



Generalized Subclasses of Analytic Functions Using Extended q -Differential Operators: Fekete-Szegő Problem

Victor Wanjala^{*1} and Beatrice Obiero Adhiambo²

¹*Department of Mathematics and Physical Sciences, Maasai Mara University, P.O. Box 861-20500, Narok, Kenya*

²*Department of Mathematics and Computing, Rongo University, P.O. Box 103-40404, Rongo, Kenya*

Abstract

In this paper, we introduce two new subclasses of analytic functions using an extended q -differential operator denoted by $\mathcal{M}_{q,\lambda,\beta}^{n,m}f(z)$. This operator generalizes many existing q -derivative operators in the literature. We obtain sharp upper bounds for the initial coefficients a_2 , a_3 and the Fekete-Szegő functional $|a_3 - \mu a_2^2|$ for functions belonging to these subclasses. The results unify and extend several known results. Strong proofs are provided using subordination principles and the Carathéodory lemma. Special cases and sharpness of the bounds are discussed.

Keywords: 30C45; 30C50; q -differential operator; analytic function; univalent function; Fekete-Szegő functional; subordination; coefficient bounds

Copyright © 2026 by Authors, Published by CAUCHY Group. This is an open access article under the CC BY-SA License (<https://creativecommons.org/licenses/by-sa/4.0>)

1. Introduction

The study of analytic univalent functions has a rich history dating back to the early development of geometric function theory. Let $\mathcal{A}$ denote the class of all analytic functions f in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ normalized by $f(0) = 0$ and $f'(0) = 1$. Each $f \in \mathcal{A}$ has a Taylor series expansion [1]:

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j, \quad a_j \in \mathbb{C}, \quad z \in \mathbb{U}. \quad (1)$$

The Fekete-Szegő functional $|a_3 - \mu a_2^2|$ is a classical coefficient problem that has attracted considerable attention in the field of geometric function theory. Many researchers have investigated this functional for various subclasses of analytic functions [2, 3, 4, 5]. This problem is fundamental because it provides insight into the geometric properties of functions and their coefficient behavior.

The q -calculus has emerged as a powerful tool in geometric function theory, offering generalizations of classical results. Jackson [6] was the first to develop the q -derivative operator systematically. The q -derivative operator D_q is defined for $0 < q < 1$ by the formula $D_q f(z) = (f(z) - f(qz))/((1-q)z)$ for $z \neq 0$, with $D_q f(0) = f'(0)$. The q -number $[j]_q = (1 - q^j)/(1 - q) = 1 + q + q^2 + \cdots + q^{j-1}$ plays a central role in this framework.

^{*}Corresponding author. E-mail: wanjalavictor421@gmail.com

Several researchers have studied q -analogues of differential operators in the context of analytic functions. Ramachandran et al. [7] introduced a q -difference operator, while Seoudy and Aouf [8] investigated q -starlike and q -convex functions. The concept of subordination, defined through the existence of a Schwarz function, is fundamental to defining new subclasses of analytic functions [9].

In this paper, we generalize the recent work of Yie and Janteng [10] by introducing an extended q -differential operator with additional parameters λ , β , and m . This operator unifies many existing q -derivative operators in the literature. We define two new subclasses $\mathcal{X}_{q,\lambda,\beta}^{n,m}(\phi)$ and $\mathcal{Y}_{q,\lambda,\beta}^{n,m}(\phi, \delta)$ and obtain sharp bounds for the Fekete-Szegő functional for functions in these classes.

The structure of this paper is as follows. Section 2 introduces the extended q -differential operator and defines the new subclasses. Section 3 presents the main results, including sharp bounds for the Fekete-Szegő functional and their proofs. Section 4 concludes the paper with a summary and suggestions for future research directions.

2. Methods

2.1. The Extended q -Differential Operator

The methodology in this paper is theoretical and proof-based, employing techniques from geometric function theory and q -calculus. We begin by defining the central operator that will be used throughout the study.

For $f \in \mathcal{A}$ given by (1), let $n, m \in \mathbb{N}_0$, $\lambda \geq 0$, and $\beta \in \mathbb{C}$ with $|\beta| < 1$. Define the extended q -differential operator $\mathcal{M}_{q,\lambda,\beta}^{n,m} : \mathcal{A} \rightarrow \mathcal{A}$ by:

$$\mathcal{M}_{q,\lambda,\beta}^{n,m}f(z) = z + \sum_{j=2}^{\infty} \left(\frac{[j]_q + \lambda}{1 + \lambda} \right)^n \left(\frac{[j]_q + \beta}{1 + \beta} \right)^m a_j z^j. \quad (2)$$

This operator generalizes several known operators in the literature. When $\lambda = 0$, $\beta = 0$, and $m = 0$, we obtain the operator $M_q^n f(z)$ as studied by Yie and Janteng [10]. When $n = 0$, $m = 1$, and $\beta = 0$, we obtain $\mathcal{M}_{q,\lambda,0}^{0,1}f(z) = z + \sum_{j=2}^{\infty} [j]_q a_j z^j = z D_q f(z)$, which is precisely the q -derivative operator multiplied by z . When $q \rightarrow 1^-$, we recover the classical differential operator.

The q -derivative of the extended operator is given by:

$$D_q \left(\mathcal{M}_{q,\lambda,\beta}^{n,m}f(z) \right) = 1 + \sum_{j=2}^{\infty} [j]_q \left(\frac{[j]_q + \lambda}{1 + \lambda} \right)^n \left(\frac{[j]_q + \beta}{1 + \beta} \right)^m a_j z^{j-1}. \quad (3)$$

For $j = 2, 3$, we define $A_j = \left(\frac{[j]_q + \lambda}{1 + \lambda} \right)^n \left(\frac{[j]_q + \beta}{1 + \beta} \right)^m$, so that (3) becomes:

$$D_q \left(\mathcal{M}_{q,\lambda,\beta}^{n,m}f(z) \right) = 1 + [2]_q A_2 a_2 z + [3]_q A_3 a_3 z^2 + \cdots. \quad (4)$$

2.2. New Subclasses

The concept of subordination is fundamental to defining our new subclasses. For two analytic functions f and g in $\mathbb{U}$, f is subordinate to g , written $f \prec g$, if there exists a Schwarz function $\omega(z)$ analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ such that $f(z) = g(\omega(z))$ for all $z \in \mathbb{U}$ [9].

Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$ with $B_1 \neq 0$ be analytic and univalent in $\mathbb{U}$ with $\text{Re}(\phi(z)) > 0$. We define the following subclasses:

Definition 1. A function $f \in \mathcal{A}$ belongs to the class $\mathcal{X}_{q,\lambda,\beta}^{n,m}(\phi)$ if:

$$D_q \left(\mathcal{M}_{q,\lambda,\beta}^{n,m}f(z) \right) \prec \phi(z). \quad (5)$$

Definition 2. For $0 \leq \delta \leq 1$, a function $f \in \mathcal{A}$ belongs to the class $\mathcal{Y}_{q,\lambda,\beta}^{n,m}(\phi, \delta)$ if:

$$(1 - \delta) \frac{z D_q \left(\mathcal{M}_{q,\lambda,\beta}^{n,m} f(z) \right)}{\mathcal{M}_{q,\lambda,\beta}^{n,m} f(z)} + \delta \left(1 + \frac{q z D_q^2 \left(\mathcal{M}_{q,\lambda,\beta}^{n,m} f(z) \right)}{D_q \left(\mathcal{M}_{q,\lambda,\beta}^{n,m} f(z) \right)} \right) \prec \phi(z). \quad (6)$$

These classes generalize several known subclasses. When $\delta = 0$, the class $\mathcal{Y}_{q,\lambda,\beta}^{n,m}(\phi, 0)$ reduces to a class related to q -starlike functions. When $\delta = 1$, it reduces to a class related to q -convex functions.

2.3. Auxiliary Lemmas

The following lemmas are essential for our proofs. The Carathéodory lemma provides bounds on the coefficients of functions with positive real part.

Lemma 1 (Carathéodory Lemma). Let $p(z) = 1 + p_1 z + p_2 z^2 + \dots$ be analytic in $\mathbb{U}$ with $\operatorname{Re}(p(z)) > 0$. Then for all $k \geq 1$, $|p_k| \leq 2$. Moreover, for any complex number ν ,

$$|p_2 - \nu p_1^2| \leq 2 \max\{1, |2\nu - 1|\}. \quad (7)$$

The bounds are sharp.

The Schwarz function representation establishes the relationship between a Schwarz function and a Carathéodory function.

Lemma 2. If $\omega(z)$ is a Schwarz function with $\omega(0) = 0$ and $|\omega(z)| < 1$ in $\mathbb{U}$, and we define $p(z) = (1 + \omega(z))/(1 - \omega(z)) = 1 + p_1 z + p_2 z^2 + \dots$, then $\operatorname{Re}(p(z)) > 0$ and

$$\omega(z) = \frac{p(z) - 1}{p(z) + 1} = \frac{1}{2} \left(p_1 z + \left(p_2 - \frac{p_1^2}{2} \right) z^2 + \left(p_3 + \frac{p_1^3}{4} - p_1 p_2 \right) z^3 + \dots \right). \quad (8)$$

These lemmas provide the analytical tools needed to derive coefficient bounds for functions in our new subclasses.

3. Results and Discussion

3.1. Results for the Class $\mathcal{X}_{q,\lambda,\beta}^{n,m}(\phi)$

We now present the main results for the class $\mathcal{X}_{q,\lambda,\beta}^{n,m}(\phi)$. The following theorem provides sharp upper bounds for the Fekete-Szegő functional.

Theorem 1. Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \dots$ with $B_1 \neq 0$, and let $f \in \mathcal{A}$ given by (1) be in the class $\mathcal{X}_{q,\lambda,\beta}^{n,m}(\phi)$. For any complex number μ , we have:

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{[3]_q A_3} \max \left\{ 1, \left| \frac{B_2}{B_1} - \frac{[3]_q A_3 \mu B_1}{[2]_q^2 A_2^2} \right| \right\}, \quad (9)$$

where A_2 and A_3 are given by $A_j = \left(\frac{[j]_q + \lambda}{1 + \lambda} \right)^n \left(\frac{[j]_q + \beta}{1 + \beta} \right)^m$ for $j = 2, 3$.

Proof. If $f \in \mathcal{X}_{q,\lambda,\beta}^{n,m}(\phi)$, then by Definition 3.1 there exists a Schwarz function $\omega(z)$ such that $D_q(\mathcal{M}_{q,\lambda,\beta}^{n,m} f(z)) = \phi(\omega(z))$.

Define the Carathéodory function $p(z)$ as in (8). From (8) and the expansion of ϕ , we

obtain:

$$\phi(\omega(z)) = 1 + \frac{B_1 p_1}{2} z + \left(\frac{B_1}{2} \left(p_2 - \frac{p_1^2}{2} \right) + \frac{B_2 p_1^2}{4} \right) z^2 + \cdots. \quad (10)$$

From (4), we have:

$$D_q \left(\mathcal{M}_{q,\lambda,\beta}^{n,m} f(z) \right) = 1 + [2]_q A_2 a_2 z + [3]_q A_3 a_3 z^2 + \cdots. \quad (11)$$

Comparing coefficients in (10) and (11) yields:

$$a_2 = \frac{B_1 p_1}{2[2]_q A_2}, \quad (12)$$

$$[3]_q A_3 a_3 = \frac{B_1}{2} \left(p_2 - \frac{p_1^2}{2} \right) + \frac{B_2 p_1^2}{4}. \quad (13)$$

From (13), we have $a_3 = \frac{B_1}{2[3]_q A_3} p_2 - \frac{B_1}{4[3]_q A_3} p_1^2 + \frac{B_2}{4[3]_q A_3} p_1^2$. Therefore,

$$a_3 - \mu a_2^2 = \frac{B_1}{2[3]_q A_3} \left(p_2 - \gamma p_1^2 \right), \quad (14)$$

where $\gamma = \frac{1}{2} \left(1 - \frac{B_2}{B_1} + \frac{[3]_q A_3 \mu B_1}{[2]_q^2 A_2^2} \right)$.

Applying the Carathéodory lemma, we have $|p_2 - \gamma p_1^2| \leq 2 \max\{1, |2\gamma - 1|\}$. Noting that $2\gamma - 1 = \frac{B_2}{B_1} - \frac{[3]_q A_3 \mu B_1}{[2]_q^2 A_2^2}$, the desired bound follows. $\square$

When $\lambda = 0$, $\beta = 0$, $m = 0$, Theorem 5.1 reduces to the following result:

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{[3]_q^{n+1}} \max \left\{ 1, \left| \frac{B_2}{B_1} - \frac{[3]_q^{n+1} \mu B_1}{[2]_q^{2n+2}} \right| \right\}, \quad (15)$$

which matches the corresponding result of Yie and Janteng [10]. This demonstrates that our generalized operator successfully unifies existing results.

3.2. Results for the Class $\mathcal{Y}_{q,\lambda,\beta}^{n,m}(\phi, \delta)$

For the class $\mathcal{Y}_{q,\lambda,\beta}^{n,m}(\phi, \delta)$, we obtain the following theorem. Define:

$$\Gamma_1 = (1 - \delta)[2]_q A_2 ([2]_q - 1) + \delta q [2]_q^2 A_2, \quad (16)$$

$$\Gamma_2 = (1 - \delta)[3]_q A_3 ([3]_q - 1) + \delta q [3]_q^2 A_3, \quad (17)$$

$$\Gamma_3 = (1 - \delta)[2]_q^2 A_2^2 ([2]_q - 1) + \delta q [2]_q^3 A_2^2. \quad (18)$$

Theorem 2. Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$ with $B_1 \neq 0$, and let $f \in \mathcal{A}$ be in the class $\mathcal{Y}_{q,\lambda,\beta}^{n,m}(\phi, \delta)$. For any complex number μ :

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{2\Gamma_2} \max \left\{ 1, \left| \frac{\Gamma_3 B_1}{\Gamma_1^2} - \frac{B_2}{B_1} - \frac{2\mu \Gamma_2}{B_1} \right| \right\}. \quad (19)$$

Proof. If $f \in \mathcal{Y}_{q,\lambda,\beta}^{n,m}(\phi, \delta)$, then by Definition 3.2 there exists a Schwarz function $\omega(z)$ such that the left-hand side of (6) equals $\phi(\omega(z))$.

Using (4), we compute each term separately. For the first term:

$$\frac{zD_q(\mathcal{M}_{q,\lambda,\beta}^{n,m}f(z))}{\mathcal{M}_{q,\lambda,\beta}^{n,m}f(z)} = 1 + [2]_q A_2([2]_q - 1)a_2 z + ([3]_q A_3([3]_q - 1)a_3 - [2]_q^2 A_2^2([2]_q - 1)a_2^2) z^2 + \dots \quad (20)$$

For the second term:

$$1 + \frac{qzD_q^2(\mathcal{M}_{q,\lambda,\beta}^{n,m}f(z))}{D_q(\mathcal{M}_{q,\lambda,\beta}^{n,m}f(z))} = 1 + q[2]_q^2 A_2 a_2 z + q([3]_q^2 A_3 a_3 - [2]_q^3 A_2^2 a_2^2) z^2 + \dots \quad (21)$$

Let $g(z)$ denote the left-hand side of (6). Using the notation in (16)-(18), we have:

$$g(z) = 1 + \Gamma_1 a_2 z + (\Gamma_2 a_3 - \Gamma_3 a_2^2) z^2 + \dots \quad (22)$$

Comparing coefficients with (10) gives:

$$a_2 = \frac{B_1 p_1}{2\Gamma_1}, \quad (23)$$

$$\Gamma_2 a_3 - \Gamma_3 a_2^2 = \frac{B_1}{2} \left(p_2 - \frac{p_1^2}{2} \right) + \frac{B_2 p_1^2}{4}. \quad (24)$$

Solving (24) for a_3 and substituting a_2 from (23) yields:

$$a_3 - \mu a_2^2 = \frac{B_1}{2\Gamma_2} (p_2 - \eta p_1^2), \quad (25)$$

where $\eta = \frac{1}{2} \left(1 - \frac{B_2}{B_1} - \frac{\Gamma_3 B_1}{\Gamma_1^2} + \frac{2\mu\Gamma_2}{B_1} \right)$.

Applying the Carathéodory lemma, we obtain $|p_2 - \eta p_1^2| \leq 2 \max\{1, |2\eta - 1|\}$. Noting that $2\eta - 1 = -\frac{B_2}{B_1} - \frac{\Gamma_3 B_1}{\Gamma_1^2} + \frac{2\mu\Gamma_2}{B_1}$, the desired bound follows. $\square$

When $\delta = 1$, $\lambda = 0$, $\beta = 0$, and $m = 0$, the theorem reduces to the corresponding result of Yie and Janteng [10]. This demonstrates the unifying nature of our extended operator.

3.3. Sharpness and Special Cases

The bounds obtained in Theorem 5.1 are sharp. To establish sharpness, consider the Schwarz function $\omega(z) = z$. The corresponding Carathéodory function is $p(z) = (1+z)/(1-z) = 1 + 2z + 2z^2 + 2z^3 + \dots$, giving $p_1 = 2$ and $p_2 = 2$.

From (12), we have $a_2 = B_1/([2]_q A_2)$. From (13), we have $a_3 = B_2/([3]_q A_3)$. For these values:

$$|a_3 - \mu a_2^2| = \left| \frac{B_2}{[3]_q A_3} - \mu \frac{B_1^2}{[2]_q^2 A_2^2} \right|. \quad (26)$$

This matches the bound in (9) when the maximum is attained by the second term. The sharpness follows from the sharpness of the Carathéodory lemma.

3.4. Discussion

The results obtained in this paper provide a significant generalization of the work of Yie and Janteng [10]. The extended operator $\mathcal{M}_{q,\lambda,\beta}^{n,m}f(z)$ with additional parameters λ , β , and m unifies several existing q -differential operators in the literature. The sharp bounds obtained for the Fekete-Szegő functional are expressed in terms of the parameters, allowing for a wide range of applications.

The class $\mathcal{X}_{q,\lambda,\beta}^{n,m}(\phi)$ generalizes classes defined by the q -derivative operator, while $\mathcal{Y}_{q,\lambda,\beta}^{n,m}(\phi, \delta)$ provides a two-parameter family that interpolates between q -starlike and q -convex functions. The parameter δ controls the convexity of the class, offering flexibility for future applications.

The methodology employed in this paper—combining subordination principles with the Carathéodory lemma—provides a robust framework for deriving sharp coefficient bounds. The proofs are self-contained and demonstrate the power of q -calculus in geometric function theory.

4. Conclusion

In this paper, we have successfully generalized the work of Yie and Janteng [10] by introducing an extended q -differential operator $\mathcal{M}_{q,\lambda,\beta}^{n,m}f(z)$ with additional parameters λ , β , and m . Using this operator, we defined two new subclasses $\mathcal{X}_{q,\lambda,\beta}^{n,m}(\phi)$ and $\mathcal{Y}_{q,\lambda,\beta}^{n,m}(\phi, \delta)$ of analytic functions. We obtained sharp upper bounds for the initial coefficients a_2 , a_3 , and the Fekete-Szegő functional $|a_3 - \mu a_2^2|$ for functions in these classes.

The results unify several existing results in the literature and provide a foundation for further research in q -analytic function theory. The parameter structure of our operator allows for a wide range of special cases, making the results applicable to many existing subclasses.

Future research directions include investigating higher-order coefficient bounds (a_4 , a_5 , etc.) for these classes, studying the Hankel determinant problems, extending the results to meromorphic functions or multivalent functions, and considering q -integral operator analogues. The framework developed in this paper can also be extended to investigate Toeplitz determinants and other coefficient-related problems in geometric function theory.

CRedit Authorship Contribution Statement

Victor Wanjala: Conceptualization, Methodology, Formal Analysis, Investigation, Writing—Original Draft Preparation, Writing—Review & Editing. **Beatrice Obiero Adhiambo:** Conceptualization, Validation, Supervision, Writing—Review & Editing.

Declaration of Generative AI and AI-assisted technologies

No generative AI or AI-assisted technologies were used during the preparation of this manuscript.

Declaration of Competing Interest

The authors declare no competing interests.

Ethics Approval and Consent to Participate

Ethics approval was not required for this study.

Funding and Acknowledgments

This research received no external funding. The authors would like to express sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions, which have significantly improved the quality of this manuscript.

Data and Code Availability

No datasets or code were generated or analyzed during the current study, as this is a purely theoretical work.

References

- [1] P. L. Duren, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften, vol. 259, Springer-Verlag, New York, 1983.
- [2] A. Alsoboh and M. Darus, Fekete-Szegő problem for certain subclass of analytic functions, *Journal of Physics: Conference Series*, vol. 1212, no. 1, pp. 012-018, 2019. DOI: 10.1088/1742-6596/1212/1/012018.
- [3] M. K. Aouf and H. Orhan, Fekete-Szegő problem for certain subclasses of analytic functions, *Turkish Journal of Mathematics*, vol. 42, no. 3, pp. 1013-1023, 2018. DOI: 10.3906/mat-1704-29.
- [4] A. Janteng and S. A. Halim, Coefficient estimates for a subclass of analytic functions, *International Journal of Mathematics and Mathematical Sciences*, vol. 2009, Article ID 829612, 8 pages, 2009. DOI: 10.1155/2009/829612.
- [5] A. Janteng, Fekete-Szegő problem for a new subclass of analytic functions, *Malaysian Journal of Mathematical Sciences*, vol. 14, no. 2, pp. 245-256, 2020.
- [6] F. H. Jackson, On q -functions and a certain difference operator, *Transactions of the Royal Society of Edinburgh*, vol. 46, no. 2, pp. 253-281, 1909. DOI: 10.1017/S0080456800002751.
- [7] C. Ramachandran, T. Soupramanien, and M. K. Aouf, New subclasses of analytic functions associated with q -difference operator, *Journal of the Korean Mathematical Society*, vol. 54, no. 4, pp. 1273-1295, 2017. DOI: 10.4134/JKMS.j160309.
- [8] T. M. Seoudy and M. K. Aouf, Coefficient estimates of new classes of q -starlike and q -convex functions of complex order, *Journal of the Egyptian Mathematical Society*, vol. 24, no. 3, pp. 378-384, 2016. DOI: 10.1016/j.joems.2015.07.005.
- [9] M. P. Jeyaraman and T. Suresh, Subordination and superordination for analytic functions, *International Journal of Mathematics and Mathematical Sciences*, vol. 2014, Article ID 316948, 5 pages, 2014. DOI: 10.1155/2014/316948.
- [10] S. Yie and A. Janteng, Fekete-Szegő problem for a subclass of analytic functions defined by q -differential operator, *Journal of Mathematics and Computing*, vol. 15, no. 1, pp. 28-37, 2024.