

Geodesic Convexity and Hop Domination in Graph Powers: The Path Family

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Abstract. Graph powers alter distances in a predictable manner, making them a useful setting for domination parameters defined through distance. We consider convex hop domination in powers of paths. The usual distance identity for graph powers is employed only as a metric reduction: distance two in the powered graph corresponds to the condition $k < d_G(u, v) \leq 2k$ in the original graph. Our contribution is the complete exact classification for path powers with $k \geq 2$, supported by matching upper and lower bounds in the complete, diameter-two transition, and stable ranges. The case $k = 1$ is not new; it is the path result of Hassan, Canoy, and Saromines (2023) and is retained only to present the classification in a unified form. For higher powers, the argument relies on k -step geodesic progressions, endpoint-span estimates, and common-neighbor structure in the transition range. We also check the formulas exhaustively for 102 pairs with $1 \leq k \leq 6$ and $2 \leq n \leq 18$. The verification procedure, complete tested values, and executable Python code are supplied to make this computational check reproducible.

Keywords: convex hop domination; path graphs; graph powers; distance transfer; geodesic convexity; domination number; metric graph theory

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1. Introduction

Domination in graphs has developed into a broad area with structural, algorithmic, and applied directions [1]. In distance-based variants, adjacency is replaced by a prescribed distance requirement. Hop domination is one example: vertices outside the chosen set must be reached from a selected vertex at distance exactly two. Bounds for trees were obtained by Ayyaswamy et al. [2], while Henning and Jafari Rad established further structural and complexity results, including NP-completeness for restricted graph classes [3].

A second ingredient comes from graph convexity. Graph distance and intervals are treated in the standard reference [4], while geodesic convexity in graphs appears in the work of Harary and Nieminen [5] and is developed more broadly by Pelayo [6]. Convex hop domination was introduced by Hassan, Canoy, and Saromines [7]. Their Proposition 1 already gives the value for ordinary paths: $\gamma_{\text{conh}}(P_n) = 2$ for $2 \leq n \leq 5$ and $\gamma_{\text{conh}}(P_n) = n - 4$ for $n \geq 6$. Related variants include weakly convex hop domination [8] and outer-convex hop domination [9]. Thus, the unpowered case $k = 1$ is an established result and is not claimed here as new.

Graph powers offer a natural way to examine how this parameter changes when the underlying metric is compressed. For a positive integer k , two distinct vertices are adjacent in G^k whenever their distance in G is at most k . Structural properties of powers of paths and cycles have been studied in [11], with related results on chordal and other graph powers in [12], [13]. Parameters investigated specifically on path powers include acyclic and star coloring [10] and graph pebbling [14]; powers of paths and cycles also arise in extremal graph theory [15]. The closest literature therefore comes from two directions: [7] supplies convex hop domination and the $k = 1$ path result, whereas [10]–[13] concern graph powers and

their standard structural or metric properties. Among these cited works, none determines $\gamma_{\text{conh}}(P_n^k)$ for $k \geq 2$.

The new part of this paper is consequently not the distance formula for graph powers. It is the exact determination of convex hop domination for every higher power of a path, together with proofs for the different parameter ranges. Lemma 3.1 records the standard metric identity because it is used repeatedly. Proposition 3.3 specializes that identity to the hop-domination condition. The path-specific arguments begin in Section 4 and lead to the stable-range result in Theorem 5.4, the diameter-two transition result in Theorem 6.3, and the complete classification in Theorem 6.5. In that final statement, the $k = 1$ case is explicitly credited to [7].

Two consequences of taking a graph power drive the proofs. Hop domination can first be translated into an annular covering condition measured in the original path. Convexity, however, must still be checked with geodesics in the powered path. In the stable range these requirements are handled through k -step arithmetic progressions and bounds on the span of a convex hop dominating set. In the transition range the decisive structure is the common-neighbor interval of the extreme selected vertices.

The remainder of the paper is arranged as follows. Section 2 fixes the terminology, and Section 3 states the distance-transfer identity and its consequence for hop domination. Section 4 develops the path-power geometry needed later. Section 5 treats the stable range, including the inequalities required by the upper-bound construction. Section 6 handles the complete and transition ranges and combines them with the known $k = 1$ result. Section 7 describes the computational verification and the certificate algorithm. Extensions are discussed in Section 8, followed by the conclusion. The complete verification code and the 102 computed values are retained as appendices so that the main mathematical development is not interrupted.

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2. Preliminaries and Definitions

Throughout the paper, graphs are finite, simple, undirected, and connected unless stated otherwise. For a graph G , its vertex and edge sets are denoted by $V(G)$ and $E(G)$. The distance $d_G(u, v)$ is the length of a shortest u - v path, and a shortest path between u and v is called a u - v geodesic. The interval $I_G[u, v]$ contains u , v , and all vertices that lie on at least one u - v geodesic. For $S \subseteq V(G)$, $G[S]$ denotes the subgraph induced by S .

Definition 2.1. A set $S \subseteq V(G)$ is a hop dominating set if for every vertex $v \in V(G) \setminus S$ there exists $s \in S$ such that $d_G(v, s) = 2$. The minimum cardinality of a hop dominating set is the hop domination number $\gamma_h(G)$.

Definition 2.2. A set $S \subseteq V(G)$ is convex if $I_G[x, y] \subseteq S$ for every $x, y \in S$. A set is convex hop dominating if it is both convex and hop dominating. The minimum cardinality of such a set is denoted by $\gamma_{\text{conh}}(G)$.

Definition 2.3. For an integer $k \geq 1$, the k th power G^k has vertex set $V(G)$, with distinct vertices u and v adjacent in G^k if and only if $d_G(u, v) \leq k$.

The definition of a graph power used here is standard [10]–[13]. Whenever k is at least $\text{diam}(G)$, the graph G^k is complete. A complete graph has no two vertices at distance two, so a hop dominating set in that case must contain every vertex [3].

Remark 2.4. Convex hop domination imposes both hop coverage and geodesic closure. Hence every convex hop dominating set is also hop dominating, and $\gamma_h(G) \leq \gamma_{\text{conh}}(G)$; the inequality need not be an equality.

3. Distance Transfer in Graph Powers

The distance relation used below is a standard property of graph powers. We record it because both notions studied in this paper depend on distance and the identity is used throughout the later arguments. Lemma 3.1 itself is not presented as a new result.

Lemma 3.1 (standard distance-transfer identity). Let G be connected and $k \geq 1$. For all $u, v \in V(G)$,

$$d_{G^k}(u, v) = \left\lceil \frac{d_G(u, v)}{k} \right\rceil.$$

Proof. Let $d = d_G(u, v)$. Take a u - v geodesic in G and divide it into consecutive segments of length at most k . Consecutive segment endpoints are adjacent in G^k , giving a walk of length $\lceil d/k \rceil$. Hence $d_{G^k}(u, v) \leq \lceil d/k \rceil$. Conversely, every edge of G^k joins vertices whose G -distance is at most k . A G^k -path of length r therefore yields $d_G(u, v) \leq rk$ by the triangle inequality, so $r \geq \lceil d/k \rceil$. Equality follows. This is the standard metric relation underlying graph powers [4], [11], [12].

Corollary 3.2. For distinct $u, v \in V(G)$, $d_{G^k}(u, v) = 2$ if and only if $k < d_G(u, v) \leq 2k$.

Proof. By Lemma 3.1, $\lceil d_G(u, v)/k \rceil = 2$ precisely when $k < d_G(u, v) \leq 2k$.

Proposition 3.3 (annular reformulation). A set $S \subseteq V(G)$ is hop dominating in G^k if and only if for every $v \in V(G) \setminus S$ there exists $s \in S$ with $k < d_G(v, s) \leq 2k$.

Proof. Apply Corollary 3.2 to each vertex outside S . This proposition is an adaptation of the standard graph-power metric identity to the hop-domination condition; it is used as a reduction device rather than as the principal novelty of the paper.

Corollary 3.4. If $k \geq \text{diam}(G)$, then $\gamma_{\text{conh}}(G^k) = |V(G)|$.

4. Geometry of Powers of Paths

Let P_n be the path with vertices $0, 1, \dots, n-1$ in their natural order. We identify each vertex with its integer label. Then $d_{P_n}(i, j) = |i-j|$ and Lemma 3.1 gives $d_{P_n^k}(i, j) = \lceil |i-j|/k \rceil$.

Lemma 4.1. Let $a \geq 0$ and $r \geq 1$ satisfy $a + (r-1)k \leq n-1$, and put $S = \{a, a+k, \dots, a+(r-1)k\}$. Then S is convex in P_n^k .

Proof. Take $x = a+ik$ and $y = a+jk$ in S with $i < j$. Their distance in P_n^k is $j-i$. Any x - y geodesic of that length consists of powered edges whose original-path lengths are at most k . Since the total original distance is $(j-i)k$, equality in the triangle inequality forces every step to have original length exactly k and to move monotonically from x to y . Hence every geodesic vertex is $a+tk$ for $i \leq t \leq j$, all of which lie in S . Therefore $I_{P_n^k}[x, y] \subseteq S$.

Lemma 4.2. Assume $r \geq 3$ and $S = \{a, a+k, \dots, a+(r-1)k\}$. Every vertex lying strictly between two consecutive vertices of S is at distance two in P_n^k from some vertex of S .

Proof. Let $a+ik < v < a+(i+1)k$. If $i \geq 1$, use $a+(i-1)k$; otherwise use $a+2k$. The original-path distance from v to the chosen selected vertex is greater than k and at most $2k$, so Corollary 3.2 applies. The symmetric endpoint choice handles the last internal gap.

Lemma 4.3. Let $r \geq 3$ and $S = \{a, a+k, \dots, a+(r-1)k\}$. If $0 \leq a \leq 2k$ and $0 \leq n-1-[a+(r-1)k] \leq 2k$, then S is hop dominating in P_n^k .

Proof. Lemma 4.2 handles vertices in the internal gaps. For a left-tail vertex $v < a$, either $a-v > k$, in which case a hop-dominates v , or $a-v \leq k$, in which case $a+k$ is at original distance $k+(a-v)$, which lies in $(k, 2k]$. The assumption $a \leq 2k$ guarantees that every left-tail vertex is covered by one of these two selected vertices. The right tail is symmetric, using the last two selected vertices. Thus every vertex outside S is at powered distance two from S .

Corollary 4.4. Under the assumptions of Lemma 4.3, S is a convex hop dominating set of P_n^k .

5. Exact Convex Hop Domination in the Stable Range

We next determine the value in the stable range. The construction for the upper bound uses a k -step arithmetic progression, whereas the lower bound follows from restrictions on how far the extreme vertices of any convex hop dominating set can be separated.

Lemma 5.1. Let S be a convex hop dominating set of P_n^k , and let $p = \min S$ and $q = \max S$. Then $p \leq 2k$ and $q \geq n-1-2k$.

Proof. If $0 \in S$ then $p = 0$. Otherwise, because S hop dominates 0 , Proposition 3.3 gives a selected vertex s with $k < s \leq 2k$, hence $p \leq 2k$. The argument at $n-1$ is symmetric.

Lemma 5.2. If S is convex in P_n^k and $p = \min S < q = \max S$, then $|S| \geq \lceil (q-p)/k \rceil + 1$.

Proof. Convexity requires S to contain every vertex of every p - q geodesic. In particular, it contains the vertices of one shortest p - q path. By Lemma 3.1 that path has length $\lceil (q-p)/k \rceil$ and therefore contains $\lceil (q-p)/k \rceil + 1$ vertices.

Lemma 5.3. If $k \geq 2$ and $n \geq 2k+1$, then every convex hop dominating set of P_n^k has at least three vertices.

Proof. A singleton cannot be a hop dominating set: if $S = \{x\}$, then any path neighbor of x that exists is at powered distance one from x , not two; when x is an endpoint, the same conclusion holds for its unique path neighbor. Now suppose $S = \{x, y\}$ with $x < y$. If $y-x > k$, then $d_{P_n^k}(x, y) \geq 2$. Every x - y geodesic therefore has at least one internal vertex, and convexity would force that vertex into S , contradicting $|S| = 2$. It remains to consider $y-x \leq k$. If $y-x \geq 2$, any integer vertex z with $x < z < y$ is outside S and satisfies $|z-x| \leq k$ and $|y-z| \leq k$; hence z is at powered distance one from both selected vertices and is not hop dominated. Thus $y = x+1$. Since $n \geq 2k+1 \geq 5$, at least one of $x-1$ or $y+1$ is a path vertex; choose such a vertex z . Because $k \geq 2$, its original-path distance from each available selected endpoint is at most $2 \leq k$, so its powered distance from every vertex of S is one. Again z is not hop dominated. Therefore no convex hop dominating set has one or two vertices, and $|S| \geq 3$.

Theorem 5.4 (stable-range exact formula). For integers $k \geq 2$ and $n \geq 2k+1$,

$$\gamma_{\text{conh}}(P_n^k) = \max\{3, \lceil (n-1)/k \rceil - 3\}.$$

Proof. Let S be any convex hop dominating set, with $p = \min S$ and $q = \max S$. Lemma 5.1 gives $p \leq 2k$ and $q \geq n-1-2k$. Therefore $q-p \geq n-1-4k$. By Lemma 5.2,

$$|S| \geq \left\lceil \frac{q-p}{k} \right\rceil + 1 \geq \left\lceil \frac{n-1}{k} \right\rceil - 3.$$

Lemma 5.3 gives the independent lower bound $|S| \geq 3$. Thus $|S| \geq r$, where $r = \max\{3, \lceil (n-1)/k \rceil - 3\}$.

For the reverse inequality, set $r = \max\{3, \lceil (n-1)/k \rceil - 3\}$ and define

$$a = \max\{0, n-1-2k-(r-1)k\},$$

$$S = \{a, a+k, \dots, a+(r-1)k\}.$$

We verify explicitly the inequalities required by Lemma 4.3. First, $a \geq 0$ by definition. If $a = 0$ then $a \leq 2k$. If $a > 0$, then $a = n-1-2k-(r-1)k$. Since $r \geq \lceil (n-1)/k \rceil - 3$, we have $r+3 \geq \lceil (n-1)/k \rceil$, and hence $(r+3)k \geq n-1$. Therefore $n-1-(r-1)k \leq 4k$, which gives $a \leq 2k$.

Second, the progression stays inside the path. If $a = 0$, then $r = \max\{3, \lceil (n-1)/k \rceil - 3\}$. In the case $r = 3$, $n \geq 2k+1$ gives $2k \leq n-1$, while $r = 3$ together with $\lceil (n-1)/k \rceil \leq 6$ gives $(r-1)k = 2k \leq n-1$. If $r = \lceil (n-1)/k \rceil - 3 > 3$, then $r-1 = \lceil (n-1)/k \rceil - 4$ and therefore $(r-1)k \leq n-1$. If $a > 0$, then $a+(r-1)k = n-1-2k \leq n-1$. Thus in all cases $a+(r-1)k \leq n-1$.

Finally, the right-tail length is at most $2k$. If $a > 0$, the preceding equality gives $n-1-[a+(r-1)k] = 2k$. If $a = 0$, its definition implies $n-1-2k-(r-1)k \leq 0$, hence $n-1-(r-1)k \leq 2k$. The right-tail length is also nonnegative because the progression lies inside the path. Therefore $0 \leq n-1-[a+(r-1)k] \leq 2k$.

We have now proved $0 \leq a \leq 2k$, $a+(r-1)k \leq n-1$, and a right tail of length at most $2k$. Since $r \geq 3$, all hypotheses of Lemma 4.3 and Corollary 4.4 hold. Hence S is convex hop dominating and $\gamma_{\text{conh}}(P_n^k) \leq r$. Combined with the lower bound, equality follows.

Corollary 5.5. If $k \geq 2$ and $2k+1 \leq n \leq 6k+1$, then $\gamma_{\text{conh}}(P_n^k) = 3$.

Corollary 5.6. For fixed $k \geq 2$, $\gamma_{\text{conh}}(P_n^k) = n/k + O(1)$ as n tends to infinity.

6. Completion of the Exact Classification

The stable range $n \geq 2k+1$ for $k \geq 2$ is settled by Theorem 5.4. We now consider the remaining complete and diameter-two ranges. The case $k = 1$ is included only to complete the final statement, since its value was already established in [7].

Proposition 6.1 (complete regime). If $k \geq 2$ and $2 \leq n \leq k+1$, then $\gamma_{\text{conh}}(P_n^k) = n$.

Proof. Here $\text{diam}(P_n) = n-1 \leq k$, so P_n^k is complete. Corollary 3.4 applies.

Lemma 6.2. Let $k \geq 2$ and $k+2 \leq n \leq 2k$. Let S be a convex hop dominating set of P_n^k , with $p = \min S$ and $q = \max S$. If $q-p > k$, then S contains p, q , and every common neighbor of p and q , namely all vertices in the interval $[q-k, p+k]$.

Proof. Since $q-p > k$ and $q-p \leq n-1 \leq 2k-1$, Lemma 3.1 gives $d_{P_n^k}(p, q) = 2$. A vertex z lies on a p - q geodesic of length two exactly when it is adjacent to both p and q . On a path power this is equivalent to $q-k \leq z \leq p+k$. Convexity requires every vertex on every p - q geodesic to belong to S , so the entire common-neighbor interval $[q-k, p+k]$ is contained in S .

Theorem 6.3 (diameter-two transition). If $k \geq 2$ and $k+2 \leq n \leq 2k$, then

$$\gamma_{\text{conh}}(P_n^k) = \min\{k+1, 2k-n+4\}.$$

Proof. For the first upper bound, $S_1 = \{0, 1, \dots, k\}$. Any two vertices of S_1 are adjacent, so S_1 is convex. If $v > k$, then $v-0 = v$ satisfies $k < v \leq n-1 \leq 2k-1$; hence 0 hop-dominates v . Thus S_1 is convex hop dominating and has size $k+1$. For the second construction, put $B = \{n-1-k, \dots, k\}$ and $S_2 = \{0, n-1\} \cup B$. Because $k+2 \leq n \leq 2k$, B is nonempty. The extreme vertices 0 and $n-1$ are at powered distance two, and their common-neighbor interval is exactly B . All other selected pairs are adjacent or have their geodesic interval contained in S_2 , so S_2 is convex. For any unselected v with $1 \leq v < n-1-k$, the distance $n-1-v$ lies in $(k, 2k]$; hence $n-1$ hop-dominates v . Symmetrically, if $k < v \leq n-2$, then $v-0$ lies in $(k, 2k]$, so 0 hop-dominates v . Therefore S_2 is convex hop dominating and $|S_2| = 2k-n+4$.

For the lower bound, let S be any convex hop dominating set, with $p = \min S$ and $q = \max S$. First suppose $q-p \leq k$. Every vertex z that is within original-path distance at

most k from both p and q is at powered distance one from every selected vertex, because every $s \in S$ lies between p and q . Such a vertex cannot remain outside S . Hence $C = [\max\{0, q-k\}, \min\{n-1, p+k\}]$ must be contained in S . Since p and q are the extreme selected vertices, C cannot extend strictly to the left of p or strictly to the right of q . Therefore either $q-k = p$ or $q \leq k$ on the left, and either $p+k = q$ or $p+k \geq n-1$ on the right. If neither equality $q-k = p$ nor $p+k = q$ held, then $q \leq k$ and $p+k \geq n-1$, so C would be the whole vertex set; this would force $p = 0$ and $q = n-1$, contradicting $q-p \leq k$ because $n-1 \geq k+1$. Thus $q-p = k$. Consequently $[p, q] \subseteq C \subseteq S$ and $|S| \geq k+1$. Now suppose $q-p > k$. By Lemma 6.2, S contains p , q , and every vertex of $[q-k, p+k]$. The interval is nonempty because $q-p \leq n-1 \leq 2k-1$, and it is disjoint from $\{p, q\}$. Hence $|S| \geq 2 + (p+k-q+k+1) = 2k-(q-p)+3 \geq 2k-n+4$. In the first case $|S| \geq k+1$, and in the second $|S| \geq 2k-n+4$. Therefore every feasible S satisfies $|S| \geq \min\{k+1, 2k-n+4\}$. Together with the two constructions above, this proves the formula.

Proposition 6.4 (known ordinary-path case [7, Proposition 1]). For $n \geq 2$,

$$\gamma_{\text{conh}}(P_n) = \max\{2, n-4\}.$$

This is exactly the ordinary-path result proved by Hassan, Canoy, and Saromines [7, Proposition 1]. We include it for completeness and do not regard it as part of the new contribution of this paper.

Theorem 6.5 (complete classification). Let $n \geq 2$ and $k \geq 1$. Then

Table 1: Complete classification of convex hop domination for path powers.

Regime	Exact value
$k = 1$	$\max\{2, n-4\}$ [7]
$k \geq 2, 2 \leq n \leq k+1$	n
$k \geq 2, k+2 \leq n \leq 2k$	$\min\{k+1, 2k-n+4\}$
$k \geq 2, n \geq 2k+1$	$\max\{3, \lceil(n-1)/k\rceil - 3\}$

Proof. The $k = 1$ line is the known result [7, Proposition 1]. The three $k \geq 2$ lines are Proposition 6.1, Theorem 6.3, and Theorem 5.4, respectively. These regimes are disjoint and cover every $n \geq 2$ and $k \geq 1$. Thus the new higher-power results, together with the established ordinary-path result, yield a unified classification.

7. Reproducible Computational Verification and Certificate Algorithm

The classification is proved analytically; the computation serves only as a small-instance check that can be reproduced independently of the proofs. For each pair (n, k) , the program forms the powered-distance matrix from $d_{P_n^k}(i, j) = \lceil |i-j|/k \rceil$ and examines candidate subsets in increasing order of cardinality. A subset passes the hop-domination test when each vertex outside it is at powered distance two from at least one selected vertex. Convexity is tested directly from graph intervals: whenever $d(x, z) + d(z, y) = d(x, y)$ for selected x and y ,

the vertex z must also be selected. The first cardinality satisfying both conditions is returned as $\gamma_{\text{conh}}(P_n^k)$. No structural pruning derived from Theorem 6.5 is used.

The test was run for all 102 pairs with $1 \leq k \leq 6$ and $2 \leq n \leq 18$. In every case, the computed minimum coincided with the value in Theorem 6.5. Table 1 gives the tested ranges and outcomes, and Appendix B lists the value obtained for each pair. The complete Python program is provided in Appendix A. Thus, the computational statement can be checked from the code and reported output rather than relying on an unavailable implementation.

Table 2: Exhaustive reproducibility check of the complete classification.

k tested	n values	Instances	Outcome
1	2–18	17	Match
2	2–18	17	Match
3	2–18	17	Match
4	2–18	17	Match
5	2–18	17	Match
6	2–18	17	Match
Total		102	No discrepancy

The proofs also provide explicit optimal certificates. For $k = 1$, the known optimal sets from [7] may be used. For $k \geq 2$ and $n \leq k+1$, the certificate is the full vertex set. In the transition range, one takes the smaller of $S_1 = \{0, \dots, k\}$ and $S_2 = \{0, n-1\} \cup \{n-1-k, \dots, k\}$. In the stable range, set $r = \max\{3, \lceil(n-1)/k\rceil - 3\}$ and $a = \max\{0, n-1-2k-(r-1)k\}$, and return $\{a, a+k, \dots, a+(r-1)k\}$.

Proposition 7.1. For every $n \geq 2$ and $k \geq 1$, an optimal convex hop dominating set of P_n^k can be generated in time $O(\gamma_{\text{conh}}(P_n^k))$ and $O(1)$ auxiliary space, apart from the returned set.

Proof. Apply the regime-specific explicit construction above. Each selected vertex is produced by constant-time arithmetic and no search or optimization routine is required.

8. Discussion and Extensions

8.1. Powers of cycles

The same distance transfer applies to powers of cycles, but their intervals do not have the linear structure available for paths. If k is at least $\text{diam}(C_n)$, the powered cycle is complete and Corollary 3.4 applies. Below that threshold, shortest paths may run in both directions around the cycle, which can enlarge the convex closure. For this reason, the path formula does not immediately extend to cycles and a separate analysis is required.

8.2. General graph classes

Lemma 3.1 and Proposition 3.3 are valid for any connected graph. They separate the problem into two tasks: hop coverage may be expressed as an annular condition in the metric of G , while convexity depends on the intervals of G^k . This separation suggests examining graph classes whose powered intervals retain enough structure to be described explicitly, in-

cluding trees, distance-hereditary graphs, block graphs, and chordal graphs.

8.3. Complexity

Hop domination remains NP-complete on restricted graph classes [3], and the complexity of convex hop domination was already identified as an open direction in [7]. The closed formulas obtained here show that path powers form a tractable structured family with direct optimal certificates. They do not, however, resolve the complexity question for more general graph powers.

9. Conclusion

We have determined the convex hop domination number for every path power with $k \geq 2$ and combined these values with the known $k = 1$ case from [7] in a single classification. The standard distance formula for graph powers is used only to translate the metric condition. The new arguments concern the convex structure of powered paths, the stable-range construction and lower bound, and the diameter-two transition range. In particular, the stable-range proof now states explicitly the inequalities needed for its progression construction. The computational check is reproducible from the supplied algorithm, Python implementation, and complete list of 102 tested values. The same distance-transfer viewpoint may be useful for studying convex hop domination in other graph-power families.

Author Contributions

Karrar Khudhair Obayes: Conceptualization, methodology, formal analysis, investigation, computational verification, writing – original draft, and writing – review and editing. Ghadeer Khudhair Obayes: Validation, literature review, verification of arguments, and writing – review and editing. Both authors read and approved the final manuscript.

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A. Python Implementation for Exhaustive Verification

The program below implements the exhaustive procedure described in Section 7 using only the Python standard library.

```
from itertools import combinations
from math import ceil

def powered_distance(i, j, k):
    return ceil(abs(i - j) / k)

def is_hop_dominating(S, n, k):
    S = set(S)
    for v in range(n):
        if v not in S and not any(powered_distance(v, s, k) == 2 for s in S):
            return False
    return True

def is_convex(S, n, k):
    S = set(S)
    for x, y in combinations(sorted(S), 2):
        dxy = powered_distance(x, y, k)
        for z in range(n):
            if powered_distance(x, z, k) + powered_distance(z, y, k) == dxy:
                if z not in S:
                    return False
    return True

def gamma_conh_bruteforce(n, k):
    V = range(n)
    for r in range(1, n + 1):
        for S in combinations(V, r):
            if is_hop_dominating(S, n, k) and is_convex(S, n, k):
                return r, S
    return n, tuple(V)

def theorem_value(n, k):
    if k == 1:
        return max(2, n - 4)
    if n <= k + 1:
        return n
    if n <= 2 * k:
        return min(k + 1, 2 * k - n + 4)
    return max(3, ceil((n - 1) / k) - 3)

for k in range(1, 7):
    for n in range(2, 19):
        value, certificate = gamma_conh_bruteforce(n, k)
        expected = theorem_value(n, k)
        assert value == expected, (n, k, value, expected)
        print(n, k, value, certificate)
```

B. Complete Values for the 102 Tested Instances

In each row, the entries correspond successively to $n = 2, 3, \dots, 18$. They are obtained by exhaustive subset enumeration, after which the program checks them against Theorem 6.5.

Table 3: Complete computed values for all 102 tested instances.

k	$n = 2, \dots, 18$: computed $\gamma_{\text{conh}}(P_n^k)$
1	2, 2, 2, 2, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14
2	2, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 4, 4, 5, 5, 6
3	2, 3, 4, 4, 4, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3
4	2, 3, 4, 5, 5, 5, 4, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3
5	2, 3, 4, 5, 6, 6, 6, 5, 4, 3, 3, 3, 3, 3, 3, 3, 3
6	2, 3, 4, 5, 6, 7, 7, 7, 6, 5, 4, 3, 3, 3, 3, 3, 3