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APOS analysis on cognitive process in mathematical proving activities

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A R T I C L E I N F O
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A B S T R A C T
Thinking is very necessary in learning mathematics, both at school and college level. Several studies have attempted to reveal students' thinking in learning mathematics at college. This article aims to describe the mental structure that occurs when constructing mathematical proofs in terms of APOS theory. The APOS theory has been widely used in analyzing the formation of mathematical concepts in universities. This research explores a thinking process in proof constructing. It uses a qualitative approach. The research was conducted on 26 students majored in mathematics education in public university at Banten, Indonesia. The consideration of that was because the students were able to think a formal proof in mathematics. Results show that there are two types of thinking process in mathematical proving activities, namely: the deductive-holistic and the inductive-partial type of thinking process. Based on the results, some suitable learning activities should be designed to support the construction of these mental categories.

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1. INTRODUCTION

Thinking is very necessary in learning mathematics, both at school and college level. According to Mason, Burton, and Stacey (2010), mathematical thinking is related with mathematical processes that include: specializing (trying on special cases or special examples), generalizing (formulating patterns of relationships), and conjecturing (assuming the form of relationships and the results), and convincing (arguing why a statement is true). In addition, mathematical thinking is important because one of the standards in the learning process of school mathematics is that students are expected to develop mathematical reasoning (Kilpatrick, Swafford, & Findell, 2001).

Several studies have attempted to reveal students' thinking in learning mathematics at college (Dreyfus, 2002; Tall, 2008; Weller, Arnon, & Dubinsky, 2011; Syamsuri, Purwanto, Subanji, & Irawati, 2016; Syamsuri & Santosa, 2017). Tall (2008) expressed the idea of a transition process leading to advanced mathematical thinking. The process of high-level mathematical thinking can be representing, visualizing, generalizing, classifying, conjecturing, inducing, analyzing, synthesizing, abstracting or formalizing (Dreyfus, 2002). Therefore, mathematical thinking in college mathematics learning is a high-level thinking process. Hence in studying math in college always involves mental confusion as a connection between perception and action, then conducting re-organization in a formal deduction, so as to build a new learning experience through formal situation.

One of Piaget's important ideas related to the mental development of human thinking is abstraction-reflective. In Arnon et al. (2013, p.6), Piaget argues that "The development of cognitive structures is due to reflective abstraction...". As for the study of mathematics, Piaget stated that reflective abstraction is a mental mechanism derived using logico-mathematical construction. According to Piaget, there are at least two features of reflective abstraction, namely: (1) the existence of reflection, in the sense of awareness in thinking about the object being studied (content) and the operation of the object, and (2) reconstruction and re-objects and operations at a higher level, so the results of those operations can be applied to objects for new operations. For example, in constructing a function, initially the function is constructed as an operation against a member of a set of domains to a set of ranges. Furthermore, at a higher level of thinking, functions can be operated in function-space so that functions are objects that are operated using the new operation.

The application of APOS theory in learning based on the following assumptions (Dubinsky & McDonald, 2001): (1) assumption on mathematical knowledge: one's mathematical knowledge is his tendency to respond and solve mathematical problems and find solutions to the problem by thinking reflection (2) hypothesis on learning, i.e.: one does not learn a mathematical concept directly. He will form a mental structure related to the concept. Learning will take place well if in the minds of learners formed a mental structure in accordance with the given mathematical concepts. If the expected mental structure is not formed, then learning about the concept will not work.

These two assumptions indicate that the purpose of teaching should contain a strategy to help students form the expected mental structure and guide them in processing the mental structure to build an understanding of a mathematical concept. According to APOS theory, the mental structure consists of: action, process, object, and scheme. The main mental mechanisms in forming the mental structure are called interiorization and encapsulation (Weller, Clark, & Dubinsky, 2003).

This APOS theory has been widely used in analyzing the formation of mathematical concepts in universities (Asiala, Cottrill, Dubinsky, & Schwingendorf, 1997; Dubinsky & McDonald, 2001; Weller et al., 2011; Syamsuri, Purwanto, Subanji, & Irawati, 2017) as well as in mathematics learning (Weller et al., 2003; Salgado & Trigueros, 2015; García-Martínez & Parraguez, 2017). In the construction of concepts, this theory describes the paths that students pass through in constructing a mathematical concept. As for learning, this theory directs how to start and apply mathematics learning so as to facilitate students in learning.

One of the characteristics of mathematics learning in universities is to emphasize the axiomatic system and the formal deduction. Therefore, there are often discussions related to mathematical proof. The study of mathematical proofs in college has been widely studied by researchers (Galbraith, 1981; Weber & Alcock, 2004; Stylianides & Stylianides, 2009; Weber, 2010; Lee, 2016; Syamsuri, Marethi, & Mutaqin, 2018). This indicates that students who enter at the college level should develop mathematical knowledge formally. Therefore, students need to be trained in mathematical proof so they can understand the formal mathematical building structures.

The ability to construct this proof must be mastered by students to understand and master mathematics in depth. The ability is a provision as a math teacher candidate in order to create a learning situation that encourages students to master reasoning and proof (reasoning and proof). This is in line with the recommendation of NCTM (2000, p. 342), "Reasoning and proof are not special activities reserved for special times or special topics in the curriculum but should be natural, on what subject is being studied".

One of the focuses in the research of mathematical proof is to investigate the difficulties and misconceptions that occurred when constructing the evidence (Moore, 1994; Baker & Campbell, 2004; Sowder & Harel, 2003). Moore (1994) revealed that there are 7 difficulties experienced by students in constructing the evidence, namely: (1) students do not know the definition of certain mathematical objects or concepts needed in the proof, (2) the students are less in understanding the concept, (3) the concept of student image is not sufficient in reconstructing the proof (4) students are unable to generalize from some case examples (5) students do not know how to use the existing definition, (6) students have difficulty in using notation and mathematical language, and (7) students do not know how to start proof. Another study, Baker and Campbell (2004) argue that student constraints in making correct mathematical proofs are usually on logical arguments and the accuracy of the mathematical language that are used.

Some misconceptions related to the forming of such mathematical proofs need to be improved. In addition to suggestions regarding the implementation of learning, these misconceptions are needed to be described through mental structures. Therefore, it is necessary to study the mental structure that occurs in the minds of students when learning the concept of probability. This article aims to describe the mental structure that occurs when constructing mathematical proofs in terms of APOS theory. Figure 1 is a diagram of mental structure and mental mechanisms in constructing knowledge based on APOS theory and abstraction-reflective.

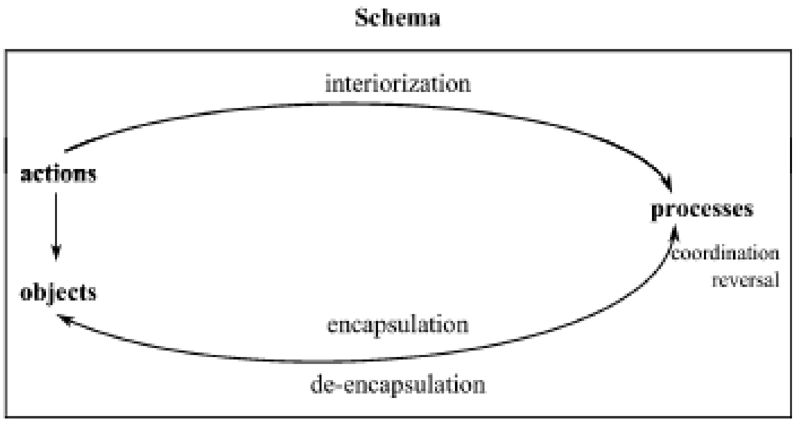

Figure 1. Mental structure and mental mechanism of mathematical knowledge construction (Adopted from Arnon et al. (2013))

2. METHOD

This research explores a thinking process in constructing a proof. It uses a qualitative approach, because there are three reasons, namely: (1) researcher as a key instrument, (2) inductive data analysis, and (3) holistic account. According to Cresswell (2012), the research that uses these research characteristics is called qualitative research. The characteristics of qualitative approach are: (1) researcher as a key instrument, (2) inductive data analysis, and (3) holistic account.

This research was conducted in public university at Banten province and involved 26 students majored in mathematics education. The consideration of that was because the students were able to think a formal proof in mathematics. The diagram of selecting research subject is presented in Figure 2.

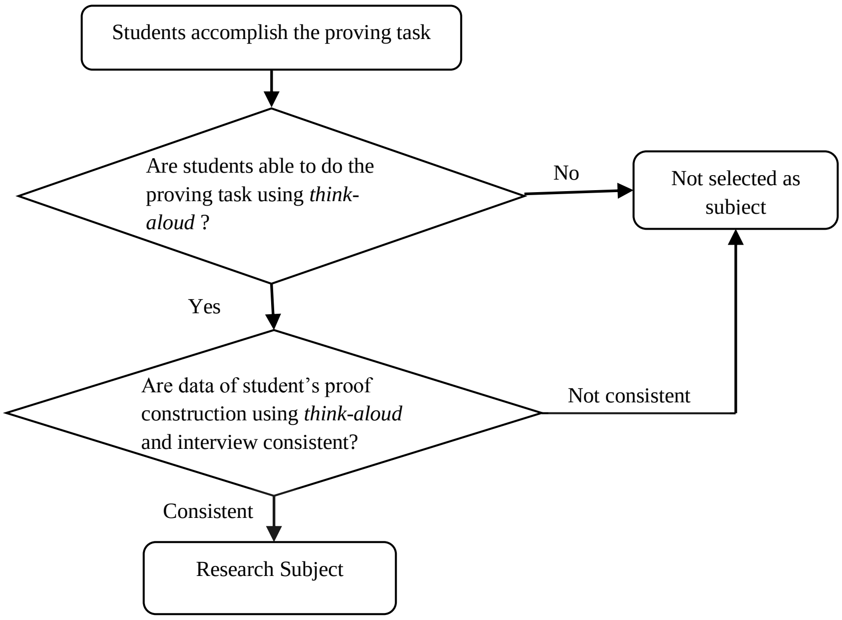

Figure 2. Process of selecting subject

The main instrument in explorative research are the researcher him/herself. The support instruments are proving-task and interview guides. These instruments were evaluated and validated from two lecturers in order to guarantee the quality of instruments. The interview is open and is needed to reveal students' response about proof comprehension. Procedures to obtain data are: 1) subject is given the task proving and asked him/her to accomplish the task by think-aloud. And then 2) subject is interviewed based on-the-task. Therefore, the scratch of proving-task and transcript of the interview is obtained. The proving-task is "for any positive integer m & n, if m^2 and n^2 are divisible by 3, then $m+n$ is divisible by 3." We used this task because some methods can be used for solving, i.e.: direct proof, contradiction, and contrapositive. Besides, we would like to test students' comprehension about mathematical induction method, because some students have an opinion that using mathematical induction to prove a number which is "divisible by 3".

The interview guides is created for confirmation and clarification about students' proof comprehension. We compile the interview questions, i.e.: (1) How do you accomplish the task? (2) Why do you argue with this step? (3) Would you like to give an example of the task?

3. RESULTS AND DISCUSSION

The subject selection process has been carried out to 26 students at one of the universities in the Banten. The process is in accordance with the previous explanation in Figure 2. The construction of formal mathematical proof from 26 students was classified into two groups. The first group of 11 students used deductive reasoning and was right in constructing proof. The second group of 15 students used inductive reasoning and was not consummate in constructing formal proof. The following describes the thinking process experienced by SA and SB. The SA subject represents the first group, and SB represents the second group.

3.1 Deductive-Holistic Proof Construction of SA

This section describes and analyzes the data think-aloud SA when constructing mathematical proof. SA began to construct proof with action in the form of reading questions carefully "here for any two positive integers m and n, if m^2 and n^2 are divisible by 3, then $m + n$ is divisible by 3. *Err…….*(look at what was read)". This also show that SA did the interiorization of the problem to be processed in his mind. SA knew that the task that must be completed is a mathematical verification task.

Hereafter, SA tried to look at the mathematical object that would be proven which is indicated by continued to think aloud by saying "firstly, m and n are members of positive integers, positive elements of natural numbers". This indicates that SA observed the mathematical object used in the proof is a positive integer consisting of m and n. The observations made by SA, raised the action by writing formal notation as shown in Figure 3.

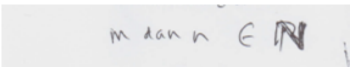

Figure 3. Formal notation that made by SA

Furthermore, SA stated "then there is m^2 divisible by 3 so that m^2 will be congruent 0 (mod 3)". This indicates that SA performed an interiorization that the information about the number m^2 is divisible by 3. In addition, SA also coordinated the information related with integer m, integer m^2 and that m^2 is divisible by 3. To continue the next process, SA performed a reversal about the concept divisible by and the concept of congruence that he/she already has. The knowledge invoked related to the definition of numbers divisible by 3 by using modulo number. SA stated that if an integer m^2 is divisible by 3 then the number will be congruent to $m^2 \equiv 0$ (mod 3).

Then SA coordinated his/her information and knowledge by specifying that $m^2 \equiv 0$ (mod 3). Determination of $m^2 \equiv 0$ (mod 3) is a mental mechanism of encapsulation so that the process of mental structure appears in the mind of SA. It is shown on Figure 4 . This also indicates that there has been coordination of the conceptual scheme of divisible with integer congruence scheme. Starting with the known information also indicates that SA realized that m^2 is divisible by 3 is this proof hypothesis. Thus, the initial structure of proof is correct.

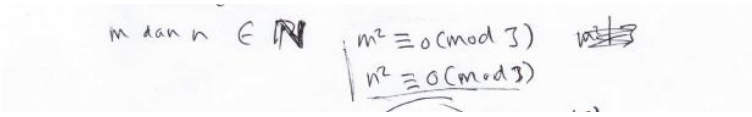

Figure 4. Encapsulation process that made by SA

Likewise, for number n, SA denoted the same thing as number n, "then n^2 too, automatically also divisible by 3, $n^2 \equiv 0$ (mod 3)". This indicates that SA did a reversal that the information about the number n^2 is divisible by 3 following the previous mechanism with the number m^2 which is divisible by 3 also. Besides that, SA also coordinated the information related with integer n, integer n^2 and that n^2 is divisible by 3. The results of the coordination of this information specify that $n^2 \equiv 0$ (mod 3). Determination of $n^2 \equiv 0$ (mod 3) is a mental mechanism of encapsulation and the coordination of the concept of divisible with the congruence of integers. Furthermore, SA tried to coordinate the two information through the mechanism of re-encapsulation by adding it up to form $m^2 + n^2 \equiv 0$ (mod 3). It is shown on Figure 5.

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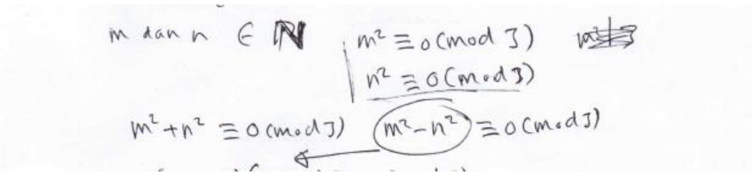

Figure 5. Coordination process that made by SA

This is indicated by SA's proof and expressions "so that $m^2 + n^2$ is also divisible by 3". SA used the term "divisible by 3" in think-aloud with the term $m^2 + n^2 \equiv 0$ (mod 3) in the proof. This shows that encapsulation and process have been formed in the mind of SA.

SA again thinked about the form $m^2 + n^2$ and realizes that $m^2 + n^2$ is a form that cannot be factored to get $m + n$. SA continued the think-aloud by saying "automatically $m^2 - n^2$ is also divisible by 3". This also implies that SA performed the encapsulation process that both of the numbers when operated with addition and subtraction will become a number that has the same properties which is also divisible by 3. This is happened because SA returned to the information that m^2 is divisible by 3 and n^2 is divisible by 3. It is shown on Figure 6.

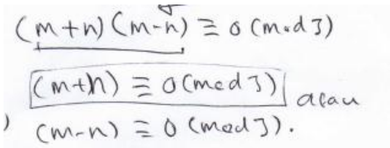

Figure 6. De-encapsulation process that made by SA

After that, SA continued with the statement "so that if we factorize $m^2 - n^2$ then it becomes $(m + n)$ $(m – n)$, this must also be divisible by 3, so that *err* .... $m + n$ will be divisible by 3 and $m – n$ will also be divisible by 3. Done". The activity of factoring $m^2 – n^2$ is a de-encapsulation mechanism, whereas declaring $m + n$ divisible by 3 is a generalization. The results of this generalization indicate that the structure of objects has appeared in the mind of SA. Thus, the mental structure of the schema that is formed by SA is a coherent collection of the actions, processes and objects that have emerged, which is then associated with the schema of congruence and divisibility of integer numbers in the mind of SA.

Based on the previous explanation, the mental structure and mental mechanism of SA in constructing the proof are complete. In addition, the structure of the proof used by SA is correct. Likewise the mathematical concept used is correct. Therefore, the proof construction of SA is a holistic construction. Figure 3 is a description of the proof construction of the SA.

Based on the previous explanation, both SA was able to construct the proof at the thought-experiment level (Balacheff, 1988; Varghese, 2011) and thinking level 2 (Van Dormolen, 1977). This level encourages students to work using accurate definitions and logic Tall (2010) and also deductive and symbolic thinking. Thus, according to Tall (2010), the thinking process of SA is already at a symbolic axiomatic development. Therefore these two subjects were able to construct formal proof correctly.

In addition, the proof construction made by SA is a proof construction that starts the initial step of proofing it well. Besides, it turned out that SA was able to connect the mathematical concepts needed to construct the proof correctly. This student difficulties expressed by Moore (1994) did not occur in these two subjects.

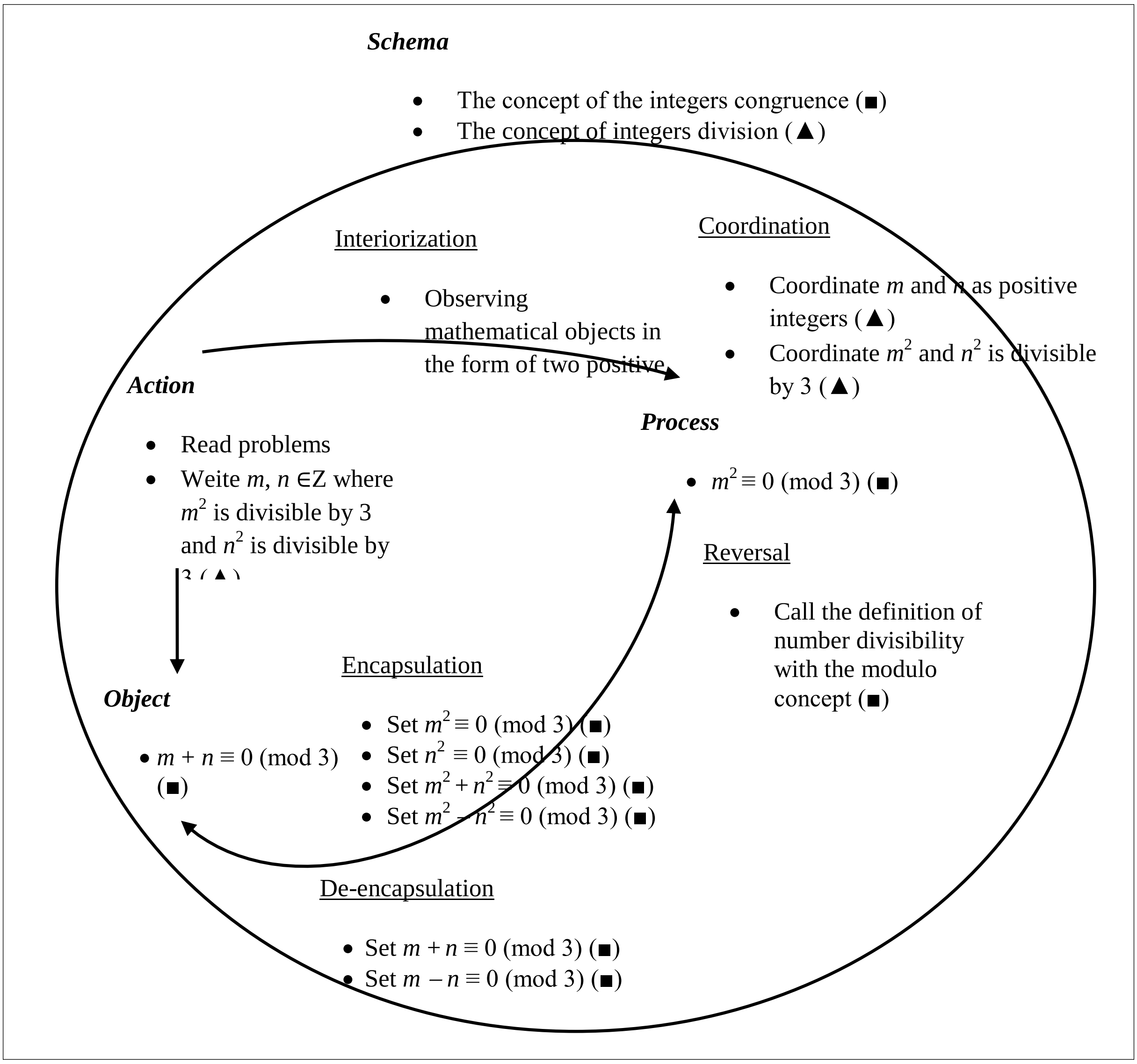

Figure 7. Deductive-holistic proof construction of SA

3.2 Inductive-Partial Thinking Construction of SB

SB began the proof construction with an action in the form of reading the task "for any two positive integers *m* and *n*, if m^2 and n^2 are divisible by 3, then $m + n$ is divisible by 3, mmmh .... (silent for a while)". This indicates that SB was trying to provide a stimulus to his thoughts about the proof task. Thus, there is an interiorization of the proof task in the mind of SB. Next, SB said "m^2 and n^2 are divisible by 3 then $m + n$ is divisible by 3". SB looked back at the proof task that he is facing, so the interiorization process of the problem still took place in SB's mind. After that, SB also coordinated the integer which is divisible by 3 as the mathematical object used in this proof task by saying "m^2 is divisible by 3, the same is n^2 is divisible by 3 also". The coordination of mental mechanism continued with "m^2 and n^2 divided by 3, *m* and *n* are positive integers". It is shown on Figure 8.

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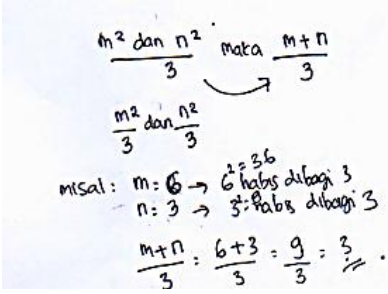
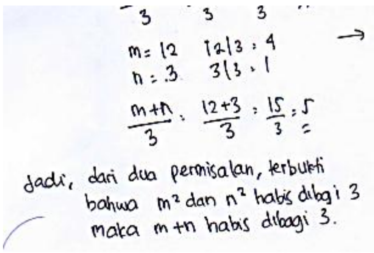


Figure 8. The first interiorization process that made by SB

The interiorization of mental mechanism still continued with the sayings "using an example, for an instance m = 2, n = 3". This shows that the mental structure of action has been formed in the mind of SB by giving examples of integers. However, after that, SB did reflective thinking by saying "Ooo.. but, which is divisible by 3". SB realized that the previous statement had something wrong by assuming m = 2. To strengthen this, SB again read the proof problem "for any two positive integers m and n, if m^2 and n^2 are divisible by 3, then m + n is divisible by 3". SB realized his mistake with "meaning divisible by 3". Therefore, SB replaced taking an integer m to 6, "suppose that m = 6, 6 is divisible by 3, 3 is divisible by 3". SB turned out to still do the interiorization of the problem by verifying whether m and n retrieval provided meets the expected conclusion or not. SB said "if from the example the theorem m + n is divisible by 3, it means (6 + 3) / 3 = 9/3 = 3. Proven". SB generalized the problem only based on the example case presented, "from the example it is proven that m + n is divisible by 3".



Translation

from 2 examples, it is proven that m^2 and n^2 are divisible by 3 then m + n is divisible by 3

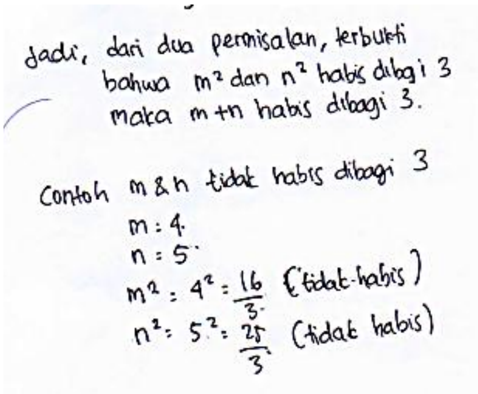
Figure 8. The second interiorization process that made by SB

After successfully creating a mental structure of action, then SB tried to convince by making another action, "another try, for example m = 12, n = 3, m = 12 is divisible by 3, 12/3 = 4. 3/3 = 1". A process like this indicates that SB thinks in the trial and error phase that fulfills the statement to be proven. After that, SB thinked "it means that (m + n) / 3 = (12 + 3) / 3 = 15/3 = 5". It was then continued by SB in generalizing the problem only based on the example case presented, "from 2 examples, it is proven that m^2 and n^2 are divisible by 3 then m + n is divisible by 3". The activity written in Figure 9 and lasted about 3 minutes and 44 seconds.

Translation

So, from 2 examples, it is proven that m^2 and n^2 are divisible by 3 then $m + n$ is divisible by 3.

Example: m & n are not divisible by 3
m=4
n=5
$m^2=4^2=16/3$ (it is not divisible by 3)
$n^2=5^2=25/3$ (it is not divisible by 3)


Jadi, dari dua permisalan, terbukti bahwa m^2 dan n^2 habis dibagi 3 maka $m+n$ habis dibagi 3.

Contoh m & n tidak habis dibagi 3
$m : 4$
$n : 5$
$m^2 : 4^2 : \frac{16}{3}$ (tidak habis)
$n^2 : 5^2 : \frac{25}{3}$ (tidak habis)

Figure 9. The third interiorization process that made by SB

Then SB again read the proof problem to confirm interiorization, "for any 2 positive integers, m^2 and n^2 are divisible by 3, then $m + n$ is divisible by 3". With this reinforcement, SB tried to encapsulate by defining an integer which is divisible by 3, "if m^2 and n^2 are divisible by 3, then m^2 divided by 3 remains 0". This encapsulation is then followed by coordination that these properties are owned by two numbers, namely m^2 and n^2, "eh ...... m^2 is the same as n^2". The result of this encapsulation specifies "it means there is a number which is 3 times k". SB then thought for a long time to continue this encapsulation process, but SB did not continue and went back to interiorizing the problem for numbers that are not multiples of 3. SB said "for example, 2 is a positive integer which is not divisible by 3, 2 × 2 = 4, 4 is not divisible by 3 " The interiorization of this problem is again reinforced by the sayings of SB "Errr ..... m and n are positive integers, if they are positive integers, it mean that they are divisible by 3". SB asked himself again which indicates that the process is still in the interiorization of the problem. Moreover, it is reinforced by the think-aloud expressed by SB "if this theorem applies for m and n positive integers which are divisible by 3, then it will be proven that $m + n$ is divisible by 3". Nevertheless, SB continued the action with "for example if m and n are not divisible by 3, m = 4, n = 5 ... $m^2 = 4^2 = 16$ is not divisible by 3 ... $n^2 = 5^2 = 25$ is not divisible by 3 ". SB worked more in action, even though he had stepped in the encapsulation process, but the encapsulation process was not able to be continued by SB. SB stated "done" in constructing the proof as shown in Figure 10.

Translation

This theorem is true for m and n are integers that divisible by 3.
And it will be proven that m+n is divisible by 3.

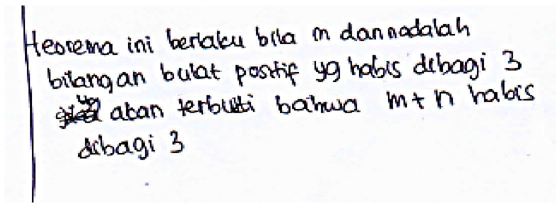

Teorema ini berlaku bila m dan n adalah bilangan bulat positif yg habis dibagi 3 akan terbukti bahwa $m+n$ habis dibagi 3

Figure 10. Encapsulation process that made by SB

Based on the think-aloud analysis, it shows that the process structure has not appeared in the mind of SB. Therefore, the object structure has not yet been formed. Thus, the schema formed by SB is a coherent collection of actions that are linked to the schema concept of integer divisibility in the mind of SB. To find out if the subject has been trying to construct the proof in accordance with his ability, the researcher asked SB.

R: Are you sure?
SB: Yes.

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Based on the explanation of the data above, it shows that the process structure has not appeared in the mind of SB. Therefore, the object structure has not yet been formed. Thus, the schema formed by SB is a coherent collection of actions that are associated with the schema concept of integer divisibility in the mind of SB. Figure 11 is a description of the proof construction by SB.

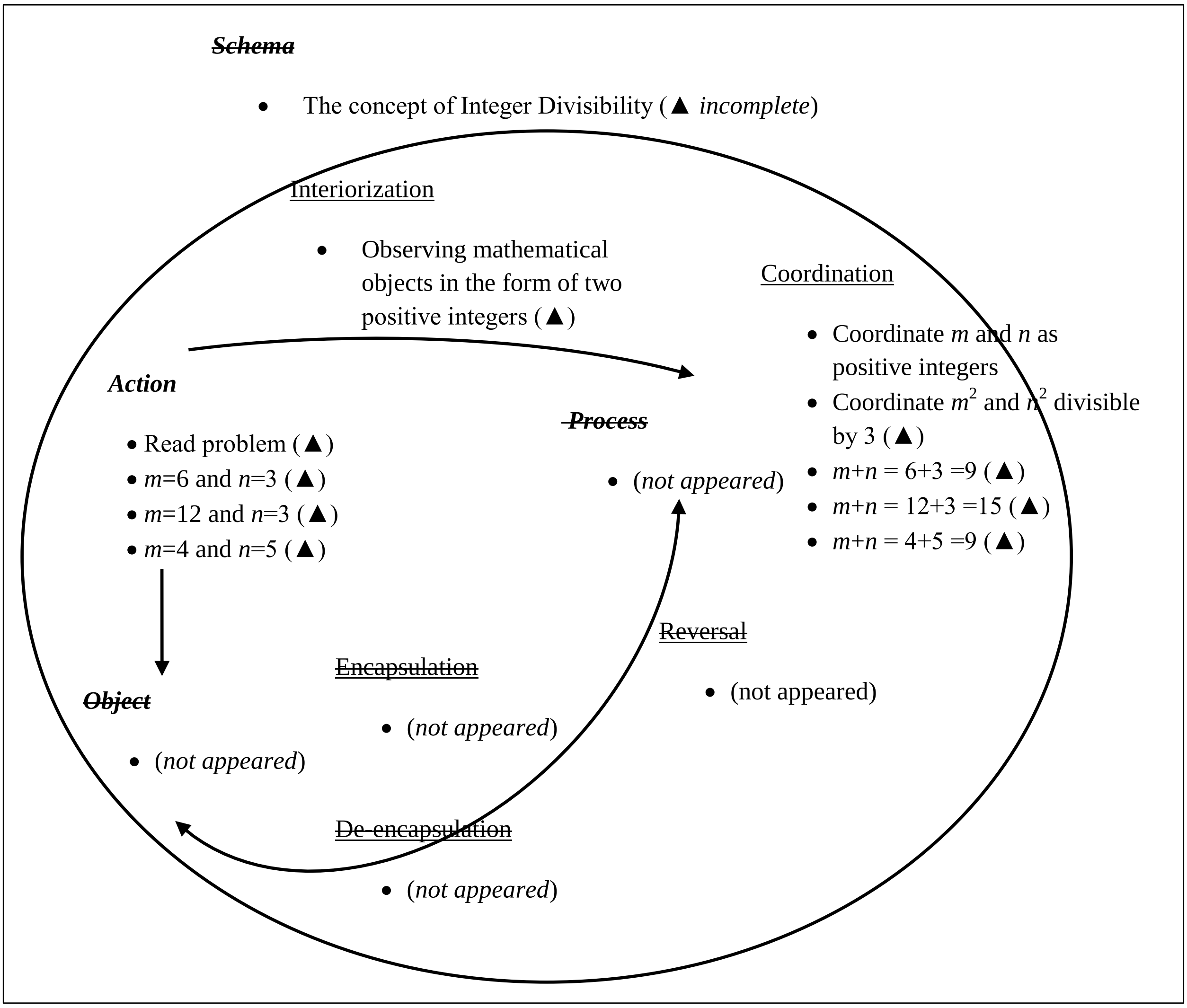

Figure 11. Inductive-partial proof construction of SB

In addition, SB constructed the proof at the naïve-empirical level (Balacheff, 1988; Varghese, 2011) and basic level thinking (ground level) (Van Dormolen, 1977). This level encourages students to work using examples of mathematical objects according to the statement. As a result, the proof construction which is expected to use the right definitions and logic did not appear (Tall, 2010). Therefore, these two subjects are still classified as inductive thinking. Thus, according to Tall (2010), the process of thinking SB is already in the development of the world of embodiment.

In addition, the proof construction made by SB is a proof construction where he was in the state of not having capability to start the initial step of proofing it well. Based on Moore (1994) opinion, SB have difficulty in understanding and using certain object definitions or mathematical concepts needed in the proof, and do not know how to begin the proof. As for

Baker and Campbell (2004), these two subjects experienced problems in making the right mathematical proofs usually in logical arguments and the accuracy of the mathematical language used. Whereas SB experienced inadequacy of students' understanding of mathematical proofs, wrong in understanding the concepts related to theorem, and lack of developing proof strategies. The difficulty reinforces that the students difficulty in proving mathematical statements is about thinking deductively (Recio & Godino, 2001). These difficulties cause difficulties in creating a mental structure "Process", which in turn results in failure in de-encapsulation and generalization in proof constructing.

4. CONCLUSION

This research found two types of cognitive process in mathematical proof. The first type is the deductive-holistic type of thinking process which is a series of activities to construct proof that uses the right deductive reasoning in the proof structure and mathematical concepts. This type of thinking process begins with action and interiorization mechanisms, followed by coordination to form the process structure. After that, there is a mechanism of encapsulation, reversal, and de-encapsulation formed object structure by generalizing. Furthermore, the object structure formed is coherently associated with other schemas. The second type is the inductive-partial type of thinking process which is a series of activities in proof constructing that uses inductive reasoning. The characteristics of this type are there are any errors in the proof structure and mathematical concepts. This type of thinking process begins with action and interiorization mechanisms, followed by coordination but it failed to form a process structure. Thus, this thinking process takes place because students are only able to form actions and perform mental mechanisms of interiorization and coordination.

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