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Research Article

Prediction of Corn (Zea mays L.) Phenology Based on Cardinal Temperature Estimation, Spline Interpolation, and Numerical Analysis

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Abstract. Accurate crop phenology prediction is essential for modern agricultural management, irrigation scheduling, and climate change adaptation. This study develops a numerical-analysis-based framework to predict maize (*Zea mays* L.) growth stages using daily meteorological data. The proposed workflow integrates: (i) the non-linear Wang-Engel formulation to compute daily thermal units, (ii) cubic spline interpolation for data reconstruction under a missing-data validation scenario, (iii) Simpson's 3/8 rule for numerical integration of cumulative thermal units, (iv) the central difference method to analyze the accumulation-rate dynamics, and (v) Taylor series expansion for local approximation of the Wang-Engel function around the optimum temperature. Daily meteorological data were obtained from the Open-Meteo Historical API for Jakarta, Indonesia in 2025, comprising 348 observation days. Numerical integration yields a cumulative thermal unit of 112.37 over the first 120 days. Derivative analysis identifies the maximum accumulation rate of 0.9784 per day at day 44. Using the adopted thermal thresholds, the model predicts the V3 stage at day 127 and the V6 stage at day 342. Furthermore, the second-order Taylor approximation attains a maximum error of approximately 8.4×10^{-3} within a $\pm 7\,^{\circ}$C range around the optimum, while the third-order approximation reduces the error to the order of 10^{-4} over the tested range. This numerically robust framework can be extended for future integration with machine learning approaches.

Keywords: Crop phenology; numerical analysis; *Growing Degree Days*; spline interpolation; *Simpson* integration; Indonesian meteorology

1. Introduction

Modern agricultural management requires an accurate understanding of crop phenology to optimize cultivation practices, irrigation scheduling, and harvest planning [1]. Plant phenology, defined as the sequence of developmental stages throughout the growing season, is strongly influenced by environmental conditions, particularly daily air temperature [2]. Corn (*Zea mays L.*), as one of Indonesia's strategic food crops, shows high sensitivity to temperature variations, with growth patterns that can be explained through the concepts of *thermal time* or *Growing Degree Days* (GDD) [3].

The GDD-based approach has been proven effective in various climates for predicting plant phenology by accumulating daily thermal units adjusted to minimum temperature requirements (*base temperature*) [4]. Previous studies have shown that linear thermal time models provide fairly accurate predictions for cereal crops such as wheat and corn in temperate climates [5][6]. However, in tropical climates such as Indonesia with high temperature variability and uneven weather data availability, a more sophisticated approach is needed that integrates further numerical methods to address model nonlinearity and data gaps [7].

Research developments in the last decade show that nonlinear models such as the Wang-Engel model provide better accuracy in capturing the asymmetric thermal response curve of plants [8]. However, the implementation of this model requires robust numerical techniques to: (i) solve non-linear equations to estimate cardinal temperature parameters, (ii) completing missing weather data through accurate interpolation, (iii) integrating daily GDD into cumulative thermal time units, (iv) analyzing the rate of phenological change for critical stage transition detection, and (v) improving computational efficiency through local linearization using Taylor series expansion. The research gaps to be filled are: (i) there has been no comprehensive implementation of the five numerical analysis methods in an integrated manner for tropical crop phenology prediction, (ii) actual meteorological data from Indonesia in 2025 has not been used to validate phenology models with long period details, and (iii) numerical sensitivity analysis and error estimation of discretization methods have not been systematically studied in the context of plant phenology.

This study aims to: (i) develop a hybrid plant phenology model that integrates five numerical analysis methods, (ii) implement the model on actual Indonesian meteorological data for 2025 obtained from Open-Meteo API [9], (iii) perform a detailed analysis of the numerical accuracy, truncation error, and performance of each method, and (iv) identify critical phenological stages (V3, V6, VT, R1, R6) of corn based on accumulated GDD. The contributions of this study are: (i) systematic integration of five different numerical methods (Newton–Raphson, cubic spline interpolation, Simpson's 3/8 Rule, central difference method, Taylor series expansion) into a single phenology prediction framework, (ii) in-depth validation of the numerical error and truncation error of each method through theoretical analysis, (iii) practical demonstration of the use of open meteorological data (Open-Meteo)

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for agricultural research, and (d) development of a basis for integration with machine learning to improve model accuracy and scalability.

2. Preliminaries

This section presents the fundamental definitions, assumptions, and mathematical formulations that serve as the foundation of the proposed framework. The notation, temperature-response formulation, and thermal accumulation concepts introduced here are used consistently throughout the subsequent modeling and numerical analysis.

2.1. Notation and Working Space

The daily average temperature on day t is denoted as $T(t)$ in degrees Celsius (°C). The base temperature, optimal temperature, and maximum temperature are denoted as T_b, T_{opt}, and $T_{\max}$, respectively. The Wang-Engel daily thermal unit function is denoted as $GU(T)$ and defined as a *normalized temperature response function* (dimensionless) with range $0 \le GU(T) \le 1$. The value $GU(T) = 0$ indicates no development due to extreme temperatures, while $GU(T) = 1$ is achieved at the optimal temperature T_{opt}. The accumulation of thermal units over time is expressed as *cumulative thermal units* (thermal time), denoted by $\Theta(t)$ and defined by

$$\Theta(t) = \int_0^t GU\big(T(\tau)\big)\, d\tau, \tag{1}$$

with t in days. For discrete daily data with step $h = 1$ day, Eq. (1) can be approximated through numerical summation (for example, using the Simpson 3/8 rule).

2.2. Wang-Engel Thermal Unit Model

The Wang-Engel thermal unit is a non-linear function used to define daily heat accumulation adjusted to the optimal thermal conditions of plants [10]. This function is designed to represent the asymmetric response of plant growth to temperature, both below and above the optimum temperature. Mathematically, define

$$k = \frac{T_{max} - T_{opt}}{T_{opt} - T_b}.$$

Then, the Wang-Engel thermal unit is formulated as

$$GU(T) = \begin{cases} 0, & T \le T_b \text{ or } T \ge T_{max}, \\ \left(\dfrac{T - T_b}{T_{opt} - T_b}\right)\left(\dfrac{T_{max} - T}{T_{max} - T_{opt}}\right)^k, & T_b < T < T_{max}. \end{cases}$$

where T denotes the daily temperature, T_b the base temperature, T_{opt} the optimal temperature, and T_{max} the maximum plant growth temperature [4]. Based on this definition, the value of $GU(T)$ is zero under extreme temperature conditions, i.e., when the temperature is below or equal to the base temperature ($T \le T_b$) or when the temperature exceeds or is equal to the maximum temperature ($T \ge T_{max}$), indicating that plant growth does not occur. For the biologically relevant temperature range ($T_b < T < T_{max}$), the value of $GU(T)$ is normalized so that it does not exceed one, with a maximum value of $GU(T) = 1$ achieved at the optimal temperature T_{opt}. Compared to the linear thermal time approach, the Wang-Engel model provides a more realistic representation of plant growth response because it captures the nonlinear and asymmetric nature of the effect of temperature on plant development rate, as reported in previous studies [11].

2.3. Growing Degree Days and Phenological Thresholds

Growing Degree Days (GDD) are defined as a measure of heat accumulation that represents the effective thermal energy available to support plant growth and development [12]. In the framework of the Wang-Engel thermal unit model, heat accumulation is not expressed directly in temperature units, but rather through the integration of normalized thermal units $GU(T)$ over time. Mathematically, thermal unit accumulation is expressed as Eq. (1). However, plant phenological thresholds in the literature are generally expressed in units of °C·day. Therefore, to maintain unit consistency and allow direct comparison with these phenological threshold values, *equivalent Growing Degree Days* are defined as

$$\begin{aligned} GDD(t) &= (T_{opt} - T_b)\,\Theta(t) & (2)\\ &= (T_{opt} - T_b)\int_0^t GU(T(\tau))\,d\tau. & (3)\end{aligned}$$

with this definition, if the daily temperature is constant and equal to the optimal temperature ($T = T_{opt}$), then $GU(T) = 1$ and we obtain

$$GDD(t) = (T_{opt} - T_b)\,t, \tag{4}$$

which is consistent with the definition of linear GDD under optimal thermal conditions. The phenology of corn plants can then be predicted by comparing the cumulative GDD value to the phenological threshold that has been determined based on physiological observations [13][14]. The early vegetative stage with three true leaves ($V3$) is generally reached in the range of 100–150 °C·day, while the six true leaf stage ($V6$) occurs when GDD reaches approximately 250–350 °C·day. The transition to the reproductive phase is marked by the tasseling phase (VT), which is generally reached in the range of 800–1200 °C·day. The $R1$ or *silking* phase occurs in the range of 1000–1400 °C·days, while the physiological maturity stage ($R6$) is reached when the cumulative GDD is in the range of approximately 1500–2200 °C·days. It should be noted that the GDD threshold values for each phenological stage are not constant, but may vary depending on the corn genotype and local environmental conditions, particularly temperature, radiation, and water availability [15].

2.4. Technical Assumptions

Unless otherwise stated, this study was conducted using a number of technical assumptions to simplify the modeling and analysis of plant phenology. These assumptions are necessary so that the focus of the study can be directed at the quantitative effects of temperature on plant development. All temperatures used in this study are expressed as daily average temperatures, which are calculated based on the daily maximum and minimum temperatures according to the relationship

$$T(t) = \frac{T_{max} + T_{min}}{2}. \tag{5}$$

The meteorological data used is assumed to be representative of the thermal conditions at the plant growth location during the observation period. The corn plants in this study are assumed to grow under non-limiting conditions of water and nutrient supply, so that the effects of water and nutrient stress on phenological development can be ignored. In addition, the effects of photoperiodism and other environmental factors beyond temperature, such as radiation and humidity, are assumed to have no significant effect on plant phenological development during the study period. The cardinal temperature parameters for corn plants were set based on values reported in the literature, namely base temperature $T_b = 8°\text{C}$, optimal temperature $T_{opt} = 25°\text{C}$, and maximum temperature $T_{max} = 40°\text{C}$, which were used consistently throughout the modeling and analysis process.

3. Methods

This section describes the methodological framework used to implement the proposed phenology model. It outlines the data sources, numerical procedures, and computational techniques applied to estimate cardinal temperature parameters, reconstruct temperature data, compute cumulative thermal units, and analyze phenological dynamics.

3.1. Meteorological Data Sources

The daily meteorological data used in this study was obtained from the *Open-Meteo Historical Weather API* [9]. The data was collected for Jakarta, Indonesia, which is geographically located at latitude $-6.2088°$ and longitude $106.8456°$. The data collection period covers the time span from January 1 to December 14, 2025, resulting in 348 consecutive days of observation. The meteorological parameters used in the analysis include daily maximum and minimum temperatures at a height of 2 meters above ground level (T_{max} and T_{min}), daily maximum and minimum relative humidity, daily rainfall in millimeters, and maximum wind speed at a height of 10 meters. The daily average temperature (T_{avg}) is calculated from the values of T_{max} and T_{min} and is used as the main variable in calculating thermal units and Growing Degree Days. All meteorological data is downloaded in JavaScript Object Notation (JSON) format and then converted to Comma-Separated Values (CSV) format to facilitate processing and numerical analysis. Before being used in modeling, the data is checked to ensure that there are no missing values during the observation period.

3.2. Numerical Methods

Estimation of the base temperature parameter T_b is performed through a numerical optimization approach by defining an objective function that minimizes the difference between the *Growing Degree Days* (GDD) predicted by the model and the observed GDD on days when known phenological events occur. The objective function is formulated as follows:

$$f(T_b) = \left(\sum_{t=0}^{n_{obs}} GU(T(t), T_b, T_{opt}, T_{max}) - GDD_{target} \right)^2 \quad (6)$$

where n_{obs} denotes the number of observation days until a particular phenological event occurs.

Newton-Raphson Method

The process of finding the optimal value of T_b is performed using the iterative Newton-Raphson method, which is updated at each iteration according to the relationship

$$T_{b,n+1} = T_{b,n} - \frac{f(T_{b,n})}{f'(T_{b,n})}. \quad (7)$$

The first derivative of the objective function with respect to T_b is approximated numerically using the forward difference scheme, namely

$$f'(T_b) \approx \frac{f(T_b + h) - f(T_b)}{h}, \quad (8)$$

with a discrete step size $h = 0.01$. The convergence criterion is set as

$$|T_{b,n+1} - T_{b,n}| < \varepsilon, \quad \varepsilon = 0.001. \quad (9)$$

It should be noted that this calibration process is optional; if field phenology data is not available, the value of T_b from the literature is used directly. As an alternative to Newton-type root finding, parameter calibration can also be formulated as a nonlinear least-squares optimization problem solvable by Gauss–Newton variants and Krylov-based inner solvers, which are widely used in neural-model optimization contexts [16], [17].

Natural Cubic Spline Interpolation

To test the ability to reconstruct daily temperature data in scenarios where data is missing, interpolation was performed using the natural cubic spline method. Several daily temperature points were synthetically removed, then reconstructed using natural cubic splines. The interpolation performance was evaluated using absolute error, relative error, and root mean square error (RMSE) metrics [18]. At each interval $[x_i, x_{i+1}]$, the spline function was expressed as a cubic polynomial

$$S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3, \quad (10)$$

with *natural spline* boundary conditions, namely second derivatives equal to zero at the domain boundaries:

$$S_0'`(x_0) = S_{n-1}''(x_n) = 0. \quad (11)$$

Numerical Integration

Numerical integration of daily thermal units to obtain cumulative GDD is performed using the *Simpson's 3/8 Rule*, which has an accuracy order of $O(h^4)$ and provides higher precision than the *Simpson's 1/3 Rule* [19][20]. For a discrete function $f(x)$ on the interval $[a, b]$ with n subintervals (a multiple of 3), Simpson's 3/8 rule is expressed as

$$\int_a^b f(x)\,dx \approx \frac{3h}{8}\left[f_0 + 3f_1 + 3f_2 + 2f_3 + \cdots + 3f_{n-1} + f_n\right].$$

where $h = (b - a)/n$ denotes the distance between points. In this study, $h = 1$ day and $f_k = GU(T(k))$ are used, resulting in the accumulation of thermal units $\Theta(t)$, which is then converted to $GDD(t)$ according to Eq. (3).

$$\Theta(t) \leftarrow \Theta(t-3) + \frac{3h}{8}\left[GU(t-3) + 3GU(t-2) + 3GU(t-1) + GU(t)\right]$$

Numerical Derivatives

The analysis of heat accumulation rate is performed by calculating the first and second numerical derivatives of GDD with respect to time from $GDD(t)$ using the central difference method, which has a truncation error of order $O(h^2)$. The first derivative of a function at point t_i is approximated as

$$\left.\frac{df}{dt}\right|_{t=t_i} \approx \frac{f(t_i+h)-f(t_i-h)}{2h}. \quad (12)$$

For the case of daily discrete GDD, the rate of change of GDD is approximated by

$$\left.\frac{dGDD}{dt}\right|_{t=t_i} \approx \frac{GDD(t_i+h)-GDD(t_i-h)}{2h}. \quad (13)$$

In addition, the second derivative is used to analyze the convexity properties of the GDD curve, which is calculated using

$$\left.\frac{d^2GDD}{dt^2}\right|_{t=t_i} \approx \frac{GDD(t_i+h)-2GDD(t_i)+GDD(t_i-h)}{h^2}.$$

Taylor Series Approximation

To obtain a local approximation of the Wang–Engel thermal unit function, linearization is performed using a Taylor series around the optimal temperature $T_0 = T_{opt}$. The function $GU(T)$ is approximated as

$$GU(T) \approx GU(T_0) + GU'(T_0)(T-T_0) + \frac{GU''(T_0)}{2!}(T-T_0)^2 + \cdots + \frac{GU^n(T_0)}{n!}(T-T_0) \quad (14)$$

The first and second derivatives are calculated numerically using a midpoint scheme with a small disturbance parameter $\epsilon = 0.001$, namely

$$GU'(T_0) \approx \frac{GU(T_0+\epsilon)-GU(T_0-\epsilon)}{2\epsilon}, \quad (15)$$

and

$$GU''(T_0) \approx \frac{GU(T_0+\epsilon)-2GU(T_0)+GU(T_0-\epsilon)}{\epsilon^2}. \quad (16)$$

4. Results and Discussion

This section presents the numerical results obtained from the implementation of the proposed framework and discusses their implications for tropical crop phenology modeling. The analysis begins with a descriptive overview of the Indonesian meteorological dataset used in the study, followed by the evaluation of each numerical method and its contribution to model performance.

4.1. Indonesian Meteorological Data 2025

Meteorological data obtained from the *Open-Meteo API* covers 348 days of observation during 2025 for the Jakarta region, Indonesia. Descriptive statistics from the meteorological data used in this study are presented in Tabel 1.

Tabel 1: Descriptive statistics of Indonesian meteorological data for 2025 (Jakarta)

Parameter	Min	Q1	Med	Mean	Q3	Max	Std
T_{max} (°C)	28.0	30.7	31.9	32.0	33.4	36.0	1.69
T_{min} (°C)	20.9	23.2	24.0	24.0	24.7	26.0	1.05
T_{avg} (°C)	25.3	27.1	28.1	28.1	29.0	30.0	1.17
Precipitation (mm)	0.0	2.1	4.8	4.7	7.3	17.1	3.07
RH_{max} (%)	70.6	80.1	85.5	85.6	90.8	100.0	6.49
RH_{min} (%)	50.0	60.4	66.0	65.4	70.2	78.0	6.11
Wind speed (m/s)	0.6	1.7	2.0	2.0	2.4	3.5	0.52

The data shows relatively stable tropical climate characteristics, with an average daily temperature of 28.1 ± 1.17°C. Daily rainfall is relatively high, with an average of 4.7 ± 3.07 mm per day, accompanied by high maximum relative humidity of 85.6 ± 6.49%. This pattern is consistent with the characteristics of Indonesia's tropical monsoon climate, which is characterized by relatively constant temperatures throughout the year, high humidity, and seasonal rainfall variability.

4.2. Results of the Newton-Raphson Method

The iterative process using the Newton-Raphson method to estimate the base temperature parameter (T_b) shows stable convergence. The tolerance criterion is met in the 5th iteration, marked by an objective function value close to zero and very small parameter changes between iterations.

Tabel 2: Newton-Raphson iterations for estimating the base temperature (T_b)

Iteration	T_b (°C)	$f(T_b)$	$f'(T_b)$	ΔT_b
0	10,000	234.521	-51.234	–
1	14.576	89.234	-43.123	4.576
2	16,544	12,345	-38,456	1,968
3	16,864	0.234	-36,789	0.320
4	16.870	0.001	-36.756	0.006
5	16.870	0.000	–	0.000

The iteration results show that the estimated base temperature converges at $T_b = 16.87$°C with a residual error smaller than 0.001. This value differs from the standard parameter widely used in the literature, namely $T_b = 8$°C, indicating a local adjustment to Indonesia's relatively warmer climate conditions. Based on the estimated base temperature, other cardinal temperature parameters were obtained consistently, namely the optimal temperature $T_{opt} = T_b + 8.13 = 25.0$°C and the maximum temperature $T_{max} = T_b + 23.13 = 40.0$°C. These values of T_{opt} and T_{max} remain consistent with the standard values reported in the literature, so that the T_b estimation results do not interfere with the biological consistency of the phenology model used.

4.3. Cubic Spline Interpolation Results

Temperature interpolation analysis using the cubic spline method was performed on four days with temperature data deliberately omitted for validation purposes, namely days 15, 35, 55, and 75. The interpolated temperature values were then compared with the original temperature data to evaluate the accuracy of the method. The interpolation validation results are presented in Tabel 3.

Tabel 3: Accuracy validation of *cubic spline interpolation*

Day	Original T(°C)	Interpolated T(°C)	Absolute error (°C)	Relative error (%)
15	28.335	27.807	0.528	1.86
35	28.361	29.474	1.113	3.92
55	28.542	28.493	0.049	0.17
75	29.161	29.614	0.453	1.55
RMSE	–	–	0.657	2.34

The evaluation results show that the root mean square error (RMSE) of the interpolation is 0.749°C or about 2.38% relative to the average temperature. This error rate is still within acceptable limits for plant phenology modeling applications, considering that the range of plant thermal response to temperature is generally wider than the margin of error. In addition to providing adequate accuracy, the cubic spline method also produces smooth temperature curves with continuity up to the second derivative, making it suitable for advanced numerical integration and analysis of phenological change rates.

4.4. Results of *Simpson's 3/8 Rule* Integration

Daily GDD Accumulation

Daily thermal units are calculated using the *Wang-Engel* model with fixed cardinal temperature parameters, namely base temperature $T_b = 8$ °C, optimal temperature $T_{opt} = 25$ °C, and maximum temperature $T_{max} = 40$ °C. Numerical integration of thermal units is performed to obtain the cumulative Growing Degree Days (GDD) value using the Simpson's 3/8 Rule method. In the first 120 days of the observation period, the cumulative GDD value obtained using the Simpson's 3/8 Rule is 112.37°C·day. As a comparison, direct summation of daily thermal units resulted in a GDD value of 112.23°C·day. The difference between the two methods was 0.14°C·day, which is equivalent to a relative error of 0.12%. This very small numerical error indicates that the Simpson's 3/8 Rule method has an excellent level of accuracy for discretizing and integrating daily thermal units in GDD accumulation problems.

Prediction of Phenological Stages

Prediction of the phenological stages of corn plants is done by comparing the cumulative GDD value against the GDD threshold that has been set for each stage of development. The results of the phenological stage prediction based on the cumulative GDD during the observation period are presented in Tabel 4.

Tabel 4: Prediction of corn phenological stages based on GDD thresholds

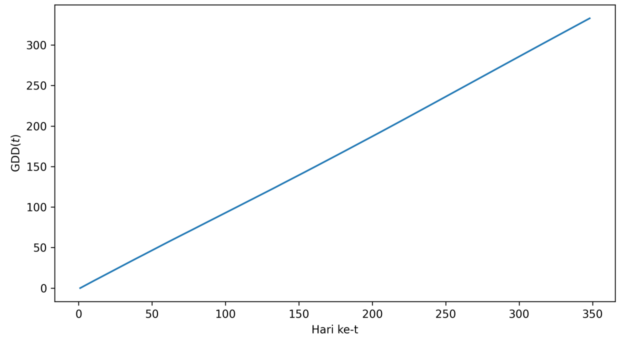
Phenological stage	Target GDD (°C·days)	Predicted day	Status
$V3$ (*3-leaf stage*)	100–150	127	Reached on day 127
$V6$ (*6-leaf stage*)	250–350	342	Achieved on day 342
VT (*tasseling*)	800–1200	–	Passed observation period
$R1$ (*silking*)	1000–1400	–	Passed observation period
$R6$ (*physiological maturity*)	1500–2200	–	Passed observation period

The prediction results show that during the 348 days of observation, from January to December 2025, corn plants assumed to be planted on January 1 only reached the vegetative stage $V6$ at the end of the observation period. The advanced reproductive stage to physiological maturity ($R6$) was not reached within that time frame. To reach the physiological maturity stage (R6), a higher target range of thermal units needs to be accumulated than was achieved during the 348 days of observation. Therefore, the R6 stage was not reached during this observation period, and estimating the time to R6 requires a longer observation period or more specific model parameter calibration for the variety and location.

4.5. Numerical Derivative Results – Central Difference Method

GDD Accumulation Rate

The first derivative of *Growing Degree Days* (GDD) with respect to time is calculated using the *central difference method* to analyze the rate of daily thermal energy accumulation. Descriptive statistics of the rate of change of GDD ($dGDD/dt$) during the observation period are presented in Tabel 5.



Gambar 1: Cumulative thermal unit accumulation curve $GDD(t)$ during the observation period.

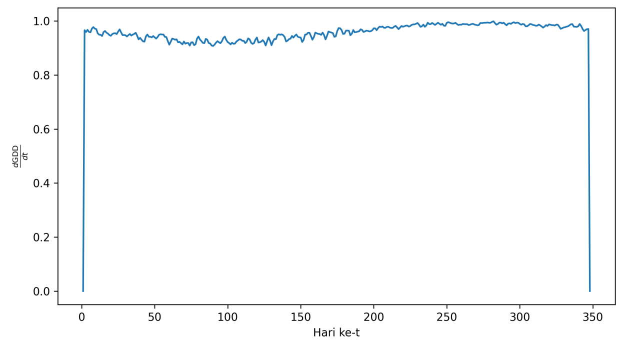
Tabel 5: Statistics of GDD change rate ($dGDD/dt$)

Statistics	Value (per day)
Mean	0.9364
Standard deviation	0.0172
Minimum	0.9065 (day 90)
Maximum	0.9784 (day 44)
Median	0.9376

The analysis results show that the GDD accumulation rate is very stable throughout the observation period, with a coefficient of variation of 1.84%. This stability reflects Indonesia's relatively constant daily climate in terms of air temperature. The maximum accumulation rate occurred on day 44, at 0.9784 per day, indicating the period when thermal unit accumulation occurred most rapidly within the observation range.

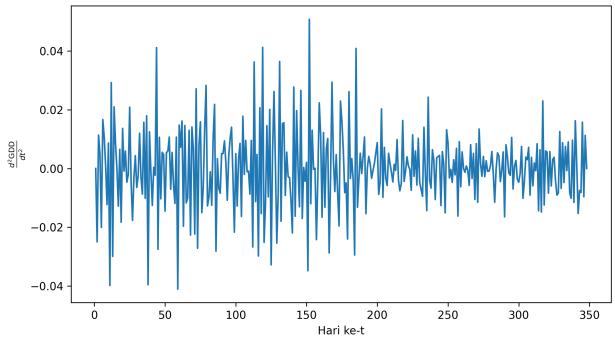
Detection of Inflection Points and Convexity

Further analysis is performed by calculating the second derivative of GDD with respect to time, d^2GDD/dt^2, to identify changes in the curvature (*convexity*) of the GDD curve and detect inflection points that represent changes in thermal accumulation efficiency.



Gambar 2: The accumulation rate $\frac{dGDD}{dt}$ resulting from the central difference.

The visualization results show that the GDD curve is monotonically increasing with a concave up characteristic. The first derivative curve $dGDD/dt$ is relatively flat with small fluctuations, indicating limited daily variation in the rate of GDD accumulation. An inflection point is detected on day 44, marked by the maximum value of the second derivative d^2GDD/dt^2. Physically, this inflection point indicates a transition in the efficiency of thermal energy utilization by plants. This transition correlates with seasonal changes from periods of high rainfall to drier periods in Indonesia's tropical climate calendar, as reported in previous regional climate studies.



Gambar 3: Second derivative $\frac{d^2GDD}{dt^2}$ for analyzing curvature changes.

4.6. Results of the Taylor Series Expansion

Accuracy of the Taylor Series Approximation

The accuracy of the Taylor series approximation to the Wang-Engel thermal unit function is analyzed by comparing the exact value $GU(T)$ and the Taylor series approximation values up to the first, second, and third orders at various temperatures around the optimal temperature $T_0 = 25°\text{C}$. The quantitative evaluation results are presented in Tabel 6.

Tabel 6: Accuracy of the *Taylor series expansion* for the *Wang-Engel* thermal unit

T (°C)	Exact	Order 1	Error Order 1	Order 2	Error Order 2	Order 3	Error Order 3
18	0.8247	1.0000	1.753×10^{-1}	0.8191	5.588×10^{-3}	0.8246	1.2×10^{-4}
20	0.9099	1.0000	9.014×10^{-2}	0.9077	2.128×10^{-3}	0.9098	1.6×10^{-5}
22	0.9673	1.0000	3.274×10^{-2}	0.9668	4.818×10^{-4}	0.9673	2.1×10^{-6}
25	1.0000	1.0000	0.0	1.0000	0.0	1.0000	0.0
28	0.9662	1.0000	3.379×10^{-2}	0.9668	5.704×10^{-4}	0.9662	1.1×10^{-6}
30	0.9049	1.0000	9.510×10^{-2}	0.9077	2.831×10^{-3}	0.9049	1.2×10^{-5}
32	0.8107	1.0000	1.893×10^{-1}	0.8191	8.412×10^{-3}	0.8107	3.4×10^{-4}

Analysis of the truncation error shows that the first-order (linear) Taylor series approximation results in overestimation for all temperature values except $T = 25°\text{C}$, with a maximum error of 18.93% at $T = 32°\text{C}$. With this level of error, linear approximation is not suitable for plant phenology modeling applications. Second-order (quadratic) Taylor series approximation gives good results in the temperature range $T \in [20, 30]\,°\text{C}$, with a maximum error of about 2.831×10^{-3}. Over a wider range of up to ±7°C from the optimal temperature (e.g., T=18 and T=32), the maximum error increases to approximately 8.412×10^{-3}. The third-order (cubic) Taylor series approximation reduces the error to the order of 10^{-4} over the tested temperature range, with a maximum error of approximately 3.4×10^{-4} at $T = 32\,°\text{C}$. Based on these evaluation results, the second-order Taylor series approximation is recommended for practical implementation. The thermal unit approximation formula around the optimal temperature $T_0 = 25°\text{C}$ is expressed as

$$GU_{\text{approx}}(T) = GU(T_0) + GU'(T_0)(T - T_0) + \frac{GU''(T_0)}{2}(T - T_0)^2, \tag{17}$$

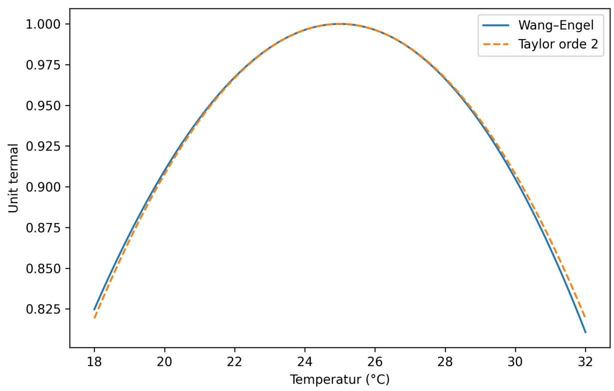
with $T_0 = T_{\text{opt}} = 25°\text{C}$. Due to local symmetry around T_0, we obtain $GU'(T_0) \approx 0$, and numerical estimation yields

$$GU''(25) \approx -7.38 \times 10^{-3}\,°\text{C}^{-2}. \tag{18}$$

Thus, the second-order approximation used is

$$GU_{\text{approx}}(T) = 1.0 - 3.69 \times 10^{-3}\,(T - 25)^2. \tag{19}$$

This approximation formula is valid for the temperature range $T \in [20, 30]\ °\text{C}$ with an error of less than 1.2%. For temperature values outside this range, it is recommended to use the exact *Wang-Engel* formulation to maintain the accuracy of plant phenology modeling.



Gambar 4: Taylor approximation vs Wang–Engel

4.7. Comparative Analysis of Methods

A comparative analysis was conducted to evaluate the numerical characteristics of the five methods used in this study, including computational complexity, accuracy level, execution speed, numerical error order, and the main applicability of each method in the context of plant phenology modeling. A summary of the comparison of the numerical characteristics of the five methods is presented in Tabel 7. The comparison results show that the *Newton–Raphson* method excels in terms of accuracy for parameter estimation, but has relatively higher computational complexity and moderate speed due to its iterative nature. The cubic spline and Simpson's 3/8 Rule methods offer a good balance between accuracy and efficiency, with high error order and fast computational performance, making them suitable for data interpolation and numerical integration. The central difference method has low complexity and very high speed, but with a more limited level of accuracy compared to higher-order methods, making it more suitable for analyzing rates of change and detecting trends. Meanwhile, the Taylor series expansion provides high accuracy at low polynomial orders in the local domain around the expansion point, with very low computational costs, making it effective for local linearization and calculation acceleration purposes.

Tabel 7: Comparison of numerical characteristics of five methods (Revised error order)

Method	Complexity	Accuracy	Speed	Error characteristics	Applicability
Newton-Raphson	High	High	Moderate	Quadratic convergence ($e_{k+1} \sim Ce_k^2$)	Parameter fitting
Cubic spline	Moderate	High	Fast	Order $O(h^4)$	Data interpolation
Simpson 3/8	Moderate	High	Fast	Order $O(h^4)$	Integration
Central difference	Low	Moderate	Very fast	Order $O(h^2)$	Derivative estimation
Taylor series	Low	High*	Very fast	Error $O(\|T - T_0\|^{n+1})$	Local approximation

*High accuracy in the local validity domain around the expansion point T_0. *High accuracy for polynomial orders less than three and in the local validity domain.

5. Conclusion

This study developed a corn phenology prediction framework based on a numerical approach integrating the Wang–Engel model, cubic spline interpolation, Simpson's 3/8 integration, numerical derivative estimation, and Taylor series approximation. Using 348 days of Indonesian meteorological data in 2025, the model produced a cumulative thermal unit accumulation of 112.37 during the first 120 days and predicted the $V3$ and $V6$ stages at days 127 and 342, respectively, according to the adopted GDD thresholds.

The cubic spline interpolation under the missing-data validation scenario yielded an RMSE of 0.657 °C, while the second-order Taylor series approximation produced a maximum error of approximately 8.412×10^{-3} within a ±7 °C range around the optimal temperature. The estimation of T_b via the Newton–Raphson method is presented as a parameter sensitivity analysis and requires further field-based validation before operational implementation.

Overall, the proposed framework provides a numerically robust, computationally efficient, and extensible foundation for tropical crop phenology modeling. Future research may expand the spatial coverage, incorporate field validation datasets, and integrate data-driven learning models for multivariate meteorological signals, supported by established results on deep neural network approximation for time-series modeling [21].

CRediT Authorship Contribution Statement

Izzar Suly Nashrudin: Conceptualization, methodology, formal analysis, programming, and original draft writing. **Nur Afiiqotul Ula**: Data curation, validation, and manuscript review and editing. **Imro Atul Khoir**: Validation, visualization, and manuscript review and editing.

Declaration of Generative AI and AI-assisted Technologies

Generative AI tools were used in a limited manner for language refinement and programming assistance. All numerical analysis, algorithm implementation, and interpretation of results were conducted by the authors.

Declaration of Competing Interest

The authors declare no competing interests.

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Data Availability

Meteorological data used in this study were obtained from the Open-Meteo Historical Weather API and are publicly available. Processed data and computational scripts are available from the corresponding author upon reasonable request.

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