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Fundamental Solution of Elliptic Equation with Positive Definite Matrix Coefficient

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ABSTRACT

We study the fundamental solution of elliptic equations with real constant coefficients

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} D_i(D_j u) = 0,$$

where a_{ij} is a positive definite matrix. We obtained by searching the radial solution such that we solved the equation into ordinary differential equations.

Keywords: fundamental solution, elliptic equation, positive definite matrix

INTRODUCTION

We consider the linear differential operator

$$Lu = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u, \qquad (1)$$

with coefficient defined in an $n-$dimensional domain D, where $b_i(x)$ and $c(x)$ are any functions that depend on x. L is said to be of elliptic type (or elliptic) at a point x^0 if the matrix $a_{ij}(x^0)$ is positive definite, i.e., if for any real vector $\xi \neq 0, \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\xi_i\xi_j > 0$ [3]. Laplace's equation $\Delta u = 0$ is the simplest and most basic example of elliptic equation, and the *Laplacian* of u is $\Delta u = \sum_{i=1}^{n} u_{x_i x_i}$ [1]. While, a symmetric matrix A is called a positive definite matrix if a quadratic form $x^T\, Ax > 0$ for all $x \neq 0, x \in \mathbb{R}$ [5].

A fundamental solution of the differential operator L in Ω is a function $K(x,z)$ defined for $x\epsilon\Omega, z\epsilon\Omega, x \neq z$ and satisfying the following property: For any bounded domain $\mathbb{R}$ with smooth boundary $\partial\mathbb{R}$ and for any $z\epsilon\mathbb{R}$,

$$u(z) = \int_{\mathbb{R}} K(x,z)\overline{L^*\overline{u(x)}}dx,$$

for every $u \in C_0^m(\mathbb{R})$, and L^* is *formal adjoint* of L [4].

Here, we define the following equation as elliptic equation with positive definite matrix

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} D_i(D_j u) = 0, \qquad (2)$$

where a_{ij} is element of positive definite matrix, $u = u(x)$ and $x \in U$, $U \subset \mathbb{R}^n$. In 2016, Fitri studied the Holder regularity of weak solutions to linear elliptic partial differential equations with continuous coefficients, Campanato type estimates are obtained for the validity of regularity of solutions (see [2]). In this paper, we study the fundamental solution of elliptic equation with positive definite matrix coefficient. Due to the symmetry of elliptic equation, radial solutions are natural to look for since the given partial differential equation can be reduced to an ordinary differential equation which is easier to solve. In this way, we can reduce the higher dimensional problems to one dimensional problems.

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RESULTS AND DISCUSSION

Let the elliptic equation with positive definite matrix as in (2) then to find a solutions u of elliptic equations , it consequently seems advisable to search for radial solutions, that is functions of r.

Theorem 3.1 Let

$$r = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}x_i x_j}\,,$$

where b_{ij} is element of coffactor matrix a_{ij}. Then partial derivative of r with respect to x_j and partial derivative of r with respect to $x_i x_j$ are defined by

$$\frac{\partial r}{\partial x_j} = \frac{1}{r}\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}x_i,$$

and

$$\frac{\partial^2 u}{\partial x_i \partial x_j} = \frac{1}{r}\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij} - \frac{1}{r^3}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\sum_{k=1}^{n} b_{ik}x_k \sum_{k=1}^{n} b_{jk}x_k\right). \tag{3}$$

Theorem 3.2 Suppose

$$u\,(x) \;=\; v\,(r)\,, \tag{4}$$

then the second derivative of u with respect to $x_i x_j$ denoted $D_i\big(D_j u\big) = \frac{\partial^2 u}{\partial x_i \partial x_j}$ is the total derivative of $v\,(r)$ respect to $x_i x_j$.

Corollary 3.3 If the second derivative of (4) is substituted into the equation (2) then we get

$$\frac{v''(r)}{r^2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\left(\sum_{k=1}^{n} b_{ik}x_k \sum_{k=1}^{n} b_{jk}x_k\right) + \frac{v'(r)}{r}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}b_{ij} - \frac{v'(r)}{r^3}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\left(\sum_{k=1}^{n} b_{ik}x_k \sum_{k=1}^{n} b_{jk}x_k\right) = 0. \tag{5}$$

Corollary 3.4 If A is positive definite matrix with a_{ij} element, and B is cofactor matrix of A with b_{ij} element then

$$I(\det A) = AB.$$

Corollary 3.5 If A matrix and B matrix are symmetric, then $\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}b_{ij}$ can be simplified to be

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}b_{ij} = n\det A.$$

Next, to simplify the sum of bracket in the first and the last terms in (5), we denoted Y_i for $i = 1,2,\dots,n$ and $j = 1,2,\dots,n$, so that we have

$$Y_i = (\det A)\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}x_i x_j = (\det A)\,r^2,$$

thus, the equation (5) can be rewrite as

$$\frac{v''}{v'} = \frac{1-n}{r},$$

then we have

$$v' = ar^{1-n}.$$

Since, the domain x is at n-dimension, $n \geq 2$ then $v(r)$ has the following general form:

$$v(r) = \begin{cases} a\log r + b, & n = 2 \\ \dfrac{c}{r^{n-2}} + b, & n \geq 3. \end{cases} \tag{6}$$

$a, b, c \in \mathbb{R}$.

By rescaling (6) to $u(x) = v(r)$, where $r = \left(\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}x_ix_j\right)^{\frac{1}{2}}$, then we get the fundamental solution of (2) is

$$v(r) = \begin{cases} a\log\left(\displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}x_ix_j\right)^{\frac{1}{2}} + b, & n = 2 \\ c\left(\displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}x_ix_j\right)^{\frac{2-n}{2}} + b, & n \geq 3 \end{cases}$$

where $a, b, c \in \mathbb{R}$.

Remarks:

We suppose the identity matrix $I_{n\times n}$ as a coefficient of (2). I is a positive definite matrix, then the elliptic equation can be formed as follows

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}D_i\left(D_ju\right) = 0, \tag{7}$$

where

$$a_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$$

Then coffactor matrix

$$b_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$$

According to a_{ij} and b_{ij} as above, equation (7) can be written as

$$\sum_{i=1}^{n} D_i(D_iu) = 0, \tag{8}$$

where (8) is equivalent to the Laplace's equation $\sum_{i=1}^{n} u_{x_ix_i}$.

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