CAUCHY – Jurnal Matematika Murni dan Aplikasi
Volume 11 (2) (2026), Pages 2832–2851
p-ISSN: 2086-0382; e-ISSN: 2477-3344

Comparative Evaluation of Boyle and Hull Trinomial Lattices for Employee Stock Option Valuation with Discrete Dividends: Convergence and Moneyness Analysis

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Abstract

Employee stock option (ESO) valuation using trinomial lattice methods may behave differently depending on the chosen parameterization and the discrete representation of the strike price. This study compares Boyle-type and Hull-type trinomial parameterizations, constructed using first-order asymptotic transition probabilities, to approximate the value of a European-style employee stock option under the Management and Employee Stock Option Plan (MESOP) Phase III.A of an anonymized Indonesian publicly listed company (referred to as PT XXX Corporindo Tbk), incorporating a discrete dividend through the escrowed dividend approach. The Black-Scholes model serves as an analytical benchmark to assess convergence, moneyness, volatility sensitivity, and strike-price position relative to lattice nodes. We examine three strike-price conditions: Rp499 as the actual contract condition, Rp1,530 as the at-the-money condition, and Rp1,630 as the out-of-the-money condition. Historical volatility is estimated at 38.1825% from 237 daily closing-price observations. The results show convergence behavior depends on lattice resolution and moneyness. Boyle reaches the prescribed error tolerance at a lower lattice resolution than Hull for the at-the-money condition. In contrast, Hull does so at a lower resolution for the out-of-the-money condition. At the same time, both parameterizations satisfy the tolerance at the same resolution for the in-the-money condition. The strike-grid analysis further reveals oscillatory error patterns and variation in relative accuracy across strike prices, suggesting that the strike position relative to lattice nodes may contribute to numerical variation beyond broad moneyness classification. These findings indicate that neither parameterization is uniformly superior and that trinomial accuracy should be assessed jointly with lattice resolution, moneyness, and strike position.

Keywords: Convergence; Discrete Dividend; Employee Stock Option; Moneyness; Trinomial Lattice

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1. Introduction

Employee stock options (ESOs) are equity-based compensation instruments that give employees the right to purchase company shares at a predetermined price within a specified period. Their valuation depends on the underlying stock price, exercise price, volatility, interest rate, maturity, and dividend payments. Lattice methods provide a numerical framework by discretizing the underlying asset's evolution. Boyle introduced a three-jump process for option valuation, followed by lattice formulations using alternative state variables and parameterizations [1, 2].

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Received: September 6, 2026 Revised: November 12, 2026 Accepted: November 18, 2026
DOI: https://doi.org/10.18860/cauchy.v11i2.46440

Hull also presents a trinomial formulation based on different jump sizes and transition probabilities [3]. In this study, "Boyle-type" and "Hull-type" denote the parameterizations obtained with the stretch parameter $\lambda = \sqrt{2}$ ($P_m = \frac{1}{2}$) and $\lambda = \sqrt{3}$ ($P_m = \frac{2}{3}$), respectively, within a common first-order moment-matching framework described in Subsection 2.5. The Black-Scholes model provides an analytical benchmark for numerical option-pricing methods [4], within the continuous-time stochastic framework described by Shreve [5]. Although both formulations approximate the same continuous-time process, their different jump sizes and transition probabilities may produce different values when a finite number of lattice periods is used.

Researchers have widely investigated the convergence and numerical accuracy of lattice methods. Wang et al. studied trinomial-lattice algorithms, while Leduc and Palmer analyzed the convergence rate of trinomial prices [6, 7]. Ratibenyakool and Neammanee further examined convergence toward the Black-Scholes formula, while Leduc investigated how lattice-node positions affect convergence behavior [8, 9]. Other studies have extended trinomial approaches through fuzzy-set formulations and empirical parameter calibration [10, 11]. Researchers have also investigated vectorized tree implementations to price path-dependent options and to select the number of time steps [12]. These studies indicate that lattice construction, parameterization, and discretization can affect numerical option valuation.

For ESO valuation, Sholikha and Artiono applied the Boyle trinomial method to European-style ESOs [13]. Meanwhile, Fitroh and Artiono examined the effects of strike price and employee exit using a binomial model [14]. Taufiqi and Artiono incorporated dilution, and Lesmana et al. considered dilution, early exercise, and employee forfeiture in a trinomial ESO framework [15, 16]. Related lattice comparisons have also been conducted for binary options and European put options under different contract and market parameters [17, 18]. These studies establish the applicability of lattice methods to ESO and related option-pricing problems.

Discrete dividends introduce an additional consideration because dividend payments affect the underlying asset during the option horizon. De Angelis et al. developed a lattice framework for assets paying discrete dividends, while Maharani and Siswanah examined European option pricing using a trinomial method with dividends [19, 20]. Shan et al. also investigated option valuation on discrete dividend-paying stocks [21]. In a related study, Elgiffary and Artiono examined discrete-dividend European options using a numerical approach with Black-Scholes as a benchmark and considered different moneyness conditions [22]. However, these studies do not directly compare Boyle and Hull trinomial parameterizations within an ESO framework with a discrete dividend.

Moneyness and volatility provide further dimensions for evaluating option-pricing behavior. Aretz et al. examined how the effect of underlying volatility on European option returns depends on moneyness [23]. Liang et al. investigated the role of overnight and intraday volatility information in option pricing [24]. Ulrich et al. analyzed implied-volatility surfaces using a large option-price dataset [25]. Tazkya et al. also incorporated volatility forecasting into trinomial-tree valuation [26]. In addition, Costabile et al. evaluated the numerical accuracy of a lattice-based derivative-pricing algorithm [27]. These studies motivate examining lattice parameterizations under different moneyness and volatility conditions.

To our knowledge, no study has directly compared Boyle and Hull trinomial parameterizations for an ESO with a discrete dividend, evaluated through lattice convergence, moneyness, and volatility sensitivity. Therefore, this study compares the Boyle and Hull trinomial parameterizations for a European-style approximation of an ESO under the Management and Employee Stock Option Plan (MESOP) Phase III.A of PT XXX Corporindo Tbk, incorporating a discrete dividend and using the Black-Scholes model as the analytical benchmark. The novelty lies in directly comparing Boyle-type and Hull-type first-order trinomial parameterizations within an ESO framework, with convergence, moneyness, and volatility sensitivity as supporting evaluation dimensions. The contribution is a numerical comparison showing how the choice between the

two parameterizations interacts with lattice discretization, moneyness, and volatility, extending lattice-based ESO studies that have not compared trinomial parameterizations. Throughout this paper, for brevity, "Boyle" and "Hull" in the text, tables, and figures refer to these Boyle-type and Hull-type first-order parameterizations, and not to the exact finite-step Boyle probabilities.

2. Methods

This section presents the methodological framework for valuing ESO using Boyle-type and Hull-type trinomial lattices. It begins with the model formulation and parameterization, followed by the option valuation procedure and numerical evaluation of convergence, moneyness, volatility sensitivity, and strike-price positioning relative to lattice nodes.

2.1. Data and Research Parameter Estimation

This study uses a numerical approach to compare trinomial lattice parameterizations for a European-style approximation of the MESOP of PT XXX Corporindo Tbk, accounting for a dividend payment at a discrete time point. The issuer is anonymized as "PT XXX Corporindo Tbk" so that the study focuses on the numerical comparison of the two trinomial parameterizations rather than on valuing a specific listed company. Anonymization affects only the issuer identification; all numerical inputs remain unchanged. The daily closing prices were obtained from Yahoo Finance and retrieved on 14 August 2026. The MESOP terms are taken from the issuer's public disclosure to the Financial Services Authority (OJK) dated 20 March 2024, and the dividend information is taken from the exchange disclosure dated 5 May 2026.

The stock price model is a continuous process driven by geometric Brownian motion (GBM). Option valuation is then performed using a trinomial lattice, which represents the process discretely over several time intervals. Dividing the option horizon T into N steps gives the following time-step size:

$$\Delta t = \frac{T}{N}$$

In each interval, the stock price is assumed to move to one of three states: up, middle, or down, with movement factors u, m, and d, where $m = 1$. The stock price in the next period is given by

$$S_{t+\Delta t} \in \{S_u, S_m, S_d\}$$

The trinomial transition parameters match the first two conditional moments of one lattice step to those of the risk-neutral GBM process asymptotically. Specifically, the moment conditions are expanded in powers of $\sqrt{\Delta t}$, and terms beyond the order required for first-order local consistency are neglected. The resulting parameterizations are then specified by the middle-movement probability adopted for the Boyle-type and Hull-type schemes.

The model-building process is carried out sequentially. The initial steps are (1) processing stock price data to obtain volatility, (2) establishing the GBM process, (3) transforming the model to the risk-neutral valuation measure, and (4) deriving the conditional moments of stock prices. These are followed by (5) matching the first two conditional moments with the trinomial process to first order in Δt, (6) determining the drift and probability parameters for the Boyle-type and Hull-type choices of the middle probability, and (7) adjusting stock prices for future dividends using the escrowed dividend approach. The final steps are (8) specifying the option payoff, (9) calculating option values via backward induction, and (10) testing convergence against the Black-Scholes model as a benchmark.

To ensure a fair comparison, the Boyle-type and Hull-type schemes use identical market inputs, maturity, dividend adjustment, and convergence tolerance. Thus, differences in the numerical results are attributable to the lattice parameterization and its discretization behavior rather than to changes in the underlying valuation assumptions. We implemented all computational steps in Python 3 on Google Colab, using NumPy for numerical testing of the trinomial model, SciPy for

the normal distribution function and log-return statistical tests, pandas for data processing, and Matplotlib for visualizing results. Table 1 summarizes the notation used throughout this study, where S_0^* denotes the stock price after the dividend adjustment in Eq. (11).

Table 1: Definition of model parameters

Parameter	Description
S_t	Stock price at time t
S_0	Stock price on the valuation date
S_0^*	Stock price after dividend adjustment
K	Exercise price
r	Risk-free rate (annualized, continuously compounded)
σ	Annual stock volatility
T	Time to maturity
Δt	Time interval
N	Number of time steps
λ	Stretch parameter ($\lambda = \sqrt{2}$ for Boyle-type, $\lambda = \sqrt{3}$ for Hull-type)
a	Movement parameter, $a = \lambda\sigma$
ν	Drift of log-price, $\nu = r - \sigma^2/2$
u, d, m	Up, down, and middle movement factors ($m = 1$, $d = 1/u$)
P_u, P_m, P_d	Probability of an up, middle, or down move
b	Drift-related parameter in $P_{u,d} = (1 - P_m)/2 \pm b\sqrt{\Delta t}$
D_j	Dividend no. j
$t_{ex,j}$	Time to the ex-dividend date of dividend no. j
$f_{i,k}$	Option value at node (i, k)

2.2. Exercise Assumptions and Scope

The MESOP Phase III.A is exercisable from 1 to 20 May 2026. This study assumes exercise on 20 May 2026 only, allowing the contract to be modeled as a European-style call with $T = 20/365$ years. This assumption provides a Black-Scholes benchmark and isolates the numerical differences between the Boyle and Hull lattices. Early exercise is not modeled because the available documents do not specify dividend entitlement or settlement timing before the 7 May 2026 ex-dividend date.

Regarding information availability, the Rp50 dividend had been approved at the Annual General Meeting on 27 April 2026, before the valuation date of 30 April 2026. In contrast, the ex-dividend date of 7 May 2026 and the payment schedule were announced only in the exchange disclosure dated 5 May 2026. The valuation therefore treats the dividend amount as known at t_0 and the ex-dividend date as an ex post assumption; the results should be read as conditional on this dividend information rather than as a strictly ex ante valuation as of 30 April 2026.

The options are treated as vested, with no forfeiture rate applied because no reliable contract-specific parameter is available. Dilution is excluded because Phase III.A uses treasury shares from the buyback, so no new shares are issued. Accordingly, the results represent the stated modeling approximation rather than the full contractual fair value of the MESOP.

2.2.1. Risk-Free Rate

The risk-free rate is proxied by the Bank Indonesia policy rate (BI-Rate) of 4.75% in force on the valuation date of 30 April 2026 (Bank Indonesia press release No. 28/84/DKom, 22 April 2026). Because the option horizon is only 20 days, a short-term money-market anchor is more appropriate than the yield of a long-dated government bond, whose term premium is not relevant for this horizon. The rate is treated as an annualized, continuously compounded rate in the discount factors $e^{-r\Delta t}$ and e^{-rT}, with $T = 20/365$. For this horizon, the continuously compounded and simple discount factors differ by only 3.4×10^{-6} in relative terms, so the compounding convention is immaterial.

As a robustness check, changing r by ± 0.5 percentage points changes the Black-Scholes value by about $\pm 0.014\%$ (in-the-money), $\pm 0.37\%$ (at-the-money), and $\pm 0.51\%$ (out-of-the-money). It changes the lattice relative errors at $N = 400$ by less than 0.001 percentage points, without altering the ranking of the two parameterizations. The BI-Rate is a policy rate and not a risk-free government yield; the use of alternative proxies (for example, short-tenor government securities) is left for future work.

2.3. Stock Price Data and Volatility Estimation

Stock volatility is a key parameter in the trinomial model because it determines the magnitude of price changes at each time interval. The data consist of the daily closing prices of PT XXX Corporindo Tbk shares from 2 May 2025 to 29 April 2026, obtained from Yahoo Finance using the Close series. The data were retrieved on 14 August 2026. The 237 price observations yield $n = 236$ daily log-returns. The closing price on the valuation date (30 April 2026) is used as S_0 and is not included in the volatility estimation.

Because the Close series represents raw closing prices, the historical returns are adjusted for cash dividends occurring within the estimation period. The log-return is defined as

$$R_t = \ln\left(\frac{S_t + D_t}{S_{t-1}}\right), \qquad t = 1, \ldots, n,$$

Here, D_t is the cash dividend per share going ex-dividend on day t. It equals $D_t = \text{Rp}50$ on 8 May 2025 and 6 August 2025 and $D_t = 0$ otherwise. The daily standard deviation is calculated as:

$$s_R = \sqrt{\frac{1}{n-1}\sum_{t=1}^{n}\left(R_t - \overline{R}\right)^2} \qquad t = 1, \ldots, n, \quad n = 236$$

After processing the data, 237 valid price observations were obtained, yielding 236 daily log-returns. The daily standard deviation of the dividend-adjusted log-returns is $s_R = 0.02405272$. Annualizing this standard deviation using 252 trading days gives $\sigma = 0.381825$, or $\sigma = 38.1825\%$. For comparison, the unadjusted return series yields an annualized volatility of 38.5920%. The difference reflects the mechanical price changes associated with dividend payments.

The dividends dated 8 May and 6 August 2025 fall within the historical estimation window and therefore affect volatility estimation rather than the future-dividend adjustment. Only the Rp50 dividend with an ex-dividend date of 7 May 2026 is treated as a future dividend in the option valuation. See Subsection 2.2 for the information availability assumption regarding the dividend.

2.4. Stock Pricing Model

2.4.1. Geometric Brownian Motion

Stock price movements are modeled using geometric Brownian motion. The model is expressed as the following stochastic differential equation under the risk-neutral probability measure.

$$dS_t = rS_t\,dt + \sigma S_t\,dW_t^{\mathbb{Q}} \tag{1}$$

Here, $W_t^{\mathbb{Q}}$ is a Wiener process under the risk-neutral probability measure.

2.4.2. Logarithmic Transformation Using Itô's Lemma

To obtain the logarithmic distribution of stock prices and the conditional moments required for moment matching, Itô's lemma is applied to the GBM process in Eq. (1) with $f(S_t) = \ln S_t$. Integration over the interval $[t, t + \Delta t]$ yields the following discrete form.

$$S_{t+\Delta t} = S_t \exp\left[\left(r - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\sqrt{\Delta t}\,Z\right], \qquad Z \sim N(0,1) \tag{2}$$

Using the moment-generating function of the normal distribution in Eq. (2), we obtain the first and second conditional moments of the process:

$$\mathbb{E}^{\mathbb{Q}}\left[S_{t+\Delta t} \mid S_t\right] = S_t e^{r\Delta t} \tag{3}$$

and

$$\mathbb{E}^{\mathbb{Q}}\left[S_{t+\Delta t}^2 \mid S_t\right] = S_t^2 e^{(2r+\sigma^2)\Delta t} \tag{4}$$

These two moments are the basis for moment matching in the construction of the trinomial lattice.

2.4.3. Construction of the Trinomial Process

At each time interval Δt, the stock price at a node can move to three states: up, middle, or down. The price movement factors are denoted by u, m and d so that

$$S_{t+\Delta t} = \begin{cases} S_t u, & \text{with a probability of } P_u, \\ S_t m, & \text{with a probability of } P_m, \\ S_t d, & \text{with a probability of } P_d. \end{cases}$$

To ensure that the tree is recombining, the following movement factors are used:

$$m = 1, \qquad u = e^{a\sqrt{\Delta t}}, \qquad d = e^{-a\sqrt{\Delta t}}$$

where a is a movement parameter determined by matching moments Eq. (3) and Eq. (4). The transition probabilities satisfy the normalization condition

$$P_u + P_m + P_d = 1$$

Consistency with the GBM process is achieved by equating the first and second moments of the trinomial process with those of the GBM process given in Eq. (3) and Eq. (4) together with the normalization $P_u + P_m + P_d = 1$. This gives

$$P_u u + P_m m + P_d d = e^{r\Delta t} + O(\Delta t^2), \tag{5}$$

and

$$P_u u^2 + P_m m^2 + P_d d^2 = e^{(2r+\sigma^2)\Delta t} + O(\Delta t^2), \tag{6}$$

where $O(\Delta t^2)$ denotes terms of order Δt^2 and higher. Eq. (5) and Eq. (6) are the moment-matching conditions for the trinomial process, imposed up to a residual of order Δt^2 per step. The conditions therefore hold to first order in Δt rather than as exact identities at finite Δt.

This system does not uniquely determine the lattice parameters. In the exact Boyle construction, additional restrictions are imposed on the node locations, namely $m = 1$, $ud = 1$, and $u = \exp(\lambda\sigma\sqrt{\Delta t})$ for a stretch parameter λ, after which the transition probabilities follow from the exact one-step moment equations. In the present study, we instead use the first-order asymptotic form of these probabilities, obtained by expanding the moment equations for small Δt and retaining the leading terms required for local consistency with the risk-neutral GBM. Boyle and Hull use different choices of the stretch parameter λ, equivalently of the middle-movement probability $P_m = 1 - 1/\lambda^2$, as part of the parameterization of their respective methods [1, 3]. Consequently, the transition probabilities used throughout this study are first-order asymptotic approximations of the moment-matching probabilities.

2.5. First-Order Asymptotic Boyle-Type and Hull-Type Parameterizations

The exact Boyle tree determines its transition probabilities by matching the first two conditional moments at a finite time step, once the recombining-node structure and the stretch parameter are specified. For the comparative analysis in this study, we use the corresponding first-order asymptotic representation. This representation provides a common local-consistency framework for examining the effect of the selected node spacing and middle-movement probability as Δt tends to zero. To obtain a probability distribution consistent with the moment-matching conditions, the probabilities of upward and downward movements are written as

$$P_u = \frac{1 - P_m}{2} + b\sqrt{\Delta t}, \qquad P_d = \frac{1 - P_m}{2} - b\sqrt{\Delta t}$$

where b is a drift-related parameter, constant in Δt, that controls the asymmetry between the up and down probabilities and is determined by the moment-matching conditions. By construction, these expressions satisfy $P_u + P_m + P_d = 1$ exactly. Expanding the exponential terms in Eq. (5) and Eq. (6) in a Taylor series, retaining terms up to order Δt, and then matching the moments of the trinomial process yields the following relationships:

$$\frac{1 - P_m}{2} a^2 + 2ab = r \tag{7}$$

and

$$2(1 - P_m)a^2 + 4ab = 2r + \sigma^2 \tag{8}$$

Solving the retained first-order coefficients gives

$$(1 - P_m)a^2 = \sigma^2, \qquad 2ab = \nu, \qquad \nu = r - \frac{\sigma^2}{2}.$$

With $\lambda = a/\sigma$, the resulting leading-order transition probabilities are

$$\begin{aligned} P_u &= \frac{1}{2\lambda^2} + \frac{\nu\sqrt{\Delta t}}{2\lambda\sigma}, \\ P_m &= 1 - \frac{1}{\lambda^2}, \\ P_d &= \frac{1}{2\lambda^2} - \frac{\nu\sqrt{\Delta t}}{2\lambda\sigma}. \end{aligned}$$

The terms neglected in the expansion are of order Δt^2 in the moment equations, so these probabilities differ from the exact finite-step Boyle probabilities by $O(\Delta t)$ for fixed λ. They are therefore asymptotic approximations, not identical to the exact probabilities at a fixed positive Δt.

For the Boyle-type parameterization, the middle-movement probability is chosen as $P_m = \frac{1}{2}$, which corresponds to $\lambda = \sqrt{2}$. Eq. (7) and Eq. (8) then yield

$$\frac{1}{4}a^2 + 2ab = r, \qquad a^2 + 4ab = 2r + \sigma^2$$

By eliminating the term b, we obtain $a^2 = 2\sigma^2$. Thus, the price movement parameter and factors are

$$a_B = \sigma\sqrt{2}, \qquad u_B = e^{\sigma\sqrt{(2\Delta t)}}, \qquad d_B = e^{-\sigma\sqrt{(2\Delta t)}}$$

and the first-order transition probabilities are:

$$\begin{aligned} P_{u,B} &= \frac{1}{4} + \left(r - \frac{1}{2}\sigma^2\right)\sqrt{\frac{\Delta t}{8\sigma^2}}, \\ P_{m,B} &= \frac{1}{2}, \\ P_{d,B} &= \frac{1}{4} - \left(r - \frac{1}{2}\sigma^2\right)\sqrt{\frac{\Delta t}{8\sigma^2}}. \end{aligned}$$

In the Hull-type parameterization, the probability of the middle (unchanged) movement is chosen as $P_m = \frac{2}{3}$ [3], which corresponds to $\lambda = \sqrt{3}$ and coincides with the Hull-White tree, i.e., the Boyle tree with $\lambda = \sqrt{3}$ up to order $O(\Delta t)$ [28]. Thus, the comparison concerns two selected first-order parameterizations with different node spacings. By substituting this value into Eq. (7) and Eq. (8), we obtain

$$\frac{1}{6}a^2 + 2ab = r, \qquad \frac{2}{3}a^2 + 4ab = 2r + \sigma^2$$

Combining these two relationships yields $a^2 = 3\sigma^2$, so

$$a_H = \sigma\sqrt{3}, \qquad u_H = e^{\sigma\sqrt{(3\Delta t)}}, \qquad d_H = e^{-\sigma\sqrt{(3\Delta t)}}$$

and the first-order transition probabilities are:

$$\begin{aligned} P_{u,H} &= \frac{1}{6} + \left(r - \frac{1}{2}\sigma^2\right)\sqrt{\frac{\Delta t}{12\sigma^2}}, \\ P_{m,H} &= \frac{2}{3}, \\ P_{d,H} &= \frac{1}{6} - \left(r - \frac{1}{2}\sigma^2\right)\sqrt{\frac{\Delta t}{12\sigma^2}}. \end{aligned}$$

Both parameterizations share the same first-order moment-matching structure but differ in the probability of the middle (unchanged) movement and the price-movement factor. Specifically, Boyle uses $P_m = 1/2$ with $\lambda = \sqrt{2}$, whereas Hull uses $P_m = 2/3$ with $\lambda = \sqrt{3}$. Write $a = \lambda\sigma$, so that

$$u = e^{\lambda\sigma\sqrt{\Delta t}}, \qquad d = e^{-\lambda\sigma\sqrt{\Delta t}}.$$

To express the transition probabilities of both parameterizations in a unified form, let $\nu = r - \frac{1}{2}\sigma^2$. Then

$$P_u = \frac{1}{2\lambda^2} + \frac{\nu\sqrt{\Delta t}}{2\lambda\sigma}, \quad P_d = \frac{1}{2\lambda^2} - \frac{\nu\sqrt{\Delta t}}{2\lambda\sigma}, \quad P_m = 1 - \frac{1}{\lambda^2}. \tag{9}$$

In Boyle's construction the stretch parameter λ is a free parameter with $\lambda > 1$, and values around $\lambda = 1.2$ have been reported to give the best results in numerical studies [29]. In this study we adopt $\lambda = \sqrt{2}$, which gives $P_m = 1/2$ in the first-order form Eq. (9), so that the Boyle-type scheme corresponds to the choice $P_m = 1/2$ attributed to Boyle [1] and can be compared with the Hull-type scheme ($\lambda = \sqrt{3}$) within the same family. The results for the Boyle-type scheme therefore refer to this specific choice of λ, and a different λ would change both the node spacing and the transition probabilities.

The probabilities in Eq. (9) are first-order asymptotic approximations to the moment-matching probabilities. If, for a fixed λ, the conditions $P_u + P_m + P_d = 1$, Eq. (5), and Eq. (6) were solved exactly at finite Δt with the same u and d Eq. (9), the probabilities would differ from Eq. (9) by a quantity of order Δt, which vanishes as $\Delta t \to 0$. For comparison, the exact one-step moment-matching probabilities of the Boyle tree are obtained by solving $P_u + P_m + P_d = 1$ together with the two moment conditions Eq. (5) and Eq. (6) exactly, for $u = e^{\lambda\sigma\sqrt{\Delta t}}$, $d = 1/u$, and $m = 1$. Writing $\rho = e^{r\Delta t}$ and $\kappa = e^{\sigma^2\Delta t}$, they are

$$\begin{aligned} P_u^{\mathrm{ex}} &= \frac{d - \rho(1+d) + \rho^2\kappa}{(u-d)(u-1)}, \\ P_d^{\mathrm{ex}} &= \frac{u - \rho(u+1) + \rho^2\kappa}{(u-d)(1-d)}, \\ P_m^{\mathrm{ex}} &= 1 - P_u^{\mathrm{ex}} - P_d^{\mathrm{ex}}. \end{aligned}$$

These probabilities satisfy the moment conditions exactly at any finite Δt. For small Δt they agree with the first-order probabilities Eq. (9) to leading order, and the difference is of order Δt.

The numerical experiments in this study use Eq. (9), not the exact probabilities above, for both parameterizations. Since $\lambda^2 \geq 1$, Eq. (9) implies $P_m \geq 0$. Both P_u and P_d are nonnegative if and only if

$$\frac{|\nu|\sqrt{\Delta t}}{\sigma} \leq \frac{1}{\lambda},$$

Because $\Delta t = T/N$, the above condition is equivalent to

$$N \geq \frac{T\lambda^2\nu^2}{\sigma^2}. \tag{10}$$

The most restrictive case occurs at the lowest volatility scenario, $\sigma = 10.8\%$. With $r = 0.0475$,

$$\nu = r - \frac{1}{2}\sigma^2 = 0.0475 - \frac{1}{2}(0.108)^2 = 0.041668,$$

and

$$\frac{\nu^2}{\sigma^2} = \frac{(0.041668)^2}{(0.108)^2} \approx 0.148853.$$

Using $T = 20/365 \approx 0.0548$ years, the lower bound for the Boyle parameterization with $\lambda^2 = 2$ is

$$N \geq \frac{20}{365}(2)(0.148853) \approx 0.01631,$$

whereas for the Hull parameterization with $\lambda^2 = 3$,

$$N \geq \frac{20}{365}(3)(0.148853) \approx 0.02447.$$

Thus, the transition probabilities are valid for all $N \geq 1$ under all volatility scenarios considered in this study. This result is consistent with the numerical verification for $N = 10$ to 1,200, where the minimum transition probabilities observed were 0.2399 for Boyle and 0.1584 for Hull. Furthermore, since

$$P_u + P_m + P_d = 1,$$

the nonnegativity of the three transition probabilities also guarantees that each probability does not exceed one. The condition in Eq. (10) is evaluated using the 20-day calendar-time horizon $T = 20/365$; if a different time convention is used, the condition must be recalculated accordingly. Therefore, no additional restriction on the minimum number of lattice periods is required. The condition in Eq. (10) is a nonnegativity condition for the asymptotic probabilities used in the numerical implementation; it does not convert them into the exact finite-step Boyle probabilities.

2.6. Treatment of Future Dividends

Future dividends are treated using the escrowed dividend approach [3]. Only one dividend falls within the option horizon, so $M = 1$, $D_1 = D = \text{Rp}50$, and $t_{ex,1} = t_{ex} = 7/365$ years, which is the time to the ex-dividend date (Table 2). The ex-dividend date is used as the dividend timing because the stock price falls by approximately the dividend amount on that date. Hence,

$$S_0^* = 1{,}580 - 50e^{-0.0475\times\frac{7}{365}} = 1{,}530.0455.$$

This approach separates the present value of known dividends from the stock price's stochastic component. For M future dividends going ex-dividend at times $t_{ex,j}$, the present value of each dividend is given by

$$PV(D_j) = D_j e^{-rt_{ex,j}},$$

so that the stock price used as the initial value in the lattice is adjusted to

$$S_0^* = S_0 - \sum_{j=1}^{M} D_j e^{-rt_{ex,j}}. \tag{11}$$

Here, S_0 denotes the stock price on the valuation date. The adjustment is applied once at the initial node so that the present value of the known dividend is not subtracted repeatedly at each lattice node [3].

2.7. Payoff and Backward Induction

Under the exercise assumption in Subsection 2.2, the MESOP is modeled as a European-style call option exercised only at maturity T (20 May 2026), with payoff

$$f_{N,k} = \max(S_{N,k} - K, 0), \qquad k = -N, ..., N$$

where K is the strike price and k is the node index. With the initial price adjusted for dividends in Eq. (11), the pricing structure in the lattice is expressed as

$$S_{i,k} = S_0^* u^k, \qquad i = 0, ..., N, \qquad k = -i, ..., i \tag{12}$$

The option value at each node is obtained through backward induction, using the node prices in Eq. (12)

$$\begin{aligned} f_{i,k} = e^{-r\Delta t} \left[P_u f_{i+1,k+1} + P_m f_{i+1,k} + P_d f_{i+1,k-1} \right], \\ i = N-1, \ldots, 0, \qquad k = -i, \ldots, i. \end{aligned} \tag{13}$$

Eq. (13) is applied to both Boyle and Hull with their respective lattice parameters.

2.8. The Black-Scholes Model as a Benchmark

The Black-Scholes model is used as the analytical benchmark. The initial stock price S_0^* is adjusted for dividends using Eq. (11). The value of a European call is given by Eq. (14).

$$C_{BS} = S_0^* N(d_1) - K e^{-rT} N(d_2) \tag{14}$$

where

$$d_1 = \frac{\ln\left(\frac{S_0^*}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T}$$

2.9. Convergence Test

Let $V_\ell^{(N)} = f_{0,0}$ denote the option value obtained from the trinomial lattice with N time steps, where $\ell \in \{\text{Boyle}, \text{Hull}\}$ and $\Delta t = T/N$. Let C_{BS} denote the corresponding Black-Scholes value given in Eq. (14). The convergence analysis is performed over the lattice grid

$$\mathcal{N} = \{10, 40, 80, 160, 400, 620, 1{,}200\},$$

for each parameterization and each strike condition considered in the study, namely $K \in \{499, 1{,}530, 1{,}630\}$. The relative error at each lattice level is defined by

$$RE_\ell^{(N)} = \frac{\left| V_\ell^{(N)} - C_{BS} \right|}{C_{BS}} \times 100\%. \tag{15}$$

Convergence is evaluated using the relative error in Eq. (15) and a predetermined tolerance of $\varepsilon = 0.05\%$. The convergence level N_ℓ^* is defined as the smallest $N \in \mathcal{N}$ such that $RE_\ell^{(N')} \leq \varepsilon$ for every $N' \in \mathcal{N}$ with $N' \geq N$. Because N^* is defined on the tested grid, a value equal to the first grid level ($N^* = 10$) means that the tolerance holds at every tested level, and a value equal to the last grid level ($N^* = 1{,}200$) means that the tolerance holds only at the last tested level; in both cases, N^* is grid-limited. If no tested level satisfies the criterion, N^* is reported as exceeding the largest tested level. This criterion prevents convergence from being declared solely because the error becomes temporarily small at a particular lattice level, which is important when the numerical error oscillates with the position of the strike relative to the lattice nodes.

The empirical order p is defined as the negative of the least-squares slope of $\ln(\text{RE})$ against $\ln(N)$ over the grid

$$N \in \{10, 40, 80, 160, 400, 620, 1{,}200\}.$$

A value close to one indicates behavior consistent with first-order convergence, namely $O(1/N)$. When the relative error decreases regularly with N, the estimated slope is interpreted as an empirical convergence order. In contrast, when the error exhibits oscillatory behavior, the resulting slope is reported only as a descriptive statistic and is not interpreted as a formal convergence rate.

The difference between the two trinomial parameterizations is evaluated at the common reference level $N_{\text{ref}} = 1{,}200$. The absolute difference is defined as

$$\Delta V = \left| V_{\text{Boyle}}^{(N_{\text{ref}})} - V_{\text{Hull}}^{(N_{\text{ref}})} \right|.$$

To provide a neutral comparison, the relative difference is normalized by the Black-Scholes benchmark rather than by either trinomial value, as shown in Eq. (16):

$$\Delta V_{\text{rel}} = \frac{\Delta V}{C_{BS}} \times 100\%. \tag{16}$$

These measures are used to evaluate the convergence behavior of the two trinomial parameterizations, their numerical sensitivity to lattice discretization, and the extent to which the resulting option values differ under the same contract and market assumptions.

2.10. Strike Position Relative to Lattice Nodes

At maturity, the log-price spacing between adjacent lattice nodes is $\lambda_\ell \sigma \sqrt{\Delta t}$, where $\lambda_\ell = \sqrt{2}$ for the Boyle parameterization and $\lambda_\ell = \sqrt{3}$ for the Hull parameterization. The position of the strike in node units is defined as

$$x_\ell(N) = \frac{\ln(K/S_0^*)}{\lambda_\ell \sigma \sqrt{\Delta t}} = \frac{\ln(K/S_0^*)\sqrt{N}}{\lambda_\ell \sigma \sqrt{T}}.$$

The distance of the strike from the nearest lattice node is measured by

$$\delta_\ell(N) = \min_{k \in \{-N,\ldots,N\}} |x_\ell(N) - k|.$$

This distance is defined when $|x_\ell(N)| \leq N$. Thus, $\delta_\ell(N) = 0$ indicates that the strike coincides exactly with a terminal lattice node. If $|x_\ell(N)| > N$, the strike lies outside the range of the terminal nodes; $\delta_\ell(N)$ is then not defined (n.a.), and the strike-position effect is absent because the payoff is linear at all terminal nodes (all terminal nodes are in-the-money for $K = \text{Rp}499$ when $N = 10$ and 40). Because both N and λ_ℓ determine the node spacing, the same strike may occupy different positions in the Boyle and Hull lattices and may move relative to the nodes as N changes. To distinguish the strike-position effect from moneyness, the relative error is additionally evaluated over a fine strike grid $K = 1{,}400, 1{,}401, \ldots, 1{,}700$ at fixed lattice levels. The association between $\delta_\ell(N)$ and the relative error is examined descriptively through the diagnostics in Table 4 and the fine strike-grid analysis, without formally quantifying the relationship.

3. Results and Discussion

The observed stock price series fluctuated throughout the estimation period, with a marked decline followed by an increase toward the valuation date. Based on 237 valid price observations, we obtained 236 dividend-adjusted daily log-returns, with a daily standard deviation of 0.02405272 and an annualized volatility of 38.1825% using 252 trading days. The adjusted log-returns exhibit skewness of 0.2465, excess kurtosis of 1.2758, and a Jarque-Bera statistic of 18.3959 with a p-value of 0.00010, indicating a statistically significant departure from normality. These results provide the empirical volatility input for the lattice models, while the non-normality of returns is acknowledged when interpreting the GBM-based valuation. The parameter values used in the valuation are presented in Table 2.

Table 2: Parameter values used in the valuation

Parameter	Value	Information
t_0	30 April 2026	Valuation date (assumed by the authors)
S_0	Rp1,580	Closing price of PT XXX Corporindo Tbk shares on the valuation date (Yahoo Finance, Close series; retrieved on 14 August 2026)
D	Rp50	Final cash dividend per share for fiscal year 2025, approved at the issuer's Annual General Meeting on 27 April 2026 (before t_0); ex-dividend date and payment schedule announced in the exchange disclosure dated 5 May 2026 (after t_0)
t_{ex}	0.01918 years	Time to the ex-dividend date of 7 May 2026 (7/365 years); announced after t_0 and treated as known at t_0 as an ex post modeling assumption.
K	Rp499	Exercise price, MESOP Phase III.A, issuer's public disclosure to the Financial Services Authority (OJK) dated 20 March 2024, Section II
r	4.75%	BI-Rate in force on the valuation date, annualized and continuously compounded (Bank Indonesia press release No. 28/84/DKom, 22 April 2026)
σ	38.1825%	Annualized historical volatility from 236 dividend-adjusted daily log-returns (237 closing prices, 2 May 2025 to 29 April 2026; 252 trading days)
T	0.0548 years	Time to maturity (20/365 years); exercise period of Phase III.A is 1-20 May 2026, and exercise is assumed on 20 May 2026

3.1. Results of the Convergence Tests for the Boyle-Type and Hull-Type Trinomial Methods

We conducted convergence tests by increasing the number of time steps N from 10 to 1,200. The option values obtained through the Boyle and Hull parameterizations were compared with the Black-Scholes values as an analytical benchmark. The tests were conducted under three strike-price assumptions: K = Rp499 as the actual contract condition, K = Rp1,530 as the "at-the-money" (ATM) condition based on the stock price after dividend adjustment S_0^*, and K = Rp1,630 as the "out-of-the-money" (OTM) condition. Table 3 presents the relative errors at several selected discretization levels to illustrate the convergence pattern.

Table 3: Relative errors of the Boyle-type and Hull-type trinomial methods

N	ITM (K = Rp499)			ATM (K = Rp1,530)			OTM (K = Rp1,630)		
	RE_{Boyle} (%)	RE_{Hull} (%)	$\lvert V_B - V_H \rvert$	RE_{Boyle} (%)	RE_{Hull} (%)	$\lvert V_B - V_H \rvert$	RE_{Boyle} (%)	RE_{Hull} (%)	$\lvert V_B - V_H \rvert$
10	2.63×10^{-5}	1.44×10^{-5}	0.000123	1.191291	2.469957	0.722467	2.145705	0.946085	0.250225
40	6.57×10^{-6}	3.59×10^{-6}	3.08×10^{-5}	0.297540	0.601665	0.171836	0.073764	0.632089	0.147232
80	3.29×10^{-6}	1.79×10^{-6}	1.54×10^{-5}	0.148149	0.298939	0.085199	0.271105	0.248744	0.004664
160	1.64×10^{-6}	8.97×10^{-7}	7.71×10^{-6}	0.073581	0.148589	0.042380	0.096720	0.056390	0.031937
400	6.57×10^{-7}	3.59×10^{-7}	3.08×10^{-6}	0.029026	0.058872	0.016863	0.072307	0.019123	0.011093
620	4.24×10^{-7}	2.31×10^{-7}	1.99×10^{-6}	0.018550	0.037755	0.010851	0.034770	0.011166	0.009582
1,200	2.19×10^{-7}	1.20×10^{-7}	1.03×10^{-6}	0.009403	0.019280	0.005581	0.013700	0.008826	0.004699

Note. Nonzero values below 10^{-4} are expressed in scientific notation, whereas all other values are reported to six decimal places. The quantity $|V_B - V_H|$ represents the absolute difference between the Boyle-type and Hull-type option values for a given N.

Table 3 shows that the relative error generally decreases as the lattice resolution N increases, although non-monotonic behavior appears at several intermediate lattice levels, particularly for the OTM condition. For the ITM condition, both Boyle and Hull produce extremely small relative errors even at $N = 10$, indicating close agreement with the Black-Scholes benchmark. In the ATM and OTM conditions, larger errors are observed at lower N and gradually decline as the lattice is refined.

For OTM, the Boyle error is non-monotonic between $N = 40$ ($RE = 0.073764\%$) and $N = 80$ ($RE = 0.271105\%$), remains above the 0.05% tolerance at $N = 400$ ($RE = 0.072307\%$), and satisfies it at $N = 620$ ($RE = 0.034770\%$). Hull, in contrast, exceeds the tolerance at $N = 160$

($RE = 0.056390\%$) and satisfies it at $N = 400$ ($RE = 0.019123\%$). Overall, the results indicate that increasing lattice resolution improves agreement with the Black-Scholes benchmark, while the magnitude and behavior of the error depend on both moneyness and trinomial parameterization.

Table 4: Convergence and strike position diagnostics

Method	Condition	K	N^*	p	$\delta_\ell(10)$	$\delta_\ell(40)$	$\delta_\ell(80)$	$\delta_\ell(160)$	$\delta_\ell(400)$	$\delta_\ell(620)$	$\delta_\ell(1{,}200)$
Boyle-type	ITM	499	10	0.999982	n.a.	n.a.	0.284468	0.125170	0.285460	0.281382	0.067424
Hull-type	ITM	499	10	1.000008	n.a.	n.a.	0.264503	0.450182	0.247028	0.215997	0.280498
Boyle-type	ATM	1,530	400	1.011109	0.000744	0.001489	0.002106	0.002978	0.004708	0.005862	0.008155
Hull-type	ATM	1,530	620	1.012583	0.000608	0.001216	0.001719	0.002431	0.003844	0.004786	0.006658
Boyle-type	OTM	1,630	620	0.862718	0.416800	0.166400	0.477965	0.332799	0.013035	0.466128	0.343085
Hull-type	OTM	1,630	400	1.129102	0.292677	0.414646	0.343757	0.170709	0.175608	0.178550	0.160569

Note: N^* denotes the first tested lattice resolution at which the relative error satisfies the 0.05% tolerance and remains within the tolerance at all subsequent tested resolutions. The value $N^* = 10$ for both ITM parameterizations corresponds to the first tested resolution. The notation n.a. indicates that $\delta_\ell(N)$ is undefined because the strike lies outside the range of terminal lattice nodes.

Table 4 summarizes the convergence levels, empirical convergence orders, and strike-position diagnostics of the Boyle and Hull parameterizations. For the ITM condition, both methods meet the 0.05% relative-error tolerance at the first tested resolution ($N^* = 10$). For ATM, Boyle and Hull first meet the tolerance at $N^* = 400$ and $N^* = 620$, respectively, while for OTM, the corresponding values are $N^* = 620$ and $N^* = 400$. Specifically, the Boyle-OTM relative error remains above the tolerance at $N = 400$ (0.072307%) but falls below it at $N = 620$ (0.034770%), whereas the Hull-OTM relative error remains above the tolerance at $N = 160$ (0.056390%) but falls below it at $N = 400$ (0.019123%). Thus, within the tested grid, Boyle reaches the specified relative-error tolerance at an earlier tested lattice resolution than Hull for the ATM condition, whereas Hull does so earlier than Boyle for the OTM condition; neither parameterization dominates across all moneyness conditions.

The empirical convergence orders p are close to one for ITM and ATM, but differ under OTM (0.862718 for Boyle and 1.129102 for Hull). Because the Boyle relative-error sequence under OTM is non-monotonic, its value is interpreted as a descriptive measure rather than a formal convergence order, whereas the Hull sequence decreases monotonically over the tested grid. Meanwhile, the values of $\delta_\ell(N)$, where defined, reflect changes in strike-node alignment across lattice resolutions. For the ITM condition at $N = 10$ and $N = 40$, $\delta_\ell(N)$ is not defined because the strike lies outside the range of terminal lattice nodes, and the payoff is linear at all terminal nodes. This diagnostic provides additional insight into the numerical behavior but does not establish a causal relationship with pricing error. Overall, numerical accuracy depends on the joint effects of moneyness, lattice resolution, parameterization, and strike-node alignment.

Fig. 1 shows that both parameterizations converge monotonically under the ITM condition, with the relative errors remaining several orders of magnitude smaller than those observed for ATM and OTM strikes. The near-identical values indicate that the actual contract condition provides limited scope for distinguishing between the two parameterizations because its payoff is dominated by intrinsic value. Thus, the close agreement in the ITM case should be interpreted as a characteristic of the contract's moneyness rather than as evidence that Boyle and Hull are numerically equivalent in general.

Fig. 2 exhibits the most regular convergence pattern among the examined non-ITM conditions. The strike $K = 1{,}530$ is almost identical to the dividend-adjusted initial price $S_0^* = 1{,}530.0455$, placing it effectively at the middle lattice level. This alignment reduces the payoff approximation's sensitivity to changes in N and produces a smooth convergence path. Boyle remains closer to the Black-Scholes benchmark than Hull throughout the tested discretization levels, indicating that the two parameterizations have different error magnitudes even when the strike is favorably aligned with the lattice.

Fig. 3(a) and Fig. 3(b) show that the OTM condition produces the clearest divergence in the

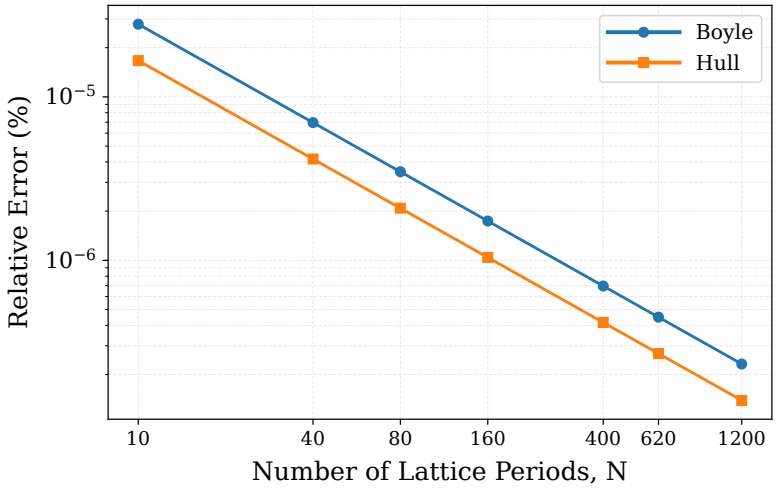


Fig. 1: Relative error to the Black-Scholes benchmark for the ITM condition (K = Rp499)

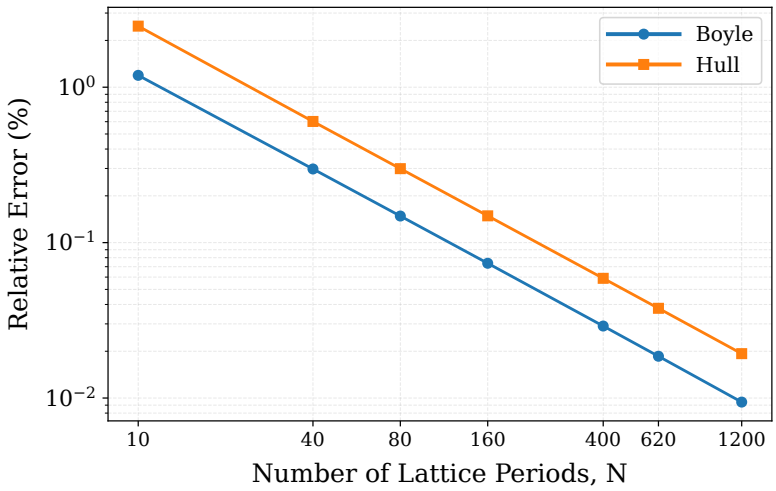


Fig. 2: Relative error to the Black-Scholes benchmark for the ATM condition (K = Rp1,530)

convergence paths of Boyle and Hull. For the OTM condition, Hull is closer to the Black-Scholes benchmark at six of the seven tested lattice resolutions, whereas Boyle is closer only at $N = 40$. Accordingly, Hull reaches the 0.05% tolerance at $N^* = 400$, whereas Boyle does so at $N^* = 620$. The non-monotonic behavior, particularly for Boyle between $N = 40$ and $N = 80$, indicates that the error depends on the strike's position relative to the discrete lattice nodes rather than on N alone. This finding matters because it shows that the apparent ranking of the two parameterizations can change with lattice resolution, making a single-N comparison insufficient to characterize their numerical performance.

3.2. Moneyness and Strike-Position Analysis

A moneyness comparison was conducted to examine how the strike price's position relative to the stock price affects valuation results. The comparison also examines the consistency of the two trinomial parameterizations. The results for $N = 1{,}200$ are summarized in Table 5.

At the fixed discretization level $N = 1{,}200$, Table 5 isolates the difference between Boyle and Hull across the examined strike conditions. The comparison shows that the parameterization ranking is not uniform: the two parameterizations are indistinguishable for the ITM condition, whereas Boyle is closer for the ATM condition. Hull is closer for the OTM condition. The larger separation observed outside the deep-ITM regime indicates that the choice of parameterization becomes more visible when the option value depends increasingly on time value. However,

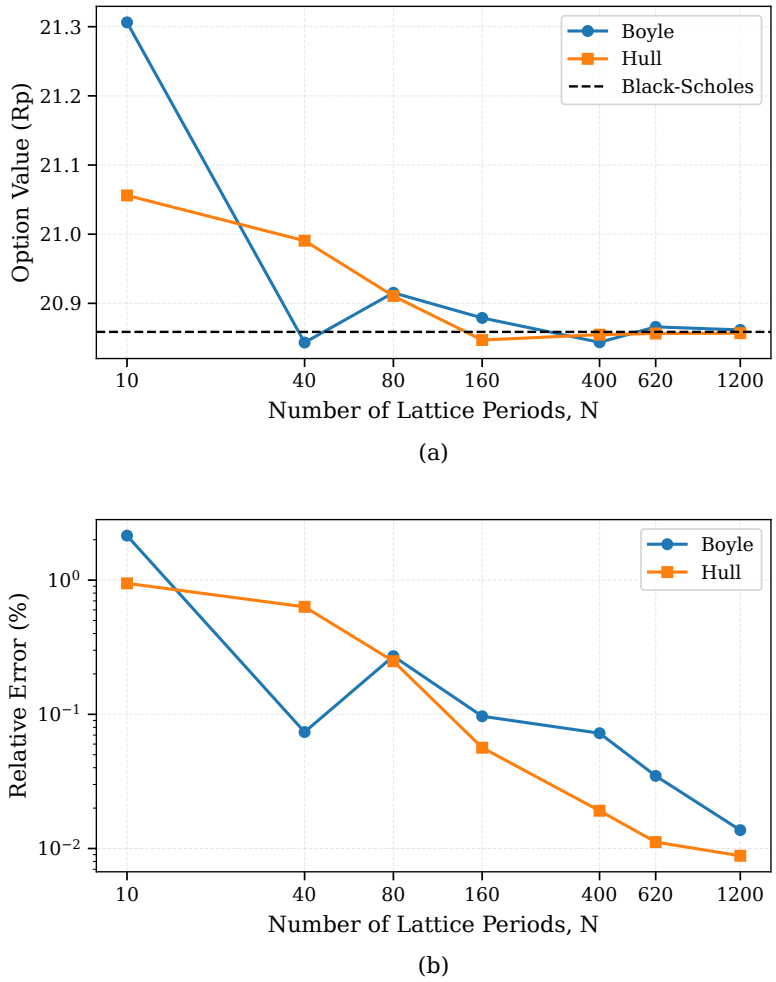


Fig. 3: Convergence for the OTM condition (K = Rp1,630): (a) option value and (b) relative error versus the number of lattice periods N.

Table 5: Option values and errors at $N = 1{,}200$

	ITM (K = Rp499)	ATM (K = Rp1,530)	OTM (K = Rp1,630)
Intrinsic value	1,031.045527	0.045527	0.000000
C_{BS}	1,032.342606	56.501657	20.858707
Boyle value	1,032.342603	56.496344	20.861565
Abs. error (Boyle)	2.26×10^{-6}	0.005313	0.002858
RE_{Boyle} (%)	2.19×10^{-7}	0.009403	0.013700
Hull value	1,032.342604	56.490763	20.856866
Abs. error (Hull)	1.23×10^{-6}	0.010894	0.001841
RE_{Hull} (%)	1.20×10^{-7}	0.019280	0.008826

Note: Nonzero values below 10^{-4} are shown in scientific notation. The intrinsic value of the OTM option is exactly zero.

moneyness alone cannot explain the observed differences because the strike also occupies different positions relative to the lattice nodes. This motivates the fine strike-grid analysis used to distinguish moneyness from strike-position effects.

3.3. Volatility Sensitivity Analysis

Sensitivity to volatility was analyzed at three moneyness levels to examine the effect of changes in σ on option values. The historical volatility in this study was 38.1825%, which was used as a reference in constructing volatility scenarios. The test volatility levels are expressed as a

proportion of historical volatility, resulting in scenarios ranging from 10.8% to 46.2673%. The historical volatility of 38.1825% was used as one of the scenarios to present results under actual conditions. The calculation results are presented in Table 6. Table 6 suggests that volatility

Table 6: Option values under different volatility levels

σ (%)	ITM Boyle	ITM Hull	ATM Boyle	ATM Hull	OTM Boyle	OTM Hull
10.8000	1,032.342602	1,032.342602	17.504211	17.502687	0.108243	0.108266
20.8678	1,032.342603	1,032.342604	31.826291	31.823281	3.960192	3.958755
30.8448	1,032.342604	1,032.342606	46.041011	46.036522	12.737723	12.738635
34.7004	1,032.342604	1,032.342605	51.534922	51.529859	16.879600	16.885768
38.1825	1,032.342603	1,032.342604	56.496344	56.490763	20.861565	20.856866
42.4117	1,032.342602	1,032.342602	62.521518	62.515307	25.919379	25.916775
46.2673	1,032.342600	1,032.342599	68.013542	68.006754	30.697712	30.697724

sensitivity varies substantially across moneyness conditions. The ITM contract remains relatively stable across the examined volatility range, whereas the ATM and OTM contracts exhibit markedly greater responses. This pattern indicates that volatility plays a more influential role in determining option value when the contract is not strongly supported by intrinsic value. The results therefore highlight moneyness as an important factor in interpreting volatility sensitivity in trinomial-based ESO valuation.

3.4. Strike-Position Sensitivity Analysis

Fig. 4 presents the relative error of the Boyle and Hull trinomial parameterizations with respect to the Black-Scholes benchmark over a fine strike-price grid from $K = 1{,}400$ to $K = 1{,}700$. The analysis is conducted at two lattice resolutions, $N = 400$ and $N = 1{,}200$. It examines whether the numerical accuracy of each parameterization depends only on the broad moneyness classification or is also affected by the strike's position relative to the discrete lattice nodes. Fig. 4 shows

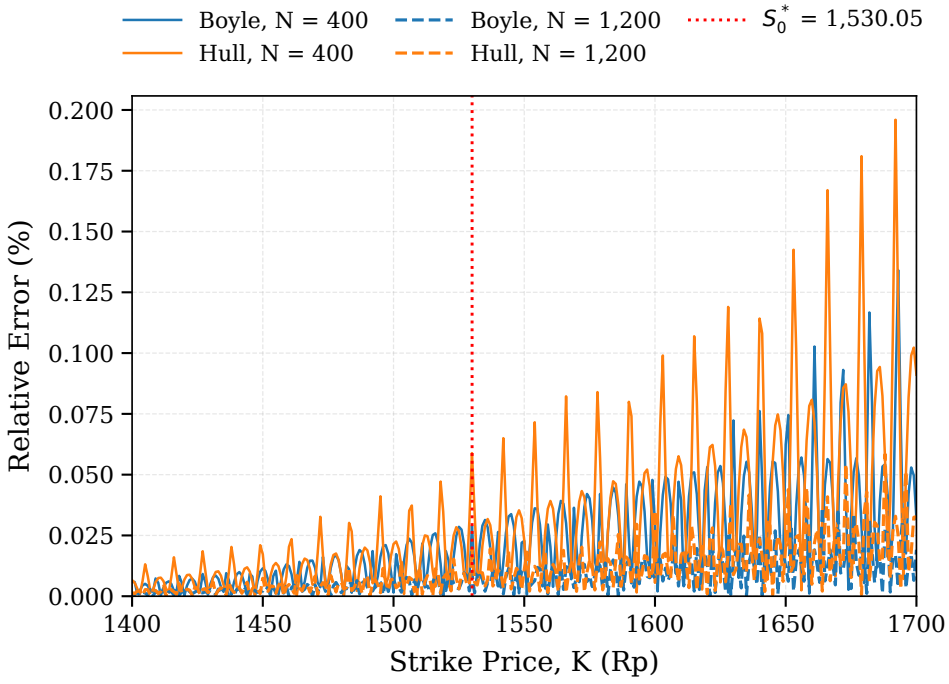


Fig. 4: Relative error with respect to the Black-Scholes benchmark across strike price

that the relative error varies substantially across the strike-price grid even when the valuation parameters and lattice resolution are fixed. The oscillatory patterns indicate that numerical

accuracy is not determined solely by broad moneyness classification. The relative ranking of the Boyle and Hull parameterizations also changes across the strike grid: Boyle produces a lower relative error for 63.46% of the 301 strikes at $N = 400$ and 64.12% at $N = 1{,}200$.

Its mean relative error is approximately 29.3% and 28.7% lower than that of Hull at $N = 400$ and $N = 1{,}200$, respectively; however, Hull remains more accurate for a substantial proportion of strikes. Therefore, the results do not support uniform superiority of either parameterization. Instead, they suggest that numerical performance varies with the interaction between lattice resolution and strike location. The observed patterns should not, however, be interpreted as direct evidence of a causal relationship between $\delta_\ell(N)$ and relative error, since the present analysis does not explicitly quantify this relationship.

3.5. Discussion

The near first-order behavior observed for the ITM and ATM conditions is consistent with the convergence rates reported for trinomial prices in [7, 8]. For the OTM condition, however, the Boyle relative error oscillates with N (Fig. 3), so its fitted slope is not a convergence order. This agrees with the role of lattice-node positions discussed in [9] and extends it by showing that the same strike can occupy different node-relative positions under the Boyle and Hull parameterizations.

Eq. (9) shows that the two parameterizations differ only through λ, which sets the node spacing $\lambda\sigma\sqrt{\Delta t}$. At the same N, the Hull lattice is therefore coarser by a factor of $\sqrt{\frac{3}{2}}$, which may explain its larger error at a given N and the change in ranking across strikes shown in Fig. 4. The numerical comparison should be interpreted within the first-order asymptotic framework adopted in this study. Since λ determines both the leading-order transition probabilities and the spacing of adjacent lattice nodes, the observed differences between the Boyle-type and Hull-type schemes cannot be attributed solely to the middle-movement probability; they reflect the combined effect of the stretch parameter, the transition probabilities, and the strike location relative to the lattice. Compared with earlier ESO and discrete-dividend lattice studies [16, 20, 22], which focus on a single parameterization or on moneyness alone, the present results indicate that the choice of parameterization and the strike position should be assessed together.

4. Conclusion

This study suggests that the Boyle and Hull trinomial parameterizations exhibit distinct numerical behavior in employee stock option valuation with a discrete dividend. Their relative performance depends on the interaction between lattice resolution and strike position: both parameterizations approach the Black-Scholes benchmark as the lattice is refined, but their convergence paths and numerical accuracy differ across the examined ITM, ATM, and OTM conditions. The results therefore show that a single lattice level is insufficient to characterize the numerical behavior of the two parameterizations. For the actual MESOP Phase III.A contract, which is deeply in-the-money under the stated assumptions, Boyle and Hull produce virtually identical option values that closely match the Black-Scholes benchmark.

In contrast, the ATM and OTM analyses reveal a more pronounced parameterization effect, while the strike-grid analysis shows that the relative advantage of Boyle or Hull can change with the strike's position within the lattice. The principal contribution of this study is therefore not to identify a universally superior trinomial method, but to demonstrate that the numerical performance of the Boyle-type and Hull-type parameterizations is jointly influenced by lattice resolution and strike position. This finding provides a more informative basis for comparing trinomial parameterizations in ESO valuation than relying on a single final option value.

Nevertheless, this study is limited by the use of historical volatility, a single issuer, a single valuation date, and a single MESOP phase. Additional limitations include the assumption that the dividend information announced on 5 May 2026 was known at the valuation date, the

European-style approximation, the escrowed-dividend treatment, and first-order asymptotic transition probabilities. The numerical findings are specific to the first-order parameterizations adopted here and should not be interpreted as a comparison against the exact finite-step Boyle probability construction. Future research should compare the first-order probabilities used here with exact moment-matching probabilities and incorporate employee exit and early exercise, alternative volatility specifications, and broader sets of issuers, valuation periods, and contract conditions to further assess the robustness of the Boyle–Hull comparison.

CRediT Authorship Contribution Statement

Maeva Wulandari contributed to conceptualization, methodology, software, validation, formal analysis, investigation, data curation, and visualization. Maeva Wulandari also contributed to writing the original draft, review, and editing. **Rudianto Artiono** contributed to supervision, validation, review, and editing.

Declaration of Generative AI and AI-assisted Technologies

The authors used ChatGPT during the preparation of this manuscript. They also used Claude. These tools were used for language refinement and proofreading.

Declaration of Competing Interest

The authors declare no competing interests. This declaration applies to both authors. Neither author reports a competing interest in relation to this study.

Funding and Acknowledgments

This research was funded by personal funds. The authors did not receive any specific grants from public funding agencies. They also received no specific grants from commercial or non-profit funding agencies.

Ethical Considerations

This study uses publicly available market data and publicly disclosed corporate documents. It does not involve human participants or animals. It also does not use personal data.

Data and Code Availability

The daily closing prices (Close series) were obtained from Yahoo Finance (https://finance.yahoo.com/) and retrieved on 14 August 2026; the 237 observations used for volatility estimation span 2 May 2025 to 29 April 2026, and the closing price on 30 April 2026 (Rp1,580) is used as S_0. To preserve the issuer's anonymity, the raw price series is not reproduced. The remaining inputs are reported in Table 2, while the additional strike-price scenarios are specified in Table 3. The valuation incorporates a single dividend of Rp50, with an ex-dividend date of 7 May 2026 ($t_{\text{ex}} = 7/365$), and uses $\sigma = 0.381825$.

The calculations are deterministic (no random numbers) and were implemented in Python 3 with NumPy, SciPy, pandas, and Matplotlib on Google Colab. Returns are computed on prices sorted from oldest to newest using

$$R_t = \ln\left(\frac{S_t + D_t}{S_{t-1}}\right).$$

The sample standard deviation uses $n - 1$ with $n = 236$, and annualization uses $\sqrt{252}$. The lattices use the first-order probabilities in Eq. (9) with $\lambda = \sqrt{2}$ and $\lambda = \sqrt{3}$, the initial price S_0^*

in Eq. (11), backward induction in Eq. (13), and the benchmark in Eq. (14). The strike grid for Fig. 4 is $K = 1{,}400, 1{,}401, \ldots, 1{,}700$. The processed data and the Python code are available from the corresponding author upon reasonable request.

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