valid for an arbitrary number of layers.

The main objective is to determine constructive lower and upper bounds for the rainbow antimagic connection numbers of these graph families. Lower bounds are obtained from vertex degrees, graph distances, and regular induced subgraphs, together with the endpoint-sum bound of Haslegrave [24]. Upper bounds are established through explicit bijective vertex labelings whose induced edge weights are counted and whose rainbow connectivity is verified. For CSF_n, a

general construction is complemented by a sharper construction for $n \geq 10$ with $n \equiv 2 \pmod 4$, where the gap between the lower and upper bounds is reduced to two colors. For $\mathrm{MCSF}_{n,k}$, an alternating labeling across successive layers yields bounds valid for every $n \geq 3$ and $k \geq 2$ and recovers the general closed sunflower bounds when $k = 2$. Except for the smallest closed sunflower graph, whose rainbow antimagic connection number is determined exactly, the results are presented as constructive bounds rather than exact characterizations.

2. Methods

This study uses a constructive and proof-based approach to investigate the rainbow antimagic connection number of closed sunflower graphs and their multilayer extensions. Unless otherwise stated, all graphs considered are finite, simple, and undirected, while graphs discussed in the context of rainbow connectivity are assumed to be connected. Standard graph-theoretic terminology follows [1]. For a graph G, its vertex set, edge set, maximum degree, and diameter are denoted by $V(G)$, $E(G)$, $\Delta(G)$, and $\mathrm{diam}(G)$, respectively.

The analysis consists of two complementary parts. The first establishes lower bounds from structural properties of the graph. We use $\mathrm{rac}(G) \geq \max\{\mathrm{rc}(G), \Delta(G)\}$, together with degree and distance information [14]. Additional lower bounds are obtained by restricting a vertex labeling to suitable regular induced subgraphs and applying the endpoint-sum bound of Haslegrave [24]. Since the restriction of a bijective labeling to an induced subgraph remains injective, this argument allows the internal structure of the selected subgraph to impose a lower bound on the number of induced edge colors of the original graph.

The second part constructs explicit bijective vertex labelings to obtain upper bounds. For a graph of order N, a bijection $f : V(G) \longrightarrow \{1, 2, \ldots, N\}$ induces the edge weight $w_f(uv) = f(u) + f(v)$ for every $uv \in E(G)$. The induced weights are determined separately for the different edge families and then organized into sets in order to identify repeated values and count the resulting number of distinct edge colors. A labeling is accepted as a rainbow antimagic labeling only after the induced edge coloring has been shown to be rainbow connected.

Two different mechanisms are used to verify rainbow connectivity. For the general constructions, we exhibit a spanning tree whose edges have pairwise distinct induced weights. The unique path between any two vertices in such a tree is therefore rainbow, so the existence of this spanning tree is sufficient to establish rainbow connectivity [7, 15]. This criterion is used only as a sufficient condition and is not part of the definition of rainbow antimagic coloring.

For the improved construction on CSF_n with $n \geq 10$ and $n \equiv 2 \pmod 4$, repeated induced weights are deliberately introduced in order to reduce the number of colors. A rainbow spanning tree is therefore no longer used. Instead, the construction is verified in three steps. First, the cyclic sequence used in the labeling is shown to contain every required element exactly once. Second, all induced edge weights are shown to belong to a prescribed normalized color set, from which the number of distinct colors is determined. Third, rainbow connectivity is established directly by constructing rainbow paths for every possible type of vertex pair, including the cyclic wrap-around cases.

The procedure is first developed for the closed sunflower graph CSF_n and then extended to the multilayer closed sunflower graph $\mathrm{MCSF}_{n,k}$. In the multilayer case, the vertex labels are assigned in alternating order on successive layers so that endpoint sums can be controlled across consecutive vertical and diagonal interfaces. The case $k = 2$ is used as a consistency check, since $\mathrm{MCSF}_{n,2} \cong \mathrm{CSF}_n$. The smallest closed sunflower graph is considered separately to obtain an exact value, whereas the remaining results provide general lower and constructive upper bounds.

3. Results and Discussion

We begin by fixing the graph-theoretic terminology, graph constructions, and supporting results used in the main arguments. Throughout this section, all graphs are finite, simple, and undirected.

Graphs considered in connection with rainbow coloring are assumed to be connected and to have at least two vertices.

Definition 1. [1] A simple graph is a pair $G = (V(G), E(G))$, where $V(G)$ is a nonempty vertex set and $E(G)$ is a set of unordered pairs of distinct vertices. We write uv for the edge $\{u, v\}$. Two vertices are adjacent if they form an edge, while two distinct edges are adjacent if they share a common endpoint. The order and size of G are $|V(G)|$ and $|E(G)|$, respectively. The degree of a vertex v is denoted by $d_G(v)$, and the maximum degree of G is $\Delta(G) = \max_{v \in V(G)} d_G(v)$. The graph G is called d-regular if every vertex has degree d.

Definition 2. [1] A path is a sequence of distinct vertices $v_0, v_1, \ldots, v_\ell$ such that $v_{j-1}v_j \in E(G)$ for $1 \leq j \leq \ell$. Its length is ℓ. A graph is connected if every pair of distinct vertices is joined by a path. For a connected graph, the distance between vertices u and v, denoted by $d_G(u, v)$, is the minimum length of a u–v path. The diameter of G is $\operatorname{diam}(G) = \max_{u,v \in V(G)} d_G(u, v)$. For $n \geq 3$, the cycle C_n has vertices $v_1, v_2, \ldots, v_n$ and edges $v_i v_{i+1}$ for $1 \leq i \leq n$, where $v_{n+1} = v_1$.

Definition 3. [1] A subgraph H of G satisfies $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. It is called a spanning subgraph if $V(H) = V(G)$. For a nonempty set $S \subseteq V(G)$, the induced subgraph $G[S]$ has vertex set S and contains exactly those edges of G whose endpoints both belong to S. A tree is a connected graph containing no cycle. A spanning tree of G is a spanning subgraph of G that is a tree.

Definition 4. [2] An edge coloring of a connected graph G is a map $c : E(G) \longrightarrow \{1, 2, \ldots, s\}$ for some positive integer s. The coloring is proper if adjacent edges receive distinct colors. A path is called a rainbow path if its edges have pairwise distinct colors. The coloring is rainbow connected if every pair of distinct vertices is joined by a rainbow path. The minimum number of colors required to make G rainbow connected is called the rainbow connection number and is denoted by $\operatorname{rc}(G)$. Properness is not required in this minimum.

Definition 5. [12] Let G be a graph of order N. A bijective vertex labeling $f : V(G) \longrightarrow \{1, 2, \ldots, N\}$ assigns to each edge $uv \in E(G)$ the induced weight $w_f(uv) = f(u) + f(v)$. The labeling f is called a local edge antimagic labeling if adjacent edges have distinct induced weights. Assigning each edge its induced weight as its color therefore produces a proper edge coloring. The set of distinct induced edge colors is denoted by $\mathcal{C}_f = \{w_f(uv) : uv \in E(G)\}$.

Definition 6. [15] A bijective vertex labeling f is called a rainbow antimagic labeling if the induced edge coloring $uv \longmapsto w_f(uv)$ is rainbow connected. The resulting coloring is called a rainbow antimagic coloring. The rainbow antimagic connection number of G is $\operatorname{rac}(G) = \min_f |\mathcal{C}_f|$, where the minimum is taken over all rainbow antimagic labelings of G. Repeated induced weights on different edges are therefore permitted, provided that every pair of distinct vertices is joined by a rainbow path.

The following graph constructions fix the notation and the orientation of the diagonal connections used throughout the paper. Subscripts on cycle vertices are interpreted modulo n, whereas layer indices are not interpreted cyclically.

Definition 7. [21, 25] For an integer $n \geq 3$, the closed sunflower graph CSF_n has vertex set $V(\mathrm{CSF}_n) = \{a\} \cup \{u_i, v_i : 1 \leq i \leq n\}$ and edge set $E(\mathrm{CSF}_n) = \{au_i : 1 \leq i \leq n\} \cup \{u_iu_{i+1}, v_iv_{i+1} : 1 \leq i \leq n\} \cup \{u_iv_i, u_{i+1}v_i : 1 \leq i \leq n\}$, where $u_{n+1} = u_1$ and $v_{n+1} = v_1$. The vertices u_i and v_i form the inner and outer cycles, respectively, while a is the central vertex. The edges au_i, u_iu_{i+1} and v_iv_{i+1}, u_iv_i, and $u_{i+1}v_i$ are referred to as spokes, cycle edges, vertical edges, and diagonal edges, respectively.

The edge description in Definition 7 gives $|V(\mathrm{CSF}_n)| = 2n + 1$, $|E(\mathrm{CSF}_n)| = 5n$, with $d(a) = n$, $d(u_i) = 5$, $d(v_i) = 4$. Figure 1 illustrates the structure of a closed sunflower graph.

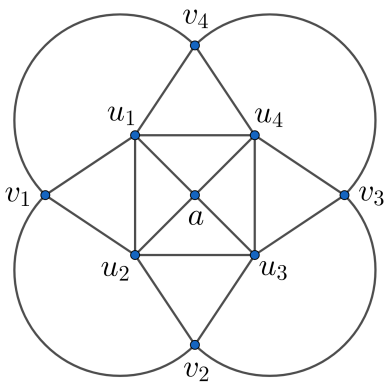


Fig. 1: The closed sunflower graph CSF_4.

Motivated by the two-layer structure of CSF_n, we now define a multilayer extension with an arbitrary number of consecutive cyclic layers.

Definition 8. Let $n \geq 3$ and $k \geq 2$ be integers. The multilayer closed sunflower graph $\mathrm{MCSF}_{n,k}$ is defined by $V(\mathrm{MCSF}_{n,k}) = \{a\} \cup \bigcup_{r=1}^{k} X_r, X_r = \left\{x_i^{(r)} : 1 \leq i \leq n\right\}$, and the sets $\{a\}, X_1, X_2, \ldots, X_k$ are pairwise disjoint. Its edge set is $E(\mathrm{MCSF}_{n,k}) = \left\{ax_i^{(1)} : 1 \leq i \leq n\right\} \cup \left\{x_i^{(r)}x_{i+1}^{(r)}, x_i^{(r)}x_i^{(r+1)}, x_{i+1}^{(r)}x_i^{(r+1)} : 1 \leq i \leq n,\ 1 \leq r \leq k\right\}$ where $x_{n+1}^{(r)} = x_1^{(r)}$ for every layer r. The four edge families are called spokes, cycle edges, vertical edges, and diagonal edges, respectively. The central vertex a is adjacent only to the vertices of the first layer X_1.

There are n spokes, kn cycle edges, $(k-1)n$ vertical edges, and $(k-1)n$ diagonal edges. Hence $|V(\mathrm{MCSF}_{n,k})| = kn + 1$ and $|E(\mathrm{MCSF}_{n,k})| = (3k-1)n$. The degrees are $d(a) = n$ and

$$d\left(x_i^{(r)}\right) = \begin{cases} 5, & r = 1, \\ 6, & 2 \leq r \leq k-1, \\ 4, & r = k. \end{cases}$$

When $k = 2$, there is no intermediate layer. Identifying $x_i^{(1)} = u_i$, $x_i^{(2)} = v_i$, and identifying the corresponding central vertices gives $\mathrm{MCSF}_{n,2} \cong \mathrm{CSF}_n$. Figure 2 illustrates the multilayer construction.

The following established results provide the lower bounds and connectivity criteria used in the subsequent proofs.

Lemma 1. *[7] Let G be a connected graph of order $N \geq 2$. Then* $\mathrm{diam}(G) \leq \mathrm{rc}(G) \leq N-1$.

Lemma 2. *[14] For every connected graph G of order at least two,* $\mathrm{rac}(G) \geq \max\{\mathrm{rc}(G), \Delta(G)\}$.

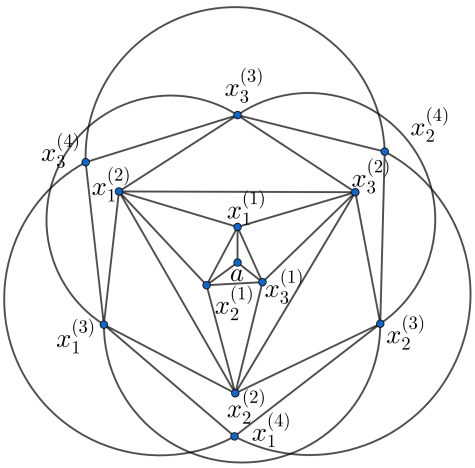


Fig. 2: The multilayer closed sunflower graph $\mathrm{MCSF}_{3,4}$.

Lemma 3. *[15] Let G be a connected graph of order at least two with $\mathrm{diam}(G) \leq 2$. Every bijection $f : V(G) \longrightarrow \{1, 2, \ldots, |V(G)|\}$ is a rainbow antimagic labeling under the induced edge weights $w_f(uv) = f(u) + f(v)$.*

The next result is a consequence of the sum-index bound and applies to arbitrary injective integer labelings rather than only to bijections onto consecutive positive integers.

Proposition 1. *[24] Let H be a finite simple graph of order $N \geq 2$, with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_N$. For every injective map $g : V(H) \longrightarrow \mathbb{Z}$, we have $|\{g(u) + g(v) : uv \in E(H)\}| \geq d_1 + d_2 - 1$. In particular, if H is d-regular, then every such labeling induces at least $2d - 1$ distinct edge weights.*

When Proposition 1 is applied to a subgraph of a vertex-labeled graph, the restriction of an injective labeling to the vertex set of that subgraph remains injective. The bound can therefore be used without relabeling the selected vertices by consecutive integers.

Lemma 4. *[1] Let T be a connected graph of order $N \geq 1$. Then T is a tree if and only if $|E(T)| = N - 1$. Moreover, every pair of distinct vertices of a tree is joined by a unique path.*

For the upper-bound constructions, we also use the standard spanning-tree criterion for rainbow connectivity. A spanning tree whose edges receive pairwise distinct colors is a rainbow spanning tree, and an edge-colored graph containing such a tree is rainbow connected [7]. Consequently, for the endpoint-sum coloring considered here, a bijection $f : V(G) \longrightarrow \{1, 2, \ldots, |V(G)|\}$ is a rainbow antimagic labeling whenever the induced weights $w_f(uv)$ are pairwise distinct on the edges of some spanning tree of G [7, 15]. This condition is used only as a sufficient criterion for rainbow connectivity and is not part of the definition of rainbow antimagic coloring.

3.1. Closed sunflower graphs

We first consider the closed sunflower family. The smallest member provides a useful illustration of how repeated endpoint sums can coexist with rainbow connectivity. Figure 3 displays a bijective vertex labeling of CSF_3. The black numbers represent the vertex labels, whereas the red numbers denote the induced edge weights $w_f(xy) = f(x) + f(y)$.

The induced weights appearing in Figure 3 are $\{5, 6, 7, 8, 9, 11, 13\}$, so exactly seven distinct edge colors occur. Moreover, both cyclic layers of CSF_3 are triangles. The central vertex is adjacent to every inner vertex, and each outer vertex is adjacent to two inner vertices. Hence

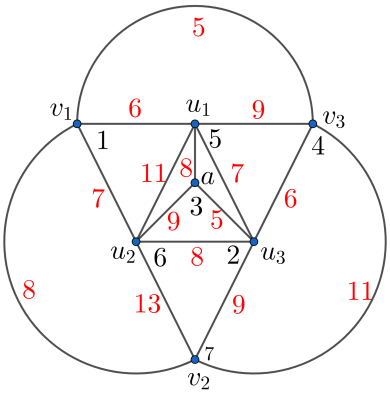


Fig. 3: A rainbow antimagic labeling of CSF_3 with seven induced edge colors.

every pair of vertices is at distance at most two, giving $\mathrm{diam}(\mathrm{CSF}_3) \leq 2$. By Lemma 3, the displayed bijection is therefore a rainbow antimagic labeling [15]. Consequently, $\mathrm{rac}(\mathrm{CSF}_3) \leq 7$.

We next establish bounds valid for every admissible closed sunflower graph.

Theorem 1. *For every integer $n \geq 3$, $\max\{n, 7\} \leq \mathrm{rac}(\mathrm{CSF}_n) \leq \left\lceil \frac{5n+4}{2} \right\rceil$.*

Proof. Let $G = \mathrm{CSF}_n$. We first establish the lower bound. Since the central vertex a has degree n, Lemma 2 gives $\mathrm{rac}(G) \geq n$. Next, consider the induced subgraph $H_0 = G[V(G) \setminus \{a\}]$. Every vertex of H_0 has two neighbors in its own cycle and two neighbors in the other cycle, so H_0 is 4-regular. The restriction of any bijective vertex labeling of G to $V(H_0)$ remains injective. Proposition 1 therefore guarantees at least $4 + 4 - 1 = 7$ distinct induced edge weights on $E(H_0)$. Since these weights also occur in G,

$$\mathrm{rac}(G) \geq \max\{n, 7\}. \tag{1}$$

For the upper bound, define $f : V(\mathrm{CSF}_n) \longrightarrow \{1, 2, \ldots, 2n+1\}$ by

$$f(a) = 1, \qquad f(u_i) = i + 1, \qquad f(v_i) = 2n + 2 - i \quad (1 \leq i \leq n). \tag{2}$$

The inner cycle receives the labels $2, 3, \ldots, n+1$, whereas the outer cycle receives $n+2, n+3, \ldots, 2n+1$ in reverse order. Together with the label 1 assigned to a, every integer from 1 to $2n+1$ is used exactly once. Hence f is bijective. Set $\mathcal{H} = \{3, 4, \ldots, n+2\}$, $\mathcal{O}_1 = \{5, 7, \ldots, 2n+1\}$, and $\mathcal{O}_2 = \{2n+5, 2n+7, \ldots, 4n+1\}$. Here $\mathcal{H}$ is the set of spoke weights, while $\mathcal{O}_1$ and $\mathcal{O}_2$ contain the nonclosing cycle-edge weights on the inner and outer cycles, respectively. Substituting the labels from Eq. (2) gives

$$\begin{aligned} \{w_f(au_i) : 1 \leq i \leq n\} &= \mathcal{H}, \qquad \{w_f(u_iu_{i+1}) : 1 \leq i \leq n\} = \mathcal{O}_1 \cup \{n+3\}, \\ \{w_f(v_iv_{i+1}) : 1 \leq i \leq n\} &= \mathcal{O}_2 \cup \{3n+3\}, \qquad \{w_f(u_iv_i) : 1 \leq i \leq n\} = \{2n+3\}, \\ \{w_f(u_{i+1}v_i) : 1 \leq i \leq n\} &= \{n+4, 2n+4\}. \end{aligned} \tag{3}$$

The edge-weight sets in Eq. (3) can be verified directly. Indeed, $w_f(u_iu_{i+1}) = 2i + 3$ $(1 \leq i < n)$, whereas the closing inner-cycle edge satisfies $w_f(u_nu_1) = n + 3$. Similarly, the closing outer-cycle edge has weight $3n + 3$. For the diagonal edges, the closing edge u_1v_n has weight $n + 4$, while every other diagonal edge has weight $2n + 4$. Both $\mathcal{O}_1$ and $\mathcal{O}_2$ contain $n - 1$ elements. The sets $\mathcal{O}_1$ and $\mathcal{O}_2$ are disjoint, and $\mathcal{O}_2$ is also disjoint from $\mathcal{H}$. Moreover, $\mathcal{H} \cap \mathcal{O}_1$ consists precisely of the odd integers from 5 to $n + 2$, and therefore $|\mathcal{H} \cap \mathcal{O}_1| = \left\lfloor \frac{n-1}{2} \right\rfloor$. Consequently,

$$|\mathcal{H} \cup \mathcal{O}_1 \cup \mathcal{O}_2| = 3n - 2 - \left\lfloor \frac{n-1}{2} \right\rfloor. \tag{4}$$

Eq. (4) gives the number of colors already contributed by $H \cup O_1 \cup O_2$. It remains to determine which closing, vertical, and diagonal weights contribute additional colors. Suppose first that $n = 2m$. The closing weights $n + 3$ and $3n + 3$ already belong to $\mathcal{O}_1$ and $\mathcal{O}_2$, respectively. The vertical weight $2n+3$ satisfies $\max \mathcal{O}_1 = 2n+1 < 2n+3 < 2n+5 = \min \mathcal{O}_2$, so it is new. The diagonal weights $n + 4$ and $2n + 4$ are distinct even integers greater than $n + 2$, and hence neither belongs to $\mathcal{H}$, $\mathcal{O}_1$, or $\mathcal{O}_2$. Thus exactly three additional colors occur, giving $|\mathcal{C}_f| = 3n - 2 - (m - 1) + 3 = 5m + 2$. Suppose now that $n = 2m + 1$. In this case, the diagonal weight $n+4$ already belongs to $\mathcal{O}_1$. The weights $n+3$, $2n+4$, and $3n+3$ are distinct even integers greater than $n + 2$, so none belongs to the sets already counted. The weight $2n+3$ is odd and satisfies $\max \mathcal{O}_1 = 2n+1 < 2n+3 < 2n+5 = \min \mathcal{O}_2$, so it is also new. Hence exactly four additional colors occur, giving $|\mathcal{C}_f| = 3n-2-m+4 = 5m+5$. Thus, for either parity,

$$|\mathcal{C}_f| = \left\lceil \frac{5n+4}{2} \right\rceil. \tag{5}$$

It remains to verify rainbow connectivity. Let T be the spanning subgraph with $E(T) = \{au_i : 1 \le i \le n\} \cup \{u_1v_1\} \cup \{v_iv_{i+1} : 1 \le i < n\}$. The spokes connect every inner vertex to a, the edges v_iv_{i+1} with $1 \le i < n$ form a path through the outer vertices, and u_1v_1 attaches that path to the remaining connected subgraph. Hence T is connected. Moreover, $|E(T)| = n + 1 + (n - 1) = 2n = |V(G)| - 1$. By Lemma 4, T is a spanning tree of G. The induced weights on the edges of T consist of the elements of $\mathcal{H}$, the weight $2n + 3$ on u_1v_1, and the elements of $\mathcal{O}_2$. These weights are pairwise distinct because $\max \mathcal{H} = n + 2 < 2n + 3 < 2n + 5 = \min \mathcal{O}_2$. Thus T is a rainbow spanning tree. The unique path in T between any two vertices of G is consequently rainbow, so f is a rainbow antimagic labeling. By Eq. (5), $\operatorname{rac}(\mathrm{CSF}_n) \le \left\lceil \frac{5n+4}{2} \right\rceil$. Together with Eq. (1), this completes the proof. □

For $n = 3$, Theorem 1 gives $\operatorname{rac}(\mathrm{CSF}_3) \ge 7$. Together with the seven-color labeling displayed in Figure 3, this yields the exact value $\operatorname{rac}(\mathrm{CSF}_3) = 7$. Theorem 1 applies to the entire parameter range $n \ge 3$. We next seek a more efficient labeling for a subclass of even closed sunflower graphs. The purpose of the following construction is not to replace the general bound, but to reduce its upper bound by allowing induced colors to be reused in a controlled way.

The refinement is based on writing $n = 2m$ and organizing the vertex labels through a signed cyclic sequence. For the sign pattern used below to alternate consistently across the two halves of the construction, m must be odd. Thus the construction does not apply to odd n, while values $n \equiv 0 \pmod 4$ correspond to even m and do not satisfy the required alternating sign structure at the junction of the two halves. Consequently, the relevant parameter class is $n \equiv 2 \pmod 4$.

Before stating the general construction, we examine the only admissible value in this congruence class below 10, namely $n = 6$. In this case $m = 3$. For a general odd m, introduce $M = 2m + 3$ and $\mathcal{A}_m = \{1, 3, 4, \ldots, m + 1\}$. We also use the auxiliary map

$$b(x) = \begin{cases} 2, & x = 1, \\ M - x, & x \in \mathcal{A}_m \setminus \{1\}. \end{cases}$$

These quantities are chosen so that the absolute values of the normalized inner-cycle, diagonal, and outer-cycle weights generated by consecutive terms of the cyclic sequence remain in $\mathcal{A}_m$.

For $m = 3$, these definitions give $M = 9$, $\mathcal{A}_3 = \{1, 3, 4\}$, $b(1) = 2$. Up to cyclic rotation, the two possible cyclic orders beginning with 1 are $(1, 3, 4)$ and $(1, 4, 3)$. For the first order, $|1 - 3| = 2 \notin \mathcal{A}_3$, whereas for the second order, $|b(1) - 4| = |2 - 4| = 2 \notin \mathcal{A}_3$. Hence neither cyclic order satisfies the containment property required by the construction. Therefore the present labeling scheme does not extend to $n = 6$. The first value for which this construction can be carried out is consequently $m = 5$, equivalently $n = 10$. This restriction is a limitation of the

present construction only; it does not rule out the possibility that a different labeling may yield a sharper upper bound for $n = 6$ or for other parameter classes.

We now develop the construction for the admissible range. Throughout the next three lemmas, let $n = 2m$, where $m \geq 5$ is odd, and set $c = 2m+1$, $M = 2m+3$, $\mathcal{A}_m = \{1, 3, 4, \ldots, m+1\}$. The three lemmas separate the verification into its essential parts: the first establishes the required cyclic ordering, the second proves that the resulting labeling is bijective and determines its complete induced color set, and the third verifies rainbow connectivity.

Arrange the elements of $\mathcal{A}_m$ in a cyclic sequence $(a_1, a_2, \ldots, a_m)$. For $m = 5$, take $(a_1, a_2, \ldots, a_5) = (1, 5, 4, 3, 6)$. For $m = 2t + 1 \geq 7$, define

$$\begin{aligned} (a_1, a_2) = (1, m), \qquad (a_{2j-3}, a_{2j-2}) = (j, m-j+2), \quad 3 \leq j \leq t, \\ (a_{m-2}, a_{m-1}, a_m) = (t+2, t+1, m+1). \end{aligned} \tag{6}$$

Lemma 5. *The cyclic sequence $(a_1, a_2, \ldots, a_m)$ defined above contains every element of $\mathcal{A}_m$ exactly once.*

Proof. For $m = 5$, the assertion follows immediately from $(a_1, \ldots, a_5) = (1, 5, 4, 3, 6)$ and $\mathcal{A}_5 = \{1, 3, 4, 5, 6\}$. Suppose now that $m = 2t + 1 \geq 7$. The first pair in Eq. (6) contributes the values 1 and m. The first coordinates of the middle block are $3, 4, \ldots, t$, while the corresponding second coordinates are $m-1, m-2, \ldots, t+3$. The final three entries are $t+2$, $t+1$, $m+1$. These collections are pairwise disjoint, and their union is $\{1\} \cup \{3, 4, \ldots, t\} \cup \{t+1, t+2\} \cup \{t+3, \ldots, m-1\} \cup \{m, m+1\} = \mathcal{A}_m$. Since the sequence contains exactly $m = |\mathcal{A}_m|$ entries, every element of $\mathcal{A}_m$ occurs exactly once. □

Using the cyclic sequence established in Lemma 5, define, for $1 \leq i \leq m$,

$$p_i = (-1)^{i+1} a_i, \qquad p_{i+m} = -p_i. \tag{7}$$

For $1 \leq i \leq 2m$, set $q_i = \operatorname{sgn}(p_i) b(|p_i|)$, where

$$\operatorname{sgn}(p_i) = \begin{cases} 1, & p_i > 0, \\ -1, & p_i < 0. \end{cases}$$

For cyclic indices, let $p_{2m+1} = p_1$, $q_{2m+1} = q_1$. The vertex labeling is then defined by

$$f(a) = c, \qquad f(u_i) = c + p_i, \qquad f(v_i) = c + q_i \quad (1 \leq i \leq 2m). \tag{8}$$

Lemma 6. *The map f defined in Eq. (8) is a bijection from $V(\mathrm{CSF}_{2m})$ onto $\{1, 2, \ldots, 4m+1\}$. Moreover, if $\widehat{w}_f(e) = w_f(e) - 2c$, then the complete normalized color set is*

$$\mathcal{P}_m = \{\pm x : x \in \mathcal{A}_m\} \cup \{-M, M\}. \tag{9}$$

Consequently, the labeling induces exactly $2m + 2 = n + 2$ distinct edge colors.

Proof. By Lemma 5, the absolute values of $p_1, \ldots, p_{2m}$ are precisely the elements of $\mathcal{A}_m$, each occurring once with each sign. Furthermore, $b(\mathcal{A}_m) = \{2, m+2, m+3, \ldots, 2m\}$. Hence the absolute values of $q_1, \ldots, q_{2m}$ are precisely the elements of this set, again each occurring once with each sign. The sets $\mathcal{A}_m$ and $\{2, m+2, m+3, \ldots, 2m\}$ are disjoint and together form $\{1, 2, \ldots, 2m\}$. Therefore the combined offsets $p_1, \ldots, p_{2m}, q_1, \ldots, q_{2m}$ use every nonzero integer from $-2m$ to $2m$ exactly once. Since $c = 2m+1$, the values $c + p_i$ and $c + q_i$ are precisely $\{1, 2, \ldots, 4m+1\} \setminus \{c\}$. Together with $f(a) = c$, this proves that

f is bijective. Every induced edge weight contains the common term $2c$. Subtracting this constant preserves equality and inequality among edge weights and therefore preserves both the number of distinct colors and the rainbow property of every path. It is consequently sufficient to work with the normalized weights $\widehat{w}_f$. The normalized spoke weights are p_i. For a vertical edge u_iv_i,

$$\mu_i := \widehat{w}_f(u_iv_i) = p_i + q_i = \begin{cases} 3p_i, & |p_i| = 1, \\ \operatorname{sgn}(p_i)M, & |p_i| \geq 3. \end{cases} \tag{10}$$

When $|p_i| = 1$, the vertical weight is ± 3, which already belongs to $\{\pm x : x \in \mathcal{A}_m\}$. All remaining vertical edges contribute only the two additional values $-M$ and M. The normalized weights on the inner-cycle, diagonal, and outer-cycle edges are, respectively, $p_i + p_{i+1}$, $q_i + p_{i+1}$, $q_i + q_{i+1}$. Since m is odd, consecutive p_i have opposite signs throughout the complete cyclic sequence, including the transitions (p_m, p_{m+1}) and (p_{2m}, p_1). Moreover, q_i has the same sign as p_i. Thus, for a consecutive cyclic pair (x, y) in $(a_1, \ldots, a_m)$, the absolute values of the three normalized weights are $d = |x - y|$, $e = |b(x) - y|$, $h = |b(x) - b(y)|$. The second half of the signed sequence reproduces the same absolute values with both signs reversed. For $m = 5$, the five cyclic pairs give $(d_1, \ldots, d_5) = (4, 1, 1, 3, 5)$, $(e_1, \ldots, e_5) = (3, 4, 6, 4, 6)$, and $(h_1, \ldots, h_5) = (6, 1, 1, 3, 5)$, all of which belong to $\mathcal{A}_5$. For $m = 2t+1 \geq 7$, every consecutive pair, including the wrap-around pair $(a_m, a_1) = (m+1, 1)$, is covered by the following table.

(x, y)	Range	d	e	h
$(1, m)$	—	$m - 1$	$m - 2$	$m + 1$
$(m - j + 2, j + 1)$	$2 \leq j \leq t - 1$	$m - 2j + 1$	m	$m - 2j + 1$
$(j, m - j + 2)$	$3 \leq j \leq t$	$m - 2j + 2$	$m + 1$	$m - 2j + 2$
$(t + 3, t + 2)$	—	1	$m - 1$	1
$(t + 2, t + 1)$	—	1	$m + 1$	1
$(t + 1, m + 1)$	—	$t + 1$	$t + 2$	$t + 1$
$(m + 1, 1)$	—	m	$m + 1$	m

Every entry in the last three columns is either 1 or lies between 3 and $m + 1$, and therefore belongs to $\mathcal{A}_m$. Hence every normalized weight on an inner-cycle, diagonal, or outer-cycle edge belongs to $\{\pm x : x \in \mathcal{A}_m\}$. Conversely, every value in this set occurs on a spoke. Since $m \geq 5$, there are indices with $|p_i| \geq 3$, and therefore both M and $-M$ occur on vertical edges by Eq. (10). Thus the complete normalized color set is exactly the set $\mathcal{P}_m$ defined in Eq. (9). Since $|\mathcal{A}_m| = m$, we obtain $|\mathcal{P}_m| = 2m + 2 = n + 2$. □

Example. Consider the first admissible case $m = 5$, so that $n = 10$, $c = 11$, $M = 13$, and $\mathcal{A}_5 = \{1, 3, 4, 5, 6\}$. Using the cyclic order $(a_1, a_2, a_3, a_4, a_5) = (1, 5, 4, 3, 6)$, the signed sequence defined in Eq. (7) is $(p_1, \ldots, p_{10}) = (1, -5, 4, -3, 6, -1, 5, -4, 3, -6)$. Since $b(1) = 2$, $b(3) = 10$, $b(4) = 9$, $b(5) = 8$, $b(6) = 7$, we obtain $(q_1, \ldots, q_{10}) = (2, -8, 9, -10, 7, -2, 8, -9, 10, -7)$. Therefore the inner-cycle labels are $(f(u_1), \ldots, f(u_{10})) = (12, 6, 15, 8, 17, 10, 16, 7, 14, 5)$, while the outer-cycle labels are $(f(v_1), \ldots, f(v_{10})) = (13, 3, 20, 1, 18, 9, 19, 2, 21, 4)$. Together with $f(a) = 11$, these values are exactly $\{1, 2, \ldots, 21\}$, confirming the bijectivity of the labeling in this case. The normalized spoke weights are the values p_i, while the normalized vertical weights are $(3, -13, 13, -13, 13, -3, 13, -13, 13, -13)$. The normalized inner-cycle, diagonal, and outer-cycle weights are, respectively, $(-4, -1, 1, 3, 5, 4, 1, -1, -3, -5)$, $(-3, -4, 6, -4, 6, 3, 4, -6, 4, -6)$, and $(-6, 1, -1, -3, 5, 6, -1, 1, 3, -5)$. Hence every normalized edge weight belongs to $\mathcal{P}_5 = \{-13, -6, -5, -4, -3, -1, 1, 3, 4, 5, 6, 13\}$. Thus $|\mathcal{P}_5| = 12 = n+2$, which illustrates the conclusion of Lemma 6.

Lemma 7. *The labeling f defined in Eq. (8) induces a rainbow-connected edge coloring of* CSF_{2m}.

Proof. All graph indices in this proof are interpreted cyclically modulo $2m$. Since translating every induced edge weight by the same constant $2c$ preserves equality and inequality among colors, it is sufficient to work with the normalized weights from Lemma 6. The center a is adjacent to every inner vertex, so each pair a, u_i is joined by a one-edge rainbow path. For a pair a, v_i, consider a, u_i, v_i. Its normalized edge weights are p_i and μ_i, which are distinct because $\mu_i - p_i = q_i \neq 0$. Two distinct inner vertices u_r and u_s are joined by u_r, a, u_s, whose edge colors p_r and p_s are distinct because the sequence (p_i) contains no repeated values. If an inner vertex and an outer vertex are adjacent, their common edge itself gives a rainbow path. It remains to consider a nonadjacent pair u_r, v_i. Since v_i is adjacent to u_i and u_{i+1}, $r \neq i$ and $r \neq i+1 \pmod{2m}$. Consider first the path u_r, a, u_i, v_i, whose normalized colors are p_r, p_i, μ_i. If $|p_i| \geq 3$, then $\mu_i = \pm M$, which is not a spoke color, and $p_r \neq p_i$. Hence the path is rainbow. Suppose instead that $p_i = s \in \{-1, 1\}$. Then $\mu_i = 3s$. The same path is rainbow unless $p_r = 3s$. In this exceptional case, use u_r, a, u_{i+1}, v_i. Because u_r and v_i are nonadjacent, this path is simple. By Lemma 5, the absolute value 1 occurs only at a_1. Hence $|p_i| = 1$ occurs only for $i = 1$ or $i = m+1$. In both cases, $p_{i+1} = -sm$ and $q_i = 2s$. Thus the normalized edge weights on the exceptional path are $3s$, $-sm$, $-s(m-2)$. The last two have sign opposite to $3s$, and they are distinct from each other because $m \geq 5$. Therefore the path is rainbow. It remains to connect two distinct outer vertices v_i and v_j. The idea is to use the center a as a common hub. Each outer vertex v_i has two two-edge routes to a, $D_i = (v_i, u_i, a)$ and $E_i = (v_i, u_{i+1}, a)$. Thus, to obtain a rainbow path between two outer vertices, it is enough to choose one route from each vertex whose normalized color sets are disjoint. The following cases verify that such a choice is always possible. Let $C(R)$ denote the set of normalized colors on a route R. Then

$$C(D_i) = \{\mu_i, p_i\}, \qquad C(E_i) = \{\delta_i, p_{i+1}\}, \qquad \delta_i = q_i + p_{i+1}. \tag{11}$$

By Eq. (11), both routes are rainbow because $\mu_i - p_i = q_i \neq 0$ and $\delta_i - p_{i+1} = q_i \neq 0$.

Case 1. Suppose $|p_i|, |p_j| \geq 3$ and p_i, p_j have opposite signs. Choose D_i and D_j. Both colors of D_i have the sign of p_i, while both colors of D_j have the opposite sign. Hence $C(D_i) \cap C(D_j) = \varnothing$.

Case 2. Suppose $|p_i|, |p_j| \geq 3$ and p_i, p_j have the same sign. Then D_i and D_j contain the same vertical color M or $-M$, so one route must be replaced. If $\delta_j \neq p_i$, choose D_i and E_j. The color $\mu_i = \pm M$ does not occur on E_j, while p_{j+1} has sign opposite to p_i. Hence the only possible common color would be $\delta_j = p_i$, which has been excluded. If $\delta_j = p_i$ but $\delta_i \neq p_j$, choose E_i and D_j, and apply the same argument with i and j interchanged. Finally, suppose $\delta_j = p_i$ and $\delta_i = p_j$. Choose E_i and E_j. Then $C(E_i) = \{p_j, p_{i+1}\}$, whereas $C(E_j) = \{p_i, p_{j+1}\}$. The values p_i and p_j have one sign, while p_{i+1} and p_{j+1} have the opposite sign. Within each sign class the values are distinct because the sequence (p_i) has no repetitions. Hence the two sets are disjoint.

Case 3. Suppose at least one of $|p_i|$ and $|p_j|$ equals 1. Interchanging i and j if necessary, let $p_i = s \in \{-1, 1\}$. Then $C(D_i) = \{3s, s\}$. If $p_j = -s$, choose D_i and D_j; the two color sets have opposite signs and are therefore disjoint. Otherwise, since the p-values are pairwise distinct, $|p_j| \geq 3$. Then $C(D_j) = \{\mathrm{sgn}(p_j)M, p_j\}$. The only possible overlap with $\{3s, s\}$ is $p_j = 3s$. If $p_j \neq 3s$, the routes D_i and D_j are already color-disjoint. If $p_j = 3s$, choose E_i and D_j. As established above, $p_{i+1} = -sm$, while $\delta_i = q_i + p_{i+1} = -s(m-2)$. Thus $C(E_i) = \{-s(m-2), -sm\}$, whereas $C(D_j) = \{sM, 3s\}$. The two sets have opposite signs and are therefore disjoint. These three cases exhaust all pairs of distinct outer vertices. In every case, the selected routes cannot contain the same inner vertex, because a common inner

vertex would contribute the same spoke color to both routes, contradicting the established disjointness of their color sets. Hence the selected routes meet only at a. Following one route to a and the other in reverse therefore gives a simple rainbow path between the two outer vertices. We have thus exhibited a rainbow path for every possible pair of vertices of CSF_{2m}. Therefore the induced edge coloring is rainbow connected. □

The preceding lemmas verify separately the three ingredients needed for the improved result. Lemma 5 establishes the cyclic ordering, Lemma 6 determines the complete induced color set, and Lemma 7 establishes rainbow connectivity.

Theorem 2. *If $n \geq 10$ and $n \equiv 2 \pmod 4$, then $n \leq \mathrm{rac}(\mathrm{CSF}_n) \leq n+2$.*

Proof. Since $n \equiv 2 \pmod 4$, we may write $n = 2m$, where $m \geq 5$ is odd. By Theorem 1, $\mathrm{rac}(\mathrm{CSF}_n) \geq n$. For the upper bound, use the labeling defined in Eq. (8). Lemma 6 shows that this labeling is bijective and induces exactly $n+2$ distinct edge colors. Lemma 7 shows that the induced edge coloring is rainbow connected. Hence the labeling is rainbow antimagic and $\mathrm{rac}(\mathrm{CSF}_n) \leq n+2$. Combining the lower and upper bounds gives $n \leq \mathrm{rac}(\mathrm{CSF}_n) \leq n+2$. □

3.2. Multilayer closed sunflower graphs

We now extend the constructive approach to $\mathrm{MCSF}_{n,k}$. Unlike the improved construction above, the multilayer result is valid over the entire parameter range in Definition 8. The labeling assigns a consecutive block of integers to each layer and reverses the order of that block on every even-indexed layer. This alternating arrangement makes the vertical weights constant on each interface and allows the remaining edge-weight families to be counted explicitly.

Theorem 3. *For all integers $n \geq 3$ and $k \geq 2$, $\max\{n, k, 7\} \leq \mathrm{rac}(\mathrm{MCSF}_{n,k}) \leq k(n+2) + \left\lfloor \frac{n-3}{2} \right\rfloor$.*

Proof. Let $G = \mathrm{MCSF}_{n,k}$. Since the central vertex a has degree n, Lemma 2 gives $\mathrm{rac}(G) \geq n$. Assign layer index 0 to a and layer index r to every vertex of X_r. Every edge of G changes the layer index by at most one. Therefore every path from a to a vertex $x_i^{(k)}$ contains at least k edges. Conversely, $a, x_i^{(1)}, x_i^{(2)}, \ldots, x_i^{(k)}$ is a path of length k. Hence $d_G(a, x_i^{(k)}) = k$, and therefore $\mathrm{diam}(G) \geq k$. By Lemma 1, $\mathrm{rc}(G) \geq k$, and Lemma 2 consequently gives $\mathrm{rac}(G) \geq k$. For the remaining lower bound, consider the induced subgraph $G[X_{k-1} \cup X_k]$. Every vertex of this induced subgraph has two neighbors in its own layer and two neighbors in the other layer, so it is 4-regular. The restriction of any bijective vertex labeling of G to $X_{k-1} \cup X_k$ remains injective. Proposition 1 therefore guarantees at least $4 + 4 - 1 = 7$ distinct induced edge weights on this subgraph. Consequently, $\mathrm{rac}(G) \geq \max\{n, k, 7\}$. We now construct a labeling for the upper bound. Define $f : V(G) \longrightarrow \{1, 2, \ldots, kn+1\}$ by $f(a) = 1$ and

$$f\left(x_i^{(r)}\right) = \begin{cases} (r-1)n + i + 1, & r \text{ odd}, \\ rn - i + 2, & r \text{ even}, \end{cases} \tag{12}$$

for $1 \leq i \leq n$ and $1 \leq r \leq k$. Layer X_r receives the block $\{(r-1)n+2, \ldots, rn+1\}$, in increasing order when r is odd and in decreasing order when r is even. These blocks are pairwise disjoint and, together with the label 1 assigned to a, contain every integer from 1 to $kn+1$ exactly once. Hence f is bijective. The spoke weights form the set $\mathcal{H} = \{3, 4, \ldots, n+2\}$. For $1 \leq i < n$, the cycle edge $x_i^{(r)} x_{i+1}^{(r)}$ has weight $2(r-1)n + 2i + 3$

when r is odd and $2rn - 2i + 3$ when r is even. In either case, the nonclosing cycle edges in layer r produce the same set

$$\mathcal{O}_r = \{2(r-1)n + 5, 2(r-1)n + 7, \ldots, 2rn + 1\}. \tag{13}$$

Let c_r denote the weight of the closing cycle edge in X_r, and let λ_r denote the common weight of the vertical edges between X_r and X_{r+1}. Substitution of Eq. (12) gives

$$\begin{aligned} c_r &= w_f\left(x_n^{(r)}x_1^{(r)}\right) = (2r-1)n + 3, \quad 1 \le r \le k, \\ \lambda_r &= w_f\left(x_i^{(r)}x_i^{(r+1)}\right) = 2rn + 3, \qquad 1 \le r < k. \end{aligned} \tag{14}$$

For the diagonal edges between X_r and X_{r+1}, let d_r denote the common weight of $x_{i+1}^{(r)}x_i^{(r+1)}$ for $1 \le i < n$, and let e_r denote the weight of the closing diagonal $x_1^{(r)}x_n^{(r+1)}$. Then

$$(d_r, e_r) = \begin{cases} (2rn + 4, (2r-1)n + 4), & r \text{ odd}, \\ (2rn + 2, (2r+1)n + 2), & r \text{ even}. \end{cases} \tag{15}$$

Every edge of G belongs to one of the families above. We now count their distinct weights. By Eq. (13), each $\mathcal{O}_r$ contains $n-1$ elements, and the sets $\mathcal{O}_1, \ldots, \mathcal{O}_k$ are pairwise disjoint because their numerical ranges do not overlap. Only $\mathcal{O}_1$ intersects $\mathcal{H}$, and $|\mathcal{H} \cap \mathcal{O}_1| = \left\lfloor \frac{n-1}{2} \right\rfloor$. Consequently,

$$B_{n,k} := \left| \mathcal{H} \cup \bigcup_{r=1}^{k} \mathcal{O}_r \right| = n + k(n-1) - \left\lfloor \frac{n-1}{2} \right\rfloor. \tag{16}$$

By Eq. (14), the $k-1$ vertical weights $\lambda_1, \ldots, \lambda_{k-1}$ are pairwise distinct and do not belong to the sets already counted. Indeed, $\lambda_1 > n + 2 = \max \mathcal{H}$, and for $1 \le r < k$,

$$\max \mathcal{O}_r = 2rn + 1 < \lambda_r = 2rn + 3 < 2rn + 5 = \min \mathcal{O}_{r+1}. \tag{17}$$

It remains to determine the new weights contributed by the closing cycle edges and diagonal edges. Suppose first that n is even. By Eq. (14), $c_r = (2r-1)n + 3$ is odd and lies in the numerical range of $\mathcal{O}_r$. Since $\mathcal{O}_r$ contains every odd integer in that range, each c_r has already been counted. Every diagonal weight in Eq. (15) is even and greater than $n + 2$. Hence no diagonal weight belongs to $\mathcal{H}$, any $\mathcal{O}_r$, the set of closing-cycle weights, or the set of vertical weights. To verify that the diagonal weights are mutually distinct, write each of them as $qn + \varepsilon$, $\varepsilon \in \{2, 4\}$. If $qn + \varepsilon = q'n + \varepsilon'$, then $(q - q')n = \varepsilon' - \varepsilon \in \{-2, 0, 2\}$. Since even $n \ge 4$, this forces $q = q'$ and $\varepsilon = \varepsilon'$. For a fixed value of ε, the coefficients occurring within each diagonal family are distinct, while the coefficients associated with d_r are even and those associated with e_r are odd. Hence all $2(k-1)$ diagonal weights are distinct and new. Together with the $k-1$ new vertical weights, $|\mathcal{C}_f| = B_{n,k} + 3(k-1)$ when n is even. Suppose now that n is odd. Each closing-cycle weight c_r is even, the values $c_1, \ldots, c_k$ are pairwise distinct, and all are greater than $n + 2$. Since every element of each $\mathcal{O}_r$ and every vertical weight λ_r is odd, none of the c_r has been counted previously. The closing diagonal weights e_r, however, already occur among the cycle-edge weights. More precisely,

$$e_r \in \begin{cases} \mathcal{O}_r, & r \text{ odd}, \\ \mathcal{O}_{r+1}, & r \text{ even}. \end{cases}$$

For odd r, the position of e_r in the increasing set $\mathcal{O}_r$ is $\frac{n+1}{2}$, while for even r, its position in $\mathcal{O}_{r+1}$ is $\frac{n-1}{2}$. Since $n \ge 3$, both positions lie between 1 and $n-1$, so the stated memberships are valid. The remaining diagonal weights d_r are even and greater than $n + 2$. They are

pairwise distinct because the sequence (d_r) is strictly increasing. Consecutive differences alternate between $2n-2$ and $2n+2$, both of which are positive. Moreover, for every $1 \le s \le k$ and $1 \le r < k$, $c_s \equiv 3 \pmod{n}$, $d_r \equiv 2$ or $4 \pmod{n}$. Since $n \ge 3$, neither $2-3=-1$ nor $4-3=1$ is divisible by n. Hence $d_r \ne c_s$ for all such r and s. Since the previously counted cycle and vertical weights are odd and $\mathcal{H}$ contains no value greater than $n+2$, every d_r contributes a new color. Thus, for odd n, the closing cycle edges contribute k new colors, the nonclosing diagonal edges contribute $k-1$ new colors, and the vertical edges contribute $k-1$ new colors. Substituting Eq. (16) into the two parity cases and simplifying gives, in both cases,

$$|\mathcal{C}_f| = k(n+2) + \left\lfloor \frac{n-3}{2} \right\rfloor. \tag{18}$$

It remains to verify rainbow connectivity. Let T be the spanning subgraph obtained by retaining all n spokes and, for each $r = 2, \ldots, k$, retaining the connector $x_1^{(r-1)}x_1^{(r)}$ together with the cycle edges $x_i^{(r)}x_{i+1}^{(r)}$ $(1 \le i < n)$. The spokes connect every vertex of X_1 to a. For each $r \ge 2$, the selected cycle edges form a path through all vertices of X_r, and the connector $x_1^{(r-1)}x_1^{(r)}$ attaches this path to the preceding layers. Hence T is connected. Moreover, $|E(T)| = n + (k-1) + (k-1)(n-1) = kn = |V(G)| - 1$. By Lemma 4, T is a spanning tree. The spoke weights in T form $\mathcal{H}$. For every $r \ge 2$, the connector entering X_r has weight λ_{r-1}, while the selected path inside X_r has weight set $\mathcal{O}_r$. The sets $\mathcal{O}_2, \ldots, \mathcal{O}_k$ are pairwise disjoint and lie entirely above $\mathcal{H}$. Eq. (17) places each connector weight strictly between two consecutive cycle-weight ranges, while $\lambda_1 > \max \mathcal{H}$. Consequently, all edges of T receive pairwise distinct induced weights. Thus T is a rainbow spanning tree. The unique path in T between any two vertices of G is rainbow, so f is a rainbow antimagic labeling. By Eq. (18), $\mathrm{rac}(G) \le k(n+2) + \left\lfloor \frac{n-3}{2} \right\rfloor$. Together with $\mathrm{rac}(G) \ge \max\{n, k, 7\}$, this completes the proof. □

3.3. Discussion of the results

The preceding results cover the complete parameter ranges introduced in Definitions 7 and 8. Theorem 1 applies to every closed sunflower graph with $n \ge 3$, while Theorem 2 improves the upper bound when $n \ge 10$ and $n \equiv 2 \pmod{4}$. Theorem 3 applies to every multilayer closed sunflower graph with $n \ge 3$ and $k \ge 2$. For $k = 2$, its upper bound becomes $2(n+2) + \left\lfloor \frac{n-3}{2} \right\rfloor = \left\lceil \frac{5n+4}{2} \right\rceil$ and its lower bound becomes $\max\{n, 7\}$, recovering the general closed sunflower bounds consistently with $\mathrm{MCSF}_{n,2} \cong \mathrm{CSF}_n$. Table 1 summarizes representative values. Here, "LB" denotes the lower bound, "General UB" the constructive upper bound valid over the full parameter range, "Best UB" the smallest upper bound proved after incorporating the exact case $n = 3$ and Theorem 2 whenever applicable, and "Gap" the difference between LB and Best UB.

Table 1: Representative lower and upper bounds established in this paper.

Graph	Parameters	LB	General UB	Best UB	Gap	Graph	Parameters	LB	General UB	Best UB	Gap
CSF_n	$n=3$	7	10	7	0	$\mathrm{MCSF}_{n,k}$	$(3,2)$	7	10	7	0
CSF_n	$n=4$	7	12	12	5	$\mathrm{MCSF}_{n,k}$	$(3,3)$	7	15	15	8
CSF_n	$n=5$	7	15	15	8	$\mathrm{MCSF}_{n,k}$	$(4,3)$	7	18	18	11
CSF_n	$n=6$	7	17	17	10	$\mathrm{MCSF}_{n,k}$	$(6,3)$	7	25	25	18
CSF_n	$n=7$	7	20	20	13	$\mathrm{MCSF}_{n,k}$	$(10,2)$	10	27	12	2
CSF_n	$n=8$	8	22	22	14	$\mathrm{MCSF}_{n,k}$	$(10,4)$	10	51	51	41
CSF_n	$n=9$	9	25	25	16						
CSF_n	$n=10$	10	27	12	2						

For the closed sunflower family, the value for $n = 3$ is exact, whereas for $n = 4, 5, 6, 7, 8,$

and 9 the table reports only the strongest bounds proved here. Although $n = 6$ satisfies $n \equiv 2 \pmod 4$, the cyclic construction of Theorem 2 fails the required containment property when $m = 3$, so the absence of a sharper bound reflects a limitation of this construction rather than an omitted case. For $n \geq 10$ with $n \equiv 2 \pmod 4$, the general upper bound $\left\lceil \frac{5n+4}{2} \right\rceil = \frac{5n+4}{2}$ is reduced to $n + 2$, an improvement of $\frac{3n}{2}$. Together with the lower bound n, this leaves only n, $n + 1$, and $n + 2$ as possible exact values; the same improvement applies to $n = 14, 18, 22, \ldots$.

For the multilayer family, Theorem 3 covers all $n \geq 3$ and $k \geq 2$. When $k = 2$, stronger closed sunflower results transfer through $\mathrm{MCSF}_{n,2} \cong \mathrm{CSF}_n$, giving $\mathrm{rac}(\mathrm{MCSF}_{3,2}) = 7$ and, for $n \geq 10$ with $n \equiv 2 \pmod 4$, $n \leq \mathrm{rac}(\mathrm{MCSF}_{n,2}) \leq n + 2$. For $k > 2$, the result remains a constructive general bound: for fixed n, the upper bound grows linearly with k, whereas $\max\{n, k, 7\}$ changes only when the layer term becomes dominant, so the gap may become large; for example, $10 \leq \mathrm{rac}(\mathrm{MCSF}_{10,4}) \leq 51$. This is a proved existence interval, not evidence that the upper endpoint is optimal.

Finally, the improved closed sunflower construction shows that a rainbow spanning tree is sufficient but not necessary for rainbow connectivity. Every spanning tree of CSF_n has $2n$ edges, while the construction in Theorem 2 uses only $n + 2$ distinct induced colors; since $n + 2 < 2n$, no spanning tree can have pairwise distinct edge colors under this labeling, and rainbow connectivity is instead established by the explicit paths in Lemma 7. Thus only $\mathrm{rac}(\mathrm{CSF}_3)$ and the isomorphic case $\mathrm{MCSF}_{3,2}$ are determined exactly, while the remaining results are constructive bounds not claimed to be optimal. Determining the remaining exact values, extending the improved construction beyond the present congruence class, and narrowing the gap for $k > 2$ remain natural directions for further investigation.

4. Conclusion

This study establishes general lower and constructive upper bounds for the rainbow antimagic connection numbers of closed sunflower graphs and their multilayer extensions. For every $n \geq 3$, $\max\{n, 7\} \leq \mathrm{rac}(\mathrm{CSF}_n) \leq \left\lceil \frac{5n+4}{2} \right\rceil$, with the exact value $\mathrm{rac}(\mathrm{CSF}_3) = 7$. For $n \geq 10$ with $n \equiv 2 \pmod 4$, a refined construction improves the upper bound to $n + 2$, leaving a gap of at most two from the lower bound; this parameter restriction belongs to the present construction, which does not extend to $n = 6$ under the same cyclic weight scheme. For the multilayer family, all $n \geq 3$ and $k \geq 2$ satisfy $\max\{n, k, 7\} \leq \mathrm{rac}(\mathrm{MCSF}_{n,k}) \leq k(n+2) + \left\lfloor \frac{n-3}{2} \right\rfloor$. The case $k = 2$ recovers the general closed sunflower bounds, while the exact and sharper results for CSF_n transfer directly to the corresponding two-layer cases through the isomorphism $\mathrm{MCSF}_{n,2} \cong \mathrm{CSF}_n$. Except for CSF_3 and the isomorphic case $\mathrm{MCSF}_{3,2}$, the results are constructive bounds rather than exact characterizations. Future work may therefore focus on determining the unresolved exact values for small closed sunflower graphs, extending the improved construction to additional parameter classes, and narrowing the gap for multilayer graphs with $k > 2$ through stronger lower bounds or more efficient color-reuse constructions.

CRediT Authorship Contribution Statement

Fransiskus Fran: Conceptualization, Analysis, Supervision, Validation, Review and Editing; **Muammar Rifqi**: Methodology, Analysis, Investigation, Visualization, Original Draft; **Evi Noviani**: Analysis, Validation, Review and Editing; **Raventino**: Methodology, Analysis, Supervision, Validation, Review and Editing.

Declaration of Generative AI and AI-assisted Technologies

Generative AI tools, including ChatGPT, were used only to support translation, language polishing, grammatical correction, and improvements in clarity and readability. All AI-assisted suggestions were carefully evaluated by the authors, and all definitions, constructions, calculations,

theorems, proofs, and conclusions were independently verified by the authors, who take full responsibility for the correctness and integrity of the manuscript.

Declaration of Competing Interest

The authors declare no competing financial interests or personal relationships that could have influenced this research.

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