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Some Properties of Operator (h, m)-Convex Functions

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ABSTRACT

Operator convex functions are continuous functions of self-adjoint operators in Hilbert space. Meanwhile, the operator (h, m)-convex functions are a new class on convexity, a generalization of operator convex functions by combining h-convexity and m-convexity classes. In this class, m is in the interval [0,1] and h is a nonnegative function with a domain that includes the interval (0,1). The definition of operator (h, m)-convex functions has already been studied by Salaş et al. in 2017. However, there are some properties of operator (h, m)-convex functions that have not yet been studied. Thus, in this research, we derive some basic properties of operator (h, m)-convex functions. We also present the definition of matrix (h, m)-convex functions, a special case, when the operator is a self-adjoint matrix. Then, we obtain some properties of matrix (h, m)-convex functions.

Keywords: operator convex functions; functions; operator (h, m)-convex functions; matrix convex functions; matrix (h, m)-convex functions; (h, m)-convexity

INTRODUCTION

The concept of convexity is central to the study of applied mathematics[1]. Convexity was initially defined and used in the study of arc length by Archimedes. The definition of convexity proposed by Archimedes is similar to the the current definition, which states that a set is called convex if it contains all line segments formed by any two points in the set. Mathematically (see [2]): $X \subseteq \mathbb{R}$ is called as a convex set if for any $x, y \in X$ and $t \in (0,1)$ holds

$$tx + (1-t)y \in X.$$

Similarly, a curve (function) is called convex if the line segment between any two points on the function's graph lies above the curve[1]. We can write mathematically (see [2]): Let X be a convex set, a functions $f: X \to \mathbb{R}$ is called as convex functions, if for any $x, y \in X$ and $t \in (0,1)$ holds

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y). \quad (1)$$

If the inequality (1) is reversed, f is called as convave function. In other words if f is a convex function then $-f$ is a concave function, vice versa.

Various studies based on the definition and classical convexity theories have been developed, leading to various variations in the definitions and convexity theories that are more general than classical convexity. In 1984, Toader[3] introduced the definition of m-convex functions, in which it must be defined on a m-convex set. In this class, when

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$m = 0$, it is called as starshaped functions[4].

In 2007, Varošanec[5] introduced a new class of convexity, which is called as h-convexity. He studied the definition of h-convex functions and their properties. Here, $h: J \to [0, \infty)$, where $J \supseteq (0,1)$. This is also the generalization of some classes in convexity, such as Godunova-Levin functions (see [6]), s-convex functions (see [7], [8]), P-functions (see [9]), and s-Godunova-Levin functions (see [10]).

Combining the concepts of h-convexity and m-convexity, Özdemir et al.[11] studied (h, m)-convex functions and their properties. In this class of convex functions, the domain should be a m-convex set. The special class that generalized from this class is (s, m)-convex functions (see [12], [13]) and (s, m)-Godunova-Levin functions (see [14]).

Several researchers studied functions of self-adjoint operators in Hilbert space, where the function is convex. It is also known as operator convex functions in Hilbert space. In this class, the operator is must be a bounded and linear operator. The operator's spectrum also must be contained in the interval of the domain of functions. Bhatia[15] studied some properties for opearator convex functions.

Operator m-convex funtions is a new class of functions of self-adjoint operators in Hilbert space. Similar with m-convex functions interval of this functions is must be m-convex set. Erdaş et al.[16] introduced the definition of operator m-convex functions. Continuing the work from Erdaş, Berbesi et al.[17] studied some properties for operator m-convex functions in 2018.

In 2017, Darvish et al.[18] studied the definition and some properties of operator h-convex functions in Hilbert space. They also obtained some results when the operator is matirce's trace. The operator h-convex functions is the generalization of some other classes such as, operator s-convex functions (see [19]), operator Godunova-Levin functions (see [20]), and operator s-Godunova-Levin-Dragomir(see [21]) functions.

Continuing the work from Özdemir et al.[11], Salaş et al.[22] introduced the definition of operator (h, m)-convex functions in Hilbert space and Hermite-Hadamard type inequalities for operator (h, m)-convex functions. However, in their article, they did not study about the properties of operator (h, m)-convex functions. The properties of this class are essential, since this class is a new class in convexity combining the operator h-convex functions and the operator m-convex functions.

Based on developments that have been presented, in this paper, we provide some basic properties for operator m-convex functions in Hilbert space. We also provide the definition and the properties of the matrix (h, m)-convex functions, a special case, when the operator is a self-adjoint matrix. We use the definition of the operator (h, m)-convex functions introduced in Salaş et al.[22].

METHODS

The method of our research is literature study. Thus, in this section, we provide several definitions and propositions that support obtaining our desired results. We also give some notations for convenience in reading this paper.

As we mentioned before, the domain of the class (h, m)-convexity is m-convex set. The definition of it can be seen in the following Definition 1.

Definition 1 (see [3], [23])**.** Let $X \subseteq \mathbb{R}^n$ be linear space. Suppose $V \subseteq X$ be a non-empty set and $m \in [0,1]$. V is called to be m-convex set if for every $x, y \in V$ and $t \in (0,1)$ holds

$$tx + m(1 - t)y \in V.$$

Using Definition 1 above, ones can easily shows that the interval $I = [0, b]$ is a m-convex

set. Next, the definition of m-convex functions can be seen in the following Definition 2.

Definition 2 (see [3]). Let $I = [0, b] \subseteq \mathbb{R}$ and $m \in [0,1]$. A function $f: I \to \mathbb{R}$ is called to be m-convex function on I if for any $x, y \in I$ and $t \in (0,1)$ holds

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y).$$

According to [4], from the Definition 2 if we take $m = 0$, we have starshaped functions.

h-Convex functions are defined in [5] as in the following Definition 3. However, we will use interval $I = [0, b]$ as the domain.

Definition 3 (see [5]). Let $h: J \supseteq (0,1) \to [0, \infty)$. A function $f: I \to \mathbb{R}$ is called to be h-convex function on I if for any $x, y \in I$ and $t \in (0,1)$ holds

$$f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y).$$

From the Definition 3, we have the Godunova-Levin functions, s-convex functions, P-functions, and s-Godunova-Levin functions if we take $h(t) = \frac{1}{t}$, $h(t) = t^s$, $h(t) = 1$, and $h(t) = \frac{1}{t^s}$ respectively. The functions h can be super-multiplicative or sub-multiplicative functions.

Definition 4 (see [5]). A function $h: J \supseteq (0,1) \to [0, \infty)$ is called to be a super-multiplicative functions if

$$h(xy) \geq h(x)h(y),$$

otherwise, h is called to be a sub-multiplicative functions.

Defined on the interval $I = [0, b] \subseteq \mathbb{R}$ as the domain, the definition of (h, m)-convex functions can be seen in the following Definition 5.

Definition 5 (see [10]). Let $h: J \supseteq (0,1) \to [0, \infty)$ and $m \in [0,1]$. A function $f: I \to \mathbb{R}$ is called to be (h, m)-convex function on I if for any $x, y \in I$ and $t \in (0,1)$ holds

$$f(tx + m(1-t)y) \leq h(t)f(x) + mh(1-t)f(y).$$

From the Definition 5, if we take $h(t) = t^s$ and $h(t) = \frac{1}{t^s}$, we have the definition of (s, m)-convex functions and (s, m)-Godunova-Levin functions respectively.

For convenience, before we go further to the definition of operator convex functions, we give some notations. We denote $\mathcal{H}$ be a Hilbert space with an inner product $\langle \cdot, \cdot \rangle$, while $\mathcal{B}(\mathcal{H}) = \{A | A: \mathcal{H} \to \mathcal{H}$ is bounded linear operator$\}$. Let $A, B \in \mathcal{B}(\mathcal{H})$ be self-adjoint operator, in this paper, the notation of operator orders

$$A \preccurlyeq B \tag{2}$$

means

$$\langle Ax, x \rangle \leq \langle Bx, x \rangle, \tag{3}$$

for any $x \in \mathcal{H}$. Thus, changing the sign in (2) with "$\prec$", "$\succcurlyeq$", "$\succ$", and "$=$" means changing the sign in (3) with "$<$", "$\geq$", "$>$", and "$=$" respectively. We also denote $\mathcal{B}^+(\mathcal{H})$ is a family of bounded linear operator in Hilbert space. Let us recall the definition of spectrum (σ) and resolvent set (ρ) below. We denote $0_{\mathcal{H}}$ be the null operator in $\mathcal{H}$. Meanwhile, $1_{\mathcal{H}}$ be identity operator in $\mathcal{H}$.

Definition 6 (see [24]). Let A be a bounded linear operator. The spectrum of A is

$$\sigma(A) = \{\mu \in \mathbb{R} | (A - \mu 1_{\mathcal{H}}) \text{ is not invertible}\}.$$

Meanwhile, the resolvent set of A is

$$\rho(A) = \mathbb{R} \setminus \sigma(A).$$

According to [26], for any self-adjoint operator $A \in \mathcal{B}(\mathcal{H})$ and $f \in C(\sigma(A))$, if $x \in \sigma(A)$, $f(x) \geq 0$, then $f(A)$ is positive operator on $\mathcal{H}$. Next, according to [25], if for every $x \in \sigma(A)$, $f(x) \leq g(x)$ $(f, g: \sigma(A) \to \mathbb{R})$, then $f(A) \preccurlyeq g(A)$ for any self-adjoint operator $A \in \mathcal{B}(\mathcal{H})$.

The convexity and monotonicity of the operator play an important role in our discussion. The operator convex and operator monotone functions is defined as follows.

Definition 7 (see [14], [25]). Let $I = [0, b] \subseteq \mathbb{R}$. A function $f: I \to \mathbb{R}$ is called to be operator convex function on I if for any self-adjoint operator $A, B \in \mathcal{B}(\mathcal{H})$ with $\sigma(A), \sigma(B) \subseteq I$ and $t \in (0,1)$ holds

$$f(tA + (1-t)B) \preccurlyeq tf(A) + (1-t)f(B).$$

Meanwhile, the functions f is called to be operator monotone if it is monotone with respect to the operator order, i.e. $A \preccurlyeq B$ with $\sigma(A), \sigma(B) \subseteq I$ imply $f(A) \preccurlyeq f(B)$.

Let us denote $\mathcal{K} \subseteq \mathcal{B}^+(\mathcal{H})$ be a m-convex set. Now, we go to the definition of operator m-convex functions.

Definition 8 (see [15], [16], [26]). Let $I = [0, b] \subseteq \mathbb{R}$ and $m \in [0,1]$. A function $f: I \to \mathbb{R}$ is called to be operator m-convex function on I if for any self-adjoint operator $A, B \in \mathcal{K}$ with $\sigma(A), \sigma(B) \subseteq I$ and $t \in (0,1)$ holds

$$f(tA + m(1-t)B) \preccurlyeq tf(A) + m(1-t)f(B).$$

The definition of operator h-convex functions are given in the following Definition 9.

Definition 9 (see [18]). Let $I = [0, b] \subseteq \mathbb{R}$, $(0,1) \subseteq J$, and $h: J \to [0, \infty)$. A function $f: I \to \mathbb{R}$ is called to be operator h-convex function on I if for any self-adjoint operator $A, B \in \mathcal{K}$ with $\sigma(A), \sigma(B) \subseteq I$ and $t \in (0,1)$ holds

$$f(tA + (1-t)B) \preccurlyeq h(t)f(A) + h(1-t)f(B).$$

From Definition 9, if we take $h(t) = t^s$, we have the definition of operator s-convex functions.

Finally, the definition of operator (h, m)-convex functions are given in the following Definition 10.

Definition 10 (see [22]). Let $I = [0, b] \subseteq \mathbb{R}$, $m \in [0,1]$, $(0,1) \subseteq J$, and $h: J \to [0, \infty)$. A function $f: I \to \mathbb{R}$ is called to be operator (h, m)-convex function on I if for any self-adjoint operator $A, B \in \mathcal{K}$ with $\sigma(A), \sigma(B) \subseteq I$ and $t \in (0,1)$ holds

$$f\big(tA + m(1-t)f(B)\big) \preccurlyeq h(t)f(A) + mh(1-t)f(B).$$

RESULTS AND DISCUSSION

In this section, we present some results on the operator (h, m)-convex function in Hilbert space. We also give the definition of matrix (h, m)-convex function as a special case of our discussions. Based on the Definition 10, we obtain basic properties of operator (h, m)-convex function. They are provided in the following propositions and theorems below.

First, we obtained the linearity property of the operator (h, m)-convex function as in the following Proposition 1 and Proposition 2.

Proposition 1. Let $I = [0, b] \subseteq \mathbb{R}$, $m \in [0,1]$, $(0,1) \subseteq J$, and $h: J \to [0, \infty)$. Suppose $f, g: I \to \mathbb{R}$ be operator (h, m)-convex functions on I. Then, $f + g$ is also operator (h, m)-convex function on I.

Proof: Let $A, B \in \mathcal{K}$ with $\sigma(A), \sigma(B) \subseteq I$. Since f, g are operator (h, m)-convex function on I, then for any $t \in (0,1)$ we have

$$\begin{aligned}(f+g)(tA + m(1-t)B) &= f(tA + m(1-t)B) + g(tA + m(1-t)B) \\ &\preccurlyeq h(t)f(A) + mh(1-t)f(B) + h(t)g(A) \\ &\quad + mh(1-t)g(B) \\ &\preccurlyeq h(t)(f+g)(A) + mh(1-t)(f+g)(B).\end{aligned}$$

That means $f + g$ is operator (h, m)-convex function on I for operators in $\mathcal{K}$.

■

Proposition 2. Let $I = [0, b] \subseteq \mathbb{R}$, $m \in [0,1]$, $(0,1) \subseteq J$, $h: J \to [0, \infty)$, and $\alpha \in (0, \infty)$. Suppose $f: I \to \mathbb{R}$ be operator (h, m)-convex function on I. Then, αf is also operator (h, m)-convex function on I.

Proof: Let $A, B \in \mathcal{K}$ with $\sigma(A), \sigma(B) \subseteq I$. Since f is operator (h, m)-convex function on I, then for any $t \in (0,1)$ we have

$$\begin{aligned}(\alpha f)(tA + m(1-t)B) &\preccurlyeq \alpha[h(t)f(A) + mh(1-t)f(B)] \\ &\preccurlyeq h(t)(\alpha f)(A) + mh(1-t)(\alpha f)(B).\end{aligned}$$

That means αf is operator (h, m)-convex function on I for operators in $\mathcal{K}$.

■

Other Properties of the operator convex functions can be seen in the following Proposition 3 up to Proposition 6 below.

Proposition 3. Let $I_1 = [0, b], I_2 = [0, c] \subseteq \mathbb{R}$, $m \in [0,1]$, $(0,1) \subseteq J$, and $h: J \to [0, \infty)$. Suppose $f: I_1 \to \mathbb{R}$ be operator (h, m)-convex functions and monotone operator on I_1 functions and $g: I_2 \to \mathbb{R}$ be operator m-convex functions on I_2, with $g(I_2) \subseteq I_1$. Then, $f \circ g$ is also operator (h, m)-convex function on I_2.

Proof: Let $A, B \in \mathcal{K}$ with $\sigma(A), \sigma(B) \subseteq I_2$. Since g is operator m-convex function on I_2 and f is operator monotone function, then for any $t \in (0,1)$ we have

$$(f \circ g)(tA + m(1-t)B) \preccurlyeq f\big(tg(A) + m(1-t)g(B)\big).$$

Next, because $g(I_2) \subseteq I_1$, also f is operator (h, m)-convex and monotone on I_1, we thus have

$$\begin{aligned}(f \circ g)(tA + m(1-t)B) &\preccurlyeq h(t)f\big(g(A)\big) + mh(1-t)f\big(g(B)\big) \\ &= h(t)\,(f \circ g)(A) + m\,h(1-t)(f \circ g)(B).\end{aligned}$$

That means $f \circ g$ is operator (h, m)-convex function on I_2 for operators in $\mathcal{K}$.

■

Next, by considering two functions, namely h_1 and h_2, also $m_1, m_2 \in [0,1]$, we obtain Proposition 4 and Proposition 5.

Proposition 4. Let $I = [0, b] \subseteq \mathbb{R}$, $m \in [0,1]$, $(0,1) \subseteq J$, and $h_1, h_2: J \to [0, \infty)$ with $h_1 \leq h_2$. If $f: I \to \mathbb{R}$ is an operator (h_1, m)-convex function on I, then $f: I \to \mathbb{R}$ is also an operator (h_2, m)-convex function on I.

Proof: Let $A, B \in \mathcal{K}$ with $\sigma(A), \sigma(B) \subseteq I$. Since f is operator (h_1, m)-convex function on I

and $h_1 \leq h_2$, then for any $t \in (0,1)$ we have

$$f(tA + m(1-t)B) \preccurlyeq h_1(t)f(A) + mh_1(1-t)f(B) \preccurlyeq h_2(t)f(A) + mh_2(1-t)f(B).$$

Hence, $f: I \to \mathbb{R}$ is also an operator (h_2, m)-convex function on I for operators in $\mathcal{K}$.

■

Proposition 5. Let $I = [0, b] \subseteq \mathbb{R}$, $(0,1) \subseteq J$, $h: J \to [0, \infty)$, and $m_1, m_2 \in [0,1]$ with $m_1 \leq m_2$. If $f: I \to \mathbb{R}$ is an operator (h, m_1)-convex function on I, then $f: I \to \mathbb{R}$ is also an operator (h, m_2)-convex function on I.

Proof: Let $A, B \in \mathcal{K}$ with $\sigma(A), \sigma(B) \subseteq I$. Since f is operator (h, m_1)-convex function on I and $m_1 \leq m_2$, then for any $t \in (0,1)$ we have

$$f(tA + m(1-t)B) \preccurlyeq h(t)f(A) + m_1h(1-t)f(B) \preccurlyeq h(t)f(A) + m_2h(1-t)f(B).$$

Hence, $f: I \to \mathbb{R}$ is also an operator (h, m_2)-convex function on I for operators in $\mathcal{K}$.

■

Proposition 6. Let $I = [0, b] \subseteq \mathbb{R}$, $(0,1) \subseteq J$, $h: J \to [0, \infty)$, and $m \in [0,1]$ such that $h(t) + mh(1-t) \leq 1$. Also suppose $A, B \in \mathcal{K}$ be commutative and comparable (i.e. $A = B$ and $A \leq B$ or $A \geq B$) with $\sigma(A), \sigma(B) \subseteq I$. If both $f, g: I \to [0, \infty)$ are operator (h, m)-convex and operator monotone functions, then fg is also operator (h, m)-convex functions.

Proof: By the assumptions $A, B \in \mathcal{K}$ be commutative and comparable with $\sigma(A), \sigma(B) \subseteq I$. Without loss the generalization, we take $A \leq B$. According to [16], it holds that $f(A)g(B) = g(B)f(A)$. Since f, g are operator monotone functions, then we have $f(B) - f(A) \succcurlyeq 0$ and $g(B) - g(A) \succcurlyeq 0$. So, by multiplication we have

$$\big(f(B) - f(A)\big)\big(g(B) - g(A)\big) \succcurlyeq 0_{\mathcal{H}}$$

or

$$f(B)g(B) - f(A)g(B) - f(B)g(A) + f(A)g(A) \succcurlyeq 0\,. \tag{4}$$

Since $f(A)g(B) = g(B)f(A)$, then (4) become

$$f(B)g(B) + f(A)g(A) \succcurlyeq f(A)g(B) + f(B)g(A)\,. \tag{5}$$

Next, because f, g are operator (h, m)-convex functions, then we get

$$\begin{aligned} fg(tA + m(1-t)B) &= f(tA + m(1-t)B) \cdot g(tA + m(1-t)B) \\ &\preccurlyeq [h(t)f(A) + mh(1-t)f(B)] \\ &\quad \cdot [h(t)g(A) + mh(1-t)g(B)] \\ &= h^2(t)f(A)g(A) + m^2h^2(1-t)f(B)g(B) \\ &\quad + mh(t)h(1-t)[f(A)g(B) + f(B)g(A)]. \end{aligned}$$

Based on (5), we thus have

$$\begin{aligned} fg(tA + m(1-t)B) &\preccurlyeq h^2(t)f(A)g(A) + m^2h^2(1-t)f(B)g(B) \\ &\quad + mh(t)h(1-t)[f(A)g(A) + f(B)g(B)]. \\ &= h(t)[h(t) + mh(1-t)]f(A)g(A) \\ &\quad + mh(1-t)[h(t) + mh(1-t)]f(B)g(B) \end{aligned}$$

By the assumption $h(t) + mh(1-t) \leq 1$. Hence, we obtain

$$fg(tA + m(1-t)B) \preccurlyeq h(t)f(A)g(A) + mh(1-t)f(B)g(B).$$

That means fg is operator (h, m)-convex functions on I for opeators in $\mathcal{K}$.

■

The characteristic of h as in the Definition 4 has several consequences to the properties of the operator (h, m)-convex functions. In the following Theorem 1 and Theorem 2, we obtain the properties of the operator (h, m)-convex functions based on

the characteristic of the h functions.

Theorem 1. Let $m \in [0,1]$, $I = [0,b] \subseteq \mathbb{R}$, $(0,1) \subseteq J$, and $h: J \to [0,\infty)$.

(a) Let $f: I \to \mathbb{R}$ be operator (h,m)-convex function on I and $f(0_{\mathcal{H}}) = 0_{\mathcal{H}}$. If h is a super-multiplicative function, then holds

$$f(\alpha A + m\beta B) \preccurlyeq h(\alpha) f(A) + mh(\beta) f(B),$$

with $A, B \in \mathcal{K}$ with $\sigma(A), \sigma(B) \subseteq I$ and $\alpha, \beta \in \mathbb{R}$ with $\alpha + \beta \leq 1$.

(b) Let $f: I \to \mathbb{R}$ such that $-f$ be operator (h,m)-convex function and $f(0_{\mathcal{H}}) = 0_{\mathcal{H}}$. If h is a sub-multiplicative function, then holds

$$f(\alpha A + m\beta B) \succcurlyeq h(\alpha) f(A) + mh(\beta) f(B),$$

with $A, B \in \mathcal{K}$ with $\sigma(A), \sigma(B) \subseteq I$ and $\alpha, \beta \in [0,\infty)$ with $\alpha + m\beta \leq 1$.

Proof: Let $A, B \in \mathcal{K}$ with $\sigma(A), \sigma(B) \subseteq I$. We separate our proof for each poin (a) and (b) as follows. (a) Let α, β be positive numbers such that $\alpha + \beta = \gamma \leq 1$ or we write $\frac{\alpha}{\gamma} + \frac{\beta}{\gamma} = 1$. Also let us define $\lambda_1 = \frac{\alpha}{\gamma}$ and $\lambda_2 = \frac{\beta}{\gamma}$. Since f be operator (h,m)-convex function on I and $f(0_{\mathcal{H}}) = 0_{\mathcal{H}}$, then have

$$\begin{aligned} f(\alpha A + m\beta B) &= f(\lambda_1 \gamma A + m\lambda_2 \gamma B) \preccurlyeq h(\lambda_1) f(\gamma A) + mh(\lambda_2) f(\gamma B) \\ &= h(\lambda_1) f(\gamma A + m(1-\gamma) 0_{\mathcal{H}}) + mh(\lambda_2) f(\gamma B + m(1-\gamma) 0_{\mathcal{H}}) \\ &\preccurlyeq h(\lambda_1)[h(\gamma) f(A) + mh(1-\gamma) f(0_{\mathcal{H}})] \\ &\quad + mh(\lambda_2)[h(\gamma) f(B) + mh(1-\gamma) f(0_{\mathcal{H}})] \\ &= h(\lambda_1) h(\gamma) f(A) + mh(\lambda_2) h(\gamma) f(B). \end{aligned}$$

Since h is a super-multiplicative function, thus we have

$$\begin{aligned} f(\alpha A + m\beta B) &\preccurlyeq h(\lambda_1 \gamma) f(A) + mh(\lambda_2 \gamma) f(B) \\ &= h(\alpha) f(A) + mh(\beta) f(B). \end{aligned}$$

The proof for case (b) in this theorem is similar to the proof for case (a).

■

Theorem 2. Let $m \in [0,1]$, $J_k \supseteq (0,1)$, $h_k: J_k \to [0,\infty)$ with $k = 1,2$, and $h_2(J_2) \subseteq J_1$ such that $h_2(t) + h_2(1-t) \leq 1$ for every $t \in (0,1)$. Suppose $f: I_1 \supseteq \{0\} \to \mathbb{R}$ and $g: I_2 \to \mathbb{R}$ with $g(I_2) \subseteq I_1$ and $f(0_{\mathcal{H}}) = 0_{\mathcal{H}}$.

(a) If h_1 is a super-multiplicative function, f is operator (h_1,m)-convex function and monotone on I_1, and g is operator (h_2,m)-convex function, then $f \circ g$ is operator (h_3,m)-convex function.

(b) If h_1 is a sub-multiplicative function, $-f$ is operator (h_1,m)-convex function and monotone on I_1, and $-g$ is operator (h_2,m)-convex function, then $-f \circ (-g)$ is operator (h_3,m)-convex function.

Here $h_3 = h_1 \circ h_2$.

Proof: Let $A, B \in \mathcal{K}$ with $\sigma(A), \sigma(B) \subseteq I_2$. We separate our proof for each case (a) and (b) as follows. (a) Since g is operator (h_2,m)-convex function on I_2 and f is monotone operator, then for any $t \in (0,1)$ we have

$$(f \circ g)(tA + m(1-t)B) \preccurlyeq f\big(h_2(t) g(A) + mh_2(1-t) g(B)\big).$$

Next, using Theorem 1 (a) and because $g(I_2) \subseteq I_1$, f is operator (h_1,m)-convex function on I_1, and $h_2(J_2) \subseteq J_1$ such that $h_2(t) + h_2(1-t) \leq 1$, then for every $t \in (0,1)$ we have

$$\begin{aligned} (f \circ g)(tA + m(1-t)B) &\preccurlyeq h_1\big(h_2(t)\big) f\big(g(A)\big) + mh_1\big(h_2(1-t)\big) f\big(g(B)\big) \\ &= h_3(t) f\big(g(A)\big) + mh_3(1-t) f\big(g(B)\big). \end{aligned}$$

The proof for case (b) in this theorem is similar to the proof for case (a).

■

Now, we introduce the definition of matrix (h,m)-convex functions, a special case in our discussions. Let us denote $\mathcal{M}_n(\mathcal{H})$ as family of matrices with n order. Thus, if $A \in \mathcal{M}_n(\mathcal{H})$, then $A: \mathcal{H}_n \to \mathcal{H}_n$, in which $\mathcal{H}_n \subseteq \mathcal{H}$ is a finite dimensional space. For more detail information about matrix convex functions can be seen in [17] and [27]. The definition of matrix (h,m)-convex functions can be seen in the following Definition 11. In [17], it stated that if $A \in \mathcal{M}_n(\mathbb{C})$ is a self-adjoint matrix, then there exists a unitary operator $U \in \mathcal{M}_n(\mathbb{C})$ and a diagonal matrix $D = \text{diag}[\lambda_1, \dots, \lambda_n] \in \mathcal{M}_n(\mathbb{C})$, with $\lambda_i \in \sigma(A)(i = 1, \dots, n)$ such that $A = UDU^*$ and $f(A) = Uf(D)U^*$. Here, $f(D) \in \mathcal{M}_n(\mathbb{C})$ it can be written as $f(D) = \text{diag}[f(\lambda_1), \dots, f(\lambda_n)]$.

Definition 11. Let $m \in [0,1]$, $n \in \mathbb{N}$, $I = [0,b] \subseteq \mathbb{R}$, $(0,1) \subseteq J$, and $h: J \to [0,\infty)$. A function $f: I \to \mathbb{R}$ is called to be matrix (h,m)-convex function if for every self-adjoint matrices $A, B \in \mathcal{M}_n(\mathcal{H})$ and $t \in (0,1)$ holds

$$f(tA + m(1-t)B) \preccurlyeq h(t)f(A) + mh(1-t)f(B).$$

Note that, the matrix (h,m)-convex functions also have properties of operator (h,m)-convex functions, since they are a special case. In the following Proposition 7 and Proposition 8, we present two special properties of the matrix (h,m)-convex functions.

Proposition 7. Let $m \in [0,1]$, $I = [0,b] \subseteq \mathbb{R}$, $(0,1) \subseteq J$, and $h: J \to [0,\infty)$. Suppose $f: I \to \mathbb{R}$ be a continuous function. Then, f is an operator (h,m)-convex function on I if only if f is a matrix (h,m)-convex function on I.

Proof: ($\Rightarrow$) Let $A, B \in \mathcal{M}_n(\mathcal{H})$ be self-adjoint matrices and $\mathcal{H}_n \subseteq \mathcal{H}$ be a finite dimensional space. Also let $P_n \in \mathcal{M}_n(\mathcal{H})$ be the projection matrix on $\mathcal{H}_n$. Define $\hat{A} := AP_n: \mathcal{H} \to \mathcal{H}_n$ and $\hat{B} := BP_n: \mathcal{H} \to \mathcal{H}_n$. Since A , B are self-adjoint matrices, then for every $x, y \in \mathcal{H}$, we can obtain

$$\langle \hat{A}x, y\rangle = \langle AP_n x, y\rangle = \langle Ax, y\rangle = \langle x, Ay\rangle = \langle x, AP_n y\rangle = \langle x, \hat{A}y\rangle.$$

So, if A, b are self-adjoint matrices, then $\hat{A}, \hat{B}$ also are self-adjoint matrices. It can be observed that $f(A) = f(\hat{A})\big|_{\mathcal{H}_n}$. Thus, we have

$$\begin{aligned} f(tA + m(1-t)B) = f\big(t\hat{A} + m(1-t)\hat{B}\big) &\preccurlyeq h(t)f\big(\hat{A}\big) + mh(1-t)f\big(\hat{B}\big) \\ &= h(t)f(A) + mh(1-t)f(B). \end{aligned}$$

($\Leftarrow$) By the assumption f is a continuous function on I, according to the Weierstrass approximation theorem (see [28, Theorem B]), then there is a sequence of polynomials, namely $\{p_n\}_{n\in\mathbb{N}}$, such that $p_n \to f$ on I. Let us consider a subspace $\mathcal{H}_{n_k} \subseteq \mathcal{H}_n$, in which it is generated by the vectors $1_{\mathcal{H}}x, Ax, A^2x, A^{n_k}x$. Also let $P_{n_k} \in \mathcal{M}_{n_k}(\mathcal{H})$ be the projection matrix on $\mathcal{H}_{n_k}$. Thus, we can define $A_n := AP_{n_k}: \mathcal{H}_n \to \mathcal{H}_{n_k}$ and $B_n := BP_{n_k}: \mathcal{H}_n \to \mathcal{H}_{n_k}$. Since

$$\langle A_n x, y\rangle = \langle AP_{n_k}x, y\rangle = \langle Ax, y\rangle = \langle x, Ay\rangle = \langle x, AP_{n_k}y\rangle = \langle x, A_n y\rangle,$$

then A_n and B_n are self-adjoint matrices. This means $p_n(A_n)x = p_n(A)x$ or

$$p_n(A_n)x - p_n(A)x = 0.$$

Because f is a continuous function on I, for any $\varepsilon > 0$, then there exist $N \in \mathbb{N}$ such that for $n > N$ holds

$$\|f(A) - p_n(A)\| = \frac{\varepsilon}{2}.$$

Hence, we can calculate

$$\begin{aligned} \|f(A_n) - f(A)\| &= \|f(A_n) - p_n(A_n) + p_n(A_n) - p_n(A) + p_n(A) - f(A)\| \\ &\le \|f(A_n) - p_n(A_n)\| + \|p_n(A_n) - p_n(A)\| + \|p_n(A) - f(A)\| \end{aligned}$$

$$\leq \frac{\varepsilon}{2} + 0 + \frac{\varepsilon}{2} = \varepsilon.$$

This means $f(A_n) \to f(A)$. Using the same way, we obtain $f(B_n) \to f(B)$ and $f(tA_n + m(1-t)B_n) \to f(tA + m(1-t)B)$. Next, since f is matrix (h,m)-convex functions, then we have

$$f(tA_n + m(1-t)B_n) \preccurlyeq h(t)f(A_n) + mh(1-t)f(B_n)$$

or

$$h(t)f(A_n) + mh(1-t)f(B_n) - f(tA_n + m(1-t)B_n) \succcurlyeq 0_{\mathcal{H}}.$$

It means

$$\langle [h(t)f(A_n) + mh(1-t)f(B_n) - f(tA_n + m(1-t)B_n)]x|x\rangle \geq 0.$$

Since n tends to infinity, then we have

$$\lim_{n\to\infty} \langle [h(t)f(A_n) + mh(1-t)f(B_n) - f(tA_n + m(1-t)B_n)]x|x\rangle \geq 0.$$

Hence, we get

$$\langle [h(t)f(A) + mh(1-t)f(B) - f(tA + m(1-t)B)]x|x\rangle \geq 0.$$

That can be expressed as

$$f(tA + m(1-t)B) \preccurlyeq h(t)f(A) + mh(1-t)f(B).$$

This is the end of our proof.

■

Proposition 8. Let $m \in [0,1]$, $n \in \mathbb{N}$, $I = [0,b] \subseteq \mathbb{R}$, $(0,1) \subseteq J$, and $h: J \to [0,\infty)$. Also let the sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ with $f_n: I \to \mathbb{R}$ be a sequence of matrix (h,m)-convex functions on I. If $f_n \to f$, then f is also matrix (h,m)-convex function on I.

Proof: Let $A, B \in \mathcal{M}_n(\mathcal{H})$ be self-adjoint matrices with $\sigma(A), \sigma(B) \subseteq I$. Thus, we can write $A = UDU^*$, where U is unitary matrix and $D = \text{diag}[\lambda_1, \dots, \lambda_n]$ with $\lambda_i (i = 1, \dots, n) \in \sigma(A)$. Since $f_n \to f$, then we have

$$f_n(A) = f_n(UDU^*) = Uf_n(D)U^* = Uf(D)U^* = f(UDU^*) = f(A).$$

That means $f_n(A) \to f(A)$. Using the same way for operator B, we obtain $f_n(B) \to f(B)$. Next, because $\{f_n\}_{n\in\mathbb{N}}$ are sequence of (h,m)-convex function on I, then it holds

$$f_n(tA + m(1-t)B) \preccurlyeq h(t)f_n(A) + mh(1-t)f_n(B)$$

or

$$h(t)f_n(A) + mh(1-t)f_n(B) - f_n(tA + m(1-t)B) \succcurlyeq 0_{\mathcal{H}}.$$

It means

$$\langle [h(t)f_n(A) + mh(1-t)f_n(B) - f_n(tA + m(1-t)B)]x|x\rangle \geq 0.$$

Since n tends to infinity, then we have

$$\lim_{n\to\infty} \langle [h(t)f_n(A) + mh(1-t)f_n(B) - f_n(tA + m(1-t)B)]x|x\rangle \geq 0.$$

Hence, we obtain

$$\langle [h(t)f(A) + mh(1-t)f(B) - f(tA + m(1-t)B)]x|x\rangle \geq 0$$

or it can be expressed as

$$f(tA + m(1-t)B) \preccurlyeq h(t)f(A) + mh(1-t)f(B).$$

This is the end of our proof.

■

Lastly, based on the results in Proposition 7 and Proposition 8 we give a Corollary below.

Corollary 1. Let $m \in [0,1]$, $b \geq 1$, $I = [0,b] \subseteq \mathbb{R}$, $(0,1) \subseteq J$, and $h: J \to [0,\infty)$. If $\{f_n\}_{n\in\mathbb{N}}$ with $f_n: I \to \mathbb{R}$ are a sequence of operator (h,m)-convex functions on I such that $f_n \to f$, then f is an of operator (h,m)-convex functions on I.

CONCLUSIONS

The properties of the operator (h, m)-convex functions have some similarities with the operator h-convex functions and m-convex functions. However, since this class is the generalization of the h-convexity and m-convexity, then it needs to fulfill some additional conditions. A special case, matrix (h, m)-convex functions also have their own unique properties.

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REFERENCES

[1] D. Henrion, "Convexity", *The Princeton Companion to Applied Mathematics*, part II, Princeton University Press, 2015.

[2] J. E. Pečarić, F. Proschan, and Y. L. Tong, Convex Functions, Partial Orderings, and Statistical Application, Academic Press, San Diego, 1992.

[3] G. H. Toader, "Some generalizations of the convexity," *J Proc. Colloq. Approx. Optim. Cluj-Naploca,* pp. 329-338, 1984.

[4] T. Lara, R. Quintero, E. Rosales, and J. Sanchez, "On strongly Jensen m-convex functions," *Journal of Pure Mathematical Sciences,* vol. 6, pp. 87-94, 2017.

[5] S. Varošanec, "On h-convexity," *Journal of Mathematical Analysis and Applications*, vol. 326, no. 1, pp. 303-311, Elsevier, 2007.

[6] E. K. Godunova and V. I. Levin, "Inequalities for functions of a broad class that contains convex, monotone and some other forms of functions," *Proceedings of (Russian) Numerical Mathematics and Mathematical Physics*, vol. 166, pp. 138-142, 1985.

[7] H. Hudzik and L. Maligranda, "Some remarks on s-convex functions," *Aequationes mathematicae*, vol. 48, pp. 100-111, 1994.

[8] S. Sezer, "The Hermite–Hadamard inequality for s-convex functions in the third sense," AIMS Math, vol. 6, no. 7, pp. 7719-7732, 2021.

[9] C. E. M. Pearce and A. M. Rubinov "P-functions, quasi-convex functions, and Hadamard-type Inequalities, *Journal of Mathematical Analysis and Applications*, vol. 240, pp. 92-104, 1999.

[10] M. E. Özdemir, "Some inequalities for the s-Godunova–Levin type functions," *Journal of Mathematical Sciences*, vol. 9, pp. 27-32, 2015.

[11] M. E. Özdemir, A. O. Akdemir, E. Set, "On (h,m)-convexity and Hadamard type inequalities," *Transylvanian Journal of Mathematical and Mechanics*, vol. 8, no. 1, pp. 51-58, 2016.

[12] B. Bayraktar and M. Gürbüz, "On some integral inequalities for (s, m)-convex functions," *TWMS J. App. Eng. Math.*, vol. 10, no. 2, pp. 288-295, 2020.

[13] M. Sertbaş and İ. Mihyaz, " Some results for (s, m)-convex function in the second sense," *Maltepe Journal of Mathematics*, vol. IV, no. 1, pp. 1-8, 2022.

[14] M. A. Noor, K. I. Noor, and M. U. Awan, " Fractional ostrowski inequalities for (s, m)-Godunova-Levin functions," *Ser. Math. Inform*, vol. 30, no. 4, pp. 489-499, 2015.

[15] R. Bhatia, *Matrix Analysis*, Springer-Verlag, New York, 1997.

[16] Y. Erdaş, E. Unluyol, S. Salaş, "The Hermite-Hadamard type inequalities for operator m-convex functions in Hilbert space," *Journal of New Theory*, vol. 5, pp. 80-91, 2015.

[17] L. Berbesi, T. Lara, P. Pena, and E. Rosales, "On operator m-convex functions in Hilbert space," *UPI Journal of Mathematics and Biostatistics*, vol. 1, no. 2, JMB11, 2018.

[18] V. Darvish, S. S. Dragomir, H. M. Nazari, and A. Taghavi, "Some inequalities associated with the Hermite-Hadamard inequalities for operator h-convex functions," *Acta Et Commentationes Universitatis Tartuensis De Mathematica*, vol. 21, no. 2, 2017.

[19] A. G. Ghazanfari, "The Hermite-Hadamard type inequalities for operator s-convex functions," *Journal of Advanced Research in Pure Mathematics*, vol. 6, no. 2, pp. 52-61, 2014.

[20] S. Salaş, E. Unluyol, Y. Erdaş,"The Hermite-Hadamard type inequalities for operator Godunova-Levin class of functions in Hilbert space," *Ordu Üniv. Bil. Tek. Derg*, vol. 6, no. 1, pp. 150-160, 2016.

[21] M. A. Noor, M. U. Awan, K. I. Noor, "Bounds involving operator *s*-Godunova-Levin-Dragomir functions," *Modern Discrete Mathematics and Analysis,* vol. 131, pp. 353-361, 2018.

[22] S. Salaş, E. Unluyol, D. Yardimciel, "Operator (h,m)-convexity and Hermite-Hadamard type inequalities," *Ordu University Journal of Science and Technology*, vol. 7, no. 2, pp. 367-377, 2017.

[23] T. Lara, N. Merentes, Z. páles, and E. Rosales, "On m-convexity of set-valued functions," *Advanced in Operator Theory*, vol. 4, no. 4, pp. 767-783, 2019.

[24] L. C. Evans, *Partial Differential Equations* (2nd Ed.), American Mathematical Society, Rhode Island, 2010.

[25] S. S. Dragomir, "Hermite–Hadamard's type inequalities for operator convex functions," *Applied Mathematics and Computation*, vol. 218, pp. 766-772, Elsevier, 2011.

[26] Y. Erdaş, E. Unluyol, S. Salaş, "Some new inequalities of operator *m*-convex functions and applications for synchronous-asynchronous functions," *Complex Analysis and Operator Theory*, vol. 13, pp. 3871-3881, 2019.

[27] T. Ando, "Trace-inequalities and matrix-convex functions," *Fixed Point Theory and Applications*, vol. 2010, Id. 241908, Hindawi, 2010.

[28] A. Pinkus, "The Weierstrass approximation theorems," *Surveys in Approximation Theory*, vol. 1, pp. 1-37, 2004.

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Mostafa Abbasi, Ali Morassaei, Farzollah Mirzapour. "Jensen–Mercer Type Inequalities for Operator h-Conv...

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Mostafa Abbasi, Ali Morassaei, Farzollah Mirzapour. "Jensen–Mercer Type Inequalities for Operator h-Conv...

→ be a

"Approximation and Computation in Science and Engineering", Springer Science and Business Media LLC, ...

1 −))

Mostafa Abbasi, Ali Morassaei, Farzollah Mirzapour. "Jensen–Mercer Type Inequalities for Operator h-Conv...

it canbe written as

"Differential and Integral Inequalities", Springer Science and Business Media LLC, 2019

( )) ( ) = ( )) (

Bo-Yan Xi, Jian Sun, Shu-Ping Bai. "On some Hermite-Hadamard-type integral inequalities for co-ordinated (...

then we have (

Shigeru Furuichi, Hamid Reza Moradi. "Advances in Mathematical Inequalities", Walter de Gruyter GmbH, ...

Grandmont, J.M.. "Capital-Labor Substitution and Competitive Nonlinear Endogenous Business Cycles", J...

0,] ⊆ , ∈ [0,1], (0,1

Bo-Yan Xi, Feng Qi. "Properties and inequalities for the (h1, h2)- and (h1, h2, m)-GA-convex functions", Cogen...

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S. S. DRAGOMIR. "OPERATOR QUASILINEARITY OF SOME FUNCTIONALS ASSOCIATED WITH DAVIS–CHOI...

∈ [0,1], (0,1

Bo-Yan Xi, Feng Qi. "Properties and inequalities for the (h1, h2)- and (h1, h2, m)-GA-convex functions", Cogen...

and

Mostafa Abbasi, Ali Morassaei, Farzollah Mirzapour. "Jensen–Mercer Type Inequalities for Operator h-Conv...

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Mostafa Abbasi, Ali Morassaei, Farzollah Mirzapour. "Jensen–Mercer Type Inequalities for Operator h-Conv...

. A function : → is

Vuk Stojiljković, Rajagopalan Ramaswamy, Fahad Alshammari, Ola A. Ashour, Mohammed Lahy Hassan Alg...

), ( ) ≤ ( ) (,

James K. Peterson. "Basic Analysis I - Functions of a Real Variable", CRC Press, 2020

studied the

Shashi Kant Mishra, Balendu Bhooshan Upadhyay. "Pseudolinear Functions and Optimization", Chapman an...

convex if

Handbook of Combinatorial Optimization, 2013.

Let X be a

Nonconvex Optimization and Its Applications, 2000.

for any

Kazuo Murota, Akiyoshi Shioura. "Relationship of M-/L-convex functions with discrete convex functions by ...

is called

Marcela V. Mihai, Flavia-Corina Mitroi-Symeonidis. "New Extensions of Popoviciu's Inequality", Mediterranea...

in this

"Nonlinear Analysis, Differential Equations, and Applications", Springer Science and Business Media LLC, ...