

The Distance Irregular Reflexive k -Labeling of Graphs

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Abstract. A total k -labeling is a function f_e from the edge set to the set $\{1, 2, \dots, k_e\}$ and a function f_v from the vertex set to the set $\{0, 2, 4, \dots, 2k_v\}$, where $k = \max\{k_e, 2k_v\}$. A *distance irregular reflexive k -labeling* of the graph G is the total k -labeling, if for every two different vertices u and u' of G , $w(u) \neq w(u')$, where $w(u) = \sum_{u_i \in N(u)} f_v(u_i) + \sum_{uv \in E(G)} f_e(uv)$. The minimum k for graph G which has a distance irregular reflexive k -labelling is called distance reflexive strength of the graph G , denoted by $\mathcal{D}_{ref}(G)$. In this paper, we determine the exact value of distance reflexive strength of some connected graphs, namely path, star, and friendship graph.

Keywords: Distance irregular reflexive k -labeling, distance reflexive strength, path, star, friendship.

1. Introduction

A simple graph G is a pair of $(V(G), E(G))$, where $V(G)$ is non empty vertex set and $E(G)$ is an unordered pair of two distinct vertices u, v in $V(G)$. The graph G is connected if there exists a path between every pair of distinct vertices of G . Among all the different kinds of problems that appear while studying graph theory, one that has been growing strongly during the last years in this area, namely labeling of graphs. Throughout this paper all graphs are simple, connected and undirected graph $G(V, E)$. The set of vertices and edges of the graph are denoted by $V(G)$ and $E(G)$, respectively [1]. A labeling of a graph is a map that carries graph elements to the numbers (usually to the positive or non-negative integers). The recent concept of graph labeling can be found in [2]. The most common choices of domain are the set of labels to the vertex set (called vertex labeling), the edge set (called edge labeling), or to the both vertices and edges (called total labeling). In many cases it is quiet interesting to consider the sum of all labels associated with a graph element is called the weight of the element which is denoted by w .

In this paper, we focus to study a new type of labeling namely irregular labeling. When we assign positive integer labels to the edges or vertices of a connected graph, with the vertex at least 3, it will become irregular if the weight (label sums) at each vertex are distinct. The minimum of the maximum weight over all such irregular assignments is known as the

irregularity strength of a graph G . The results of irregularity strength of a graph can be seen on [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. Baca *et al* obtained the lower bound and the upper bound of total vertex irregularity strength on graph G , see [15].

By a vertex irregular reflexive labeling, we define, for a graph G , a k -labeling assign the numbers $\{1, 2, \dots, k\}$ to the elements of graph. Let k be a natural number, a function $f : V(G) \cup E(G) \rightarrow \{1, 2, 3, \dots, k\}$ is called total k irregular labeling. Irregular k -labeling extends to many types, one of them is a vertex irregular reflexive total k -labeling. Tanna *et al* [14] give the detailed definition of vertex irregular reflexive labeling. Total k -labeling is a function f_e from the edge set to the first natural number k_e and a function f_v from the vertex set to the non-negative even number up to $2k_v$, where $k = \max\{k_e, 2k_v\}$. A *vertex irregular reflexive k -labeling* of the graph G is the total k -labeling, if for every two different vertices x and x' of G , $wt(x) \neq wt(x')$, where $wt(x) = f_v(x) + \sum_{xy \in E(G)} f_e(xy)$. The minimum k for graph G which has a vertex irregular reflexive k -labelling is called the reflexive vertex strength of the graph G , denoted by $rvs(G)$.

The minimum number of edges between two vertices u and v is called the **distance** between these vertices and is denoted as $d(u, v)$.

Motivated by definition of **distance irregular labeling** [16] and **irregular reflexive labeling** of graphs, we develop a new concept namely **distance irregular reflexive labeling**. We have given a formal definition as follows: A total k -labeling is a function f_e from the edge set to the set $\{1, 2, \dots, k_e\}$ and a function f_v from the vertex set to the set $\{0, 2, 4, \dots, 2k_v\}$, where $k = \max\{k_e, 2k_v\}$. A *distance irregular reflexive k -labeling* of the graph G is the total k -labeling, if for every two different vertices u and u' of G , $w(u) \neq w(u')$, where $w(u) = \sum_{u_i \in N(u)} f_v(u_i) + \sum_{uv \in E(G)} f_e(uv)$. The minimum k for graph G which has a distance irregular reflexive k -labelling is called distance reflexive strength of the graph G , denoted by $\mathcal{D}_{ref}(G)$.

In this paper, we determine the exact value of distance reflexive strength of some connected graphs, namely path, star, and friendship graph.

2. Basic Results on Lower Bound of Distance Reflexive Strength

To prove the theorem, the following lemma were used to determine the lower bound. The Lemma is as follows.

Lemma 1 *Let G be a graph of order p , the minimum degree δ , and the maximum degree Δ . The distance reflexive strength is*

$$\mathcal{D}_{ref}(G) \geq \left\lceil \frac{p + \delta - 1}{2\Delta} \right\rceil.$$

Proof. Let G be a graph of order p , the minimum degree δ , and the maximum degree Δ . Let $k' = \mathcal{D}_{ref}(G)$ is the largest label on graph under any distance irregular reflexive k -labeling. We choose u as a vertex of degree δ , such that the vertex weight of u at least:

$$w(u) = \sum_{u_i \in N(u)} f_v(u_i) + \sum_{uv \in E(G)} f_e(uv) = 0 + 1(\delta) = \delta.$$

Furthermore, to get the minimum k' the vertex weight must form arithmetic sequence, namely $\delta, \delta + 1, \delta + 2, \dots, \delta + p - 1$.

In other hand, we choose u' as a vertex of degree Δ , such that

$$w(u') = \sum_{u'_i \in N(u')} f_v(u'_i) + \sum_{u'v' \in E(G)} f_e(u'v') = 2k_v(\Delta) + k_e(\Delta).$$

Since $k' = \max\{k_e, 2k_v\}$, then

$$w(u') = 2k_v(\Delta) + k_e(\Delta) \leq k'(2\Delta).$$

It is obtained the largest vertex weight is $p + \delta - 1$. Since $p + \delta - 1 \leq wt(u')$, then $k'(2\Delta) \geq p + \delta - 1$, such that $k' \geq \frac{p+\delta-1}{(2\Delta)}$. Since $\mathcal{D}_{ref}(G)$ should be integer and we need a sharpest lower bound, it implies $k' \geq \left\lceil \frac{(p+\delta-1)}{(2\Delta)} \right\rceil$ or $\mathcal{D}_{ref}(G) \geq \left\lceil \frac{(p+\delta-1)}{(2\Delta)} \right\rceil$. $\square$

From Lemma 1, we determine the following corollary.

Corollary 1 *Let G be a graph of order p , the minimum degree δ , and the maximum degree Δ . If $p + \delta - 1 > \Delta(2t - 1)$, then*

$$\mathcal{D}_{ref}(G) \geq \left\lceil \frac{p + \delta - 1}{2\Delta} \right\rceil + 1,$$

where t is odd natural number.

Proof. Let G be a graph of order p , the minimum degree δ , the maximum degree Δ . Let $p + \delta - 1 = t(2\Delta)$. Based on Lemma 1 we have $\mathcal{D}_{ref}(G) \geq \left\lceil \frac{p+\delta-1}{2\Delta} \right\rceil > \left\lceil \frac{\Delta(2t-1)}{2\Delta} \right\rceil = \left\lceil \frac{2t-1}{2} \right\rceil = t$. Furthermore, suppose t is $\mathcal{D}_{ref}(G)$, t will be the largest label under any distance irregular reflexive k -labeling on graph G . The largest weight can be obtained on the vertex with the maximum degree Δ . Let u' be the vertex of maximum degree, such that the vertex weight of u' is

$$w(u') = \sum_{u'_i \in N(u')} f_v(u'_i) + \sum_{u'v' \in E(G)} \leq (t-1)(\Delta) + t(\Delta) = \Delta(2t-1).$$

Based on Lemma 1, we obtain $p + \delta - 1 \leq \Delta(2t - 1)$. It contradicts with $p + \delta - 1 > \Delta(2t - 1)$. Hence $t \neq \mathcal{D}_{ref}(G)$, such that $\mathcal{D}_{ref}(G) > \left\lceil \frac{p+\delta-1}{2\Delta} \right\rceil > \left\lceil \frac{\Delta(2t-1)}{2\Delta} \right\rceil = \left\lceil \frac{2t-1}{2} \right\rceil = t$. Thus, it concludes that $\mathcal{D}_{ref}(G) \geq \left\lceil \frac{p+\delta-1}{2\Delta} \right\rceil + 1$. $\square$

We use those lemma and corollary above to determined the distance reflexive strength of some connected graphs, namely path, star, and friendship graph.

3. Main Results of Distance Reflexive Strength of Some Connected Graphs

Theorem 1 *Let P_n be a path graph. For every natural number $n \geq 3$,*

$$\mathcal{D}_{ref}(P_n) = \begin{cases} \left\lceil \frac{n}{4} \right\rceil + 1, & \text{if } n \equiv 3, 4(\text{mod } 8) \\ \left\lceil \frac{n}{4} \right\rceil, & \text{if } n \equiv 0, 5, 6, 7(\text{mod } 8) \end{cases}$$

Proof. The path graph P_n , $n \geq 3$ is a connected graph, whose vertex set $V(P_n) = \{a_i; 1 \leq i \leq n\}$ and edge set $E(P_n) = \{a_i a_{i+1}; 1 \leq i \leq n\}$. Since $\delta(P_n) = 1$ and $\Delta(P_n) = 2$, for $n \equiv 0, 5, 6, 7(\text{mod } 8)$ base on Lemma 1 we have

$$\mathcal{D}_{ref}(P_n) \geq \left\lceil \frac{p + \delta - 1}{2\Delta} \right\rceil = \left\lceil \frac{n + 1 - 1}{2(2)} \right\rceil = \left\lceil \frac{n}{4} \right\rceil.$$

For $n \equiv 3(\text{mod } 8)$, it gives $\mathcal{D}_{ref}(P_n) \geq \left\lceil \frac{n}{4} \right\rceil = \left\lceil \frac{8b+3}{4} \right\rceil = 2b + 1$ and for $n \equiv 4(\text{mod } 8)$, it gives $\mathcal{D}_{ref}(P_n) \geq \left\lceil \frac{n}{4} \right\rceil = \left\lceil \frac{8b+4}{4} \right\rceil = 2b + 1$, where b is natural number. Since $2b + 1$ is odd number, it satisfies the condition in Corollary 1, such that we have

$$\mathcal{D}_{ref}(P_n) \geq \left\lceil \frac{n}{4} \right\rceil + 1.$$

Now, to show the upper bound of the reflexive vertex strength on P_n , we construct the following Algorithm:

- (i) Give the label on vertices with the function as follows. For $1 \leq i \leq 2k + 1$,

$$f_v(a_i) = k,$$

where

$$k = \begin{cases} \lceil \frac{n}{4} \rceil + 1, & \text{if } n \equiv 3, 4 \pmod{8} \\ \lceil \frac{n}{4} \rceil, & \text{if } n \equiv 0, 5, 6, 7 \pmod{8} \end{cases}$$

- (ii) Give the label on edges with the function as follows. For $1 \leq i \leq 2k$,

$$f_e(a_i a_{i+1}) = k + 1 - \left\lceil \frac{i}{2} \right\rceil.$$

- (iii) We have the vertex weight, $w(a_1) = 2k$ and for $2 \leq i \leq 2k$, $w(a_i) = 4k + 2 - i$.
 (iv) Next, we determine $w(a_{2k+1}) = 2k + 1$, and for $2k + 2 \leq i \leq n - 1$, $w(a_i) = 4k + 1 - i$.
 (v) Furthermore, for $2k + 2 \leq i \leq n$, label each vertex a_i with $f_v(a_{i-1})$ and label edge $a_{i-1}a_i$ with $w(a_{i-1}) - f_v(a_{i-2}) - f_e(a_{i-2}a_{i-1}) - f_v(a_i)$.
 (vi) If edge label of point [4] is 0, then label each vertex a_i with $f_v(a_{i-1}) - 2$ and label edge $a_{i-1}a_i$ with $w(a_{i-1}) - f_v(a_{i-2}) - f_e(a_{i-2}a_{i-1}) - f_v(a_i)$.
 (vii) The vertex weight of a_n as follows.

$$w(a_n) = f_v(a_{n-1}) + f_e(a_{n-1}a_n),$$

since $w(a_n)$ is the sum of two labels (vertex and edge), then $w(a_n)$ must be different from the others.

Since the all vertices are distinct, $\mathcal{D}_{ref}(P_n) \geq k$ and $\mathcal{D}_{ref}(P_n) \leq k$, it concludes that $\mathcal{D}_{ref}(P_n) = k$ for $n \geq 3$. $\square$

Theorem 2 Let S_n be a star graph. For every natural number $n \geq 3$,

$$\mathcal{D}_{ref}(S_n) = n$$

Proof. The star graph S_n , $n \geq 3$ is a connected graph which has vertex set $V(S_n) = \{\alpha, x_i; 1 \leq i \leq n\}$ and edge set $E(S_n) = \{\alpha x_i; 1 \leq i \leq n\}$. The star graph has n vertices of degree 1 and one dominant vertex. Since $\delta(S_n) = 1$ and the dominant vertex always contribute the constant label to other vertices, the smallest vertex weight is at least 1 and the largest vertex weight on vertices of degree 1 is at least n . Since α is the dominant vertex and we can label it with 0, the weight of each vertex on vertices of degree 1 depends on 1 label (only edge), such that $\mathcal{D}_{ref}(S_n) \geq n$. Since $\Delta(S_n) = n$, then $\mathcal{D}_{ref}(S_n) \geq n \geq \lceil \frac{n+\delta-1}{2\Delta} \rceil$ and satisfies the Lemma1.

Now, to show the upper bound of the reflexive vertex strength on S_n , we define the following function:

$$\begin{aligned} f_v(\alpha) &= f_v(x_i) = 0 \\ f_e(\alpha x_i) &= i : 1 \leq i \leq n \end{aligned}$$

By those vertex and edge labels, we have the vertex weight as follows.

$$\begin{aligned} w(\alpha) &= \sum_{i=1}^n [f_v(x_i) + f_e(\alpha x_i)] = \frac{n(n+1)}{2} \\ w(x_i) &= i : 1 \leq i \leq n \end{aligned}$$

We know that the all vertex weights of graph S_n are distinct and it gives a distance reflexive strength on star graph. Since $\mathcal{D}_{ref}(S_n) \leq n$ and $\mathcal{D}_{ref}(S_n) \geq n$, it conclude that $\mathcal{D}_{ref}(S_n) = n$ for $n \geq 3$. $\square$

Theorem 3 Let $\mathcal{F}_n$ be a friendship graph. For every natural number $n \geq 3$,

$$\mathcal{D}_{ref}(\mathcal{F}_n) = \begin{cases} \lceil \frac{2n+1}{3} \rceil + 1, & \text{if } n \equiv 1(\text{mod } 3) \\ \lceil \frac{2n+1}{3} \rceil, & \text{otherwise} \end{cases}$$

Proof. The friendship graph $\mathcal{F}_n$, $n \geq 3$ is a connected graph which has vertex set $V(\mathcal{F}_n) = \{a, u_i, v_i; 1 \leq i \leq n\}$ and edge set $E(\mathcal{F}_n) = \{au_i; 1 \leq i \leq n\} \cup \{av_i; 1 \leq i \leq n\} \cup \{u_i v_i; 1 \leq i \leq n\}$. The friendship graph is an $(2n, 2)$ -almost regular graph []. The friendship graph has $2n$ vertices of degree 2 and one dominant vertex. Since $\delta(\mathcal{F}_n) = 2$ and the dominant vertex always contribute the constant label to other vertices, the smallest vertex weight is at least 2 and the largest vertex weight on vertices of degree 2 is at least $2n + 1$. Since a is the dominant vertex and we can label it with 0, the weight of each vertex on vertices of degree 2 depends on 3 labels (2 edges and 1 adjacent vertex). Since label of vertex must be an even number, such that $\mathcal{D}_{ref}(\mathcal{F}_n) \geq \lceil \frac{2n+1}{3} \rceil$. Since $\Delta(\mathcal{F}_n) = 2n$, then $\mathcal{D}_{ref}(\mathcal{F}_n) \geq \lceil \frac{2n+1}{3} \rceil \geq \lceil \frac{p+\delta-1}{2\Delta} \rceil$ and satisfies the Lemma 1. For $n \equiv 1(\text{mod } 3)$, there is an integer s such that $2n = 2(3s + 1) = 6s + 2$, then $\lceil \frac{2n+1}{3} \rceil = \lceil \frac{6s+3}{3} \rceil = 2s + 1$. The number $2s + 1$ is an odd number. The vertex weight $2n + 1 = 6s + 3$ is the result of the sum of 3 labels where each label is not more than $2s + 1$, such that $2n + 1 = 6s + 3 = (2s + 1) + (2s + 1) + (2s + 1)$. Since the vertex label must be an even number, for $n \equiv 1(\text{mod } 3)$ we get $\mathcal{D}_{ref}(\mathcal{F}_n) \geq \lceil \frac{2n+1}{3} \rceil + 1$.

Now, to show the upper bound of the reflexive vertex strength on $\mathcal{F}_n$, we define the following function:

$$\begin{aligned} f_v(a) &= 0 \\ f(u_i) &= \begin{cases} 0, & \text{if } i = 1 \\ 2\lceil \frac{i-1}{3} \rceil, & \text{if } 2 \leq i \leq n \end{cases} \\ f_v(u_i) &= f_v(v_i) \\ f_e(u_i v_i) &= \begin{cases} \frac{2}{3}i, & \text{if } i \equiv 0(\text{mod } 3) \\ \frac{1}{3}(2i + 1), & \text{if } i \equiv 1(\text{mod } 3) \\ \frac{1}{3}(2i - 1), & \text{if } i \equiv 2(\text{mod } 3) \end{cases} \\ f_e(au_i) &= f_e(u_i v_i) \\ f_e(av_i) &= \begin{cases} \frac{2}{3}i + 1, & \text{if } i \equiv 0(\text{mod } 3) \\ \frac{1}{3}(2i + 1) + 1, & \text{if } i \equiv 1(\text{mod } 3) \\ \frac{1}{3}(2i - 1) + 1, & \text{if } i \equiv 2(\text{mod } 3) \end{cases} \end{aligned}$$

By those vertex and edge labels, we have the vertex weight as follows.

$$\begin{aligned} w(a) &= \sum_{i=1}^n [f_v(u_i) + f_v(v_i) + f_e(au_i) + f_e(av_i)] \\ w(u_i) &= 2i : 1 \leq i \leq n \end{aligned}$$

$$w(v_i) = 2i + 1 : 1 \leq i \leq n$$

We know that the all vertex weights of graph $\mathcal{F}_n$ are distinct. Thus, it gives a distance reflexive strength on $\mathcal{F}_n$ for $n \geq 3$. $\square$

4. Concluding Remark

We determined the lower bound of graph distance reflexive strength and the exact values of distance reflexive strength for several graphs, including the path, star, and friendship graph. We have made significant efforts to characterize the exact values of reflexive vertex strength in any of the above graphs, but it remains difficult, despite the fact that this problem has grown in complexity. As a result, determining the distance reflexive strength of any graph classified as one of the above graphs remains an open problem. As a result, we propose the open problems listed below.

5. Open Problem

There are two open problems that can be carried out for future results

- (i) Determine the exact value of distance reflexive strength of P_n for $n \equiv 1, 2 \pmod{8}$ and any graphs apart from those families.
- (ii) Determine the sharpest upper bound of reflexive vertex strength of any operation graphs.

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